Total Radiated Light Per Frequency from the Cooled Black Body with Zero Chemical Potential

By Kadir Aydoğdu
student from
MIDDLE EAST TECHNICAL UNIVERSITY – DEPARTMENT OF PHYSICS
e-mail: kadir.aydogdu@metu.edu.tr

Non-Equilibrium Thermodynamic System
Temperature Dependency of the Planck's Law and Radiation Constant
Time Integral of the Planck's Law
Integral of Inverse of Exponential of Exponential Function
Numerical Analysis of the Function
Series Solution of the Function
Discussions about Empirical Function and Unknown Exact Solution

11 June 2017 (Sunday)

ABSTRACT

To understand the relation between temperature and black body radiation which is continuous photon radiation, we are using the Planck’s Law and Stefan-Boltzmann Law, to model the heat transfer. Moreover to find the total energy of the free vacuum we are using the radiation constant which is only dependent to the temperature and the volume. However in this project to understand the mechanism behind the vacuum energy, our aim is to find the total radiated light from the black body until it lost all the energy. By deriving this function we will be able to speak about the total radiation potential of non-zero temperature free space.

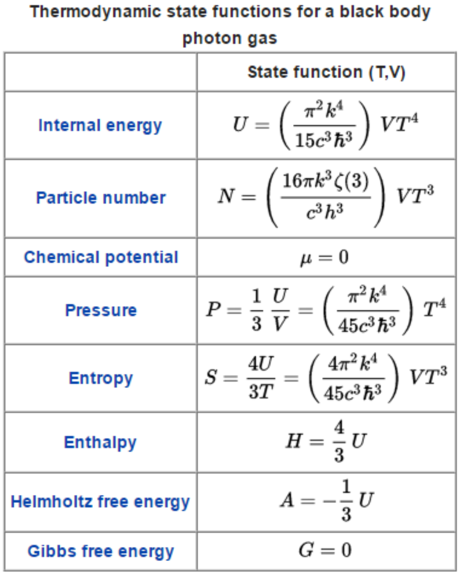
We will start with analyzing Planck’s Law and its temperature dependency then we will write our function as a time dependent integral. Afterwards, we will try to solve it with numeric analysis and series solution to find the function we need.

INDEX

INFORMATION

The first important equation of the quantum mechanics is the ***Planck Law*** which gives the intensity or energy per frequency with respect to the temperature only. By using the law we can calculate how much energy produces a hot object at any frequency and how heat flows as light form hot to cold. Then by integrating the Planck law at every wavelength and in every direction there is ***Stefan-Boltzmann law*** which gives the total radiation power of a surface area ***A*** at temperature ***T***. Afterwards multiplying the Stefan-Boltzmann law by ***4/c*** there is ***radiation constant*** which gives the total radiation energy or internal energy of a constant volume even if there is no particle except light inside the volume.

In thermodynamics, these laws result a new topic which is ***photon gas***. In photon gas theory, all the radiation inside the box or volume modeled as a light which carries energy and momentum, no mass and goes with speed of light. We know all the information about the photon gas because it is modeled as a simple ideal gas. First thing we have to know about photon gas, it is a collection of photons inside a box and we assume that box has no chemical potential energy, moreover, it has no chemical interaction and for this reason we are assuming the box as a finite vacuumed volume. As we said before, we know that radiation constant gives us the total radiation energy or internal energy inside the constant volume at constant temperature and we can say that this internal energy is the amount of heat. So, in our question we should calculate everything only with respect to temperature and volume but the problem is, we have a non-equilibrium thermodynamic system and we can't calculate everything exactly because only in constant temperature systems it is possible. Because we are changing the temperature with time we need to find all the radiation with respect to time which results decreasing the power. The result should be again the distributed light which has continuous property means it includes all wavelength photons.

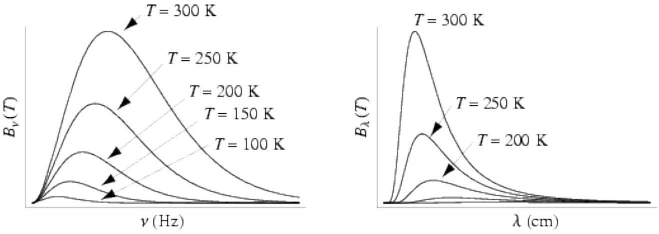


Thermodynamic state functions for a black body photon gas

	State function (T,V)
Internal energy	$U = \left(\frac{\pi^2 k^4}{15 c^3 \hbar^3}\right) V T^4$
Particle number	$N = \left(\frac{16 \pi k^3 \zeta(3)}{c^3 h^3}\right) V T^3$
Chemical potential	$\mu = 0$
Pressure	$P = \frac{1}{3}\frac{U}{V} = \left(\frac{\pi^2 k^4}{45 c^3 \hbar^3}\right) T^4$
Entropy	$S = \frac{4U}{3T} = \left(\frac{4\pi^2 k^4}{45 c^3 \hbar^3}\right) V T^3$
Enthalpy	$H = \frac{4}{3} U$
Helmholtz free energy	$A = -\frac{1}{3} U$
Gibbs free energy	$G = 0$

For photon gas theory we have instant photon distribution which is dependent to the internal energy which is dependent to the ***absolute temperature*** of the box. But in our question to find a relation between the temperature and the total radiated light ***we should decrease the internal energy with time*** and we add or sum up all the radiation with time. So, decrease in internal energy result decrease in power and distribution and we should calculate it until there is no internal energy left.

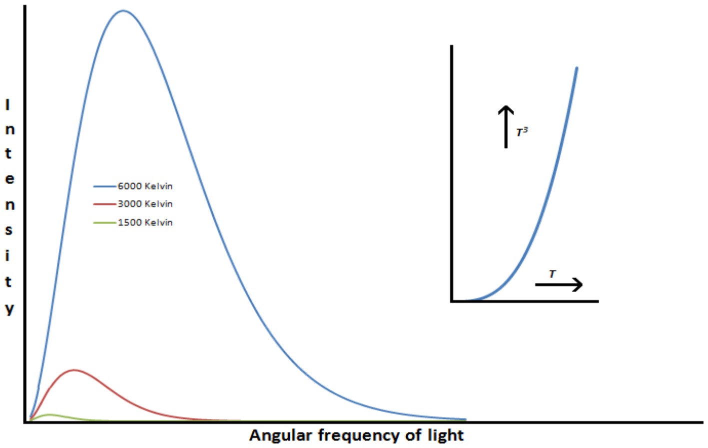
The important thing for us is to see how the energies or intensities of photons change in Planck's law at every wavelength with respect to the temperature. The solution comes as in a constant temperature, in black body distribution every photons energies proportional to the temperature and the number of the photon is proportional to the $\boldsymbol{T^3}$. Because total energy comes as a $\boldsymbol{T^4}$ and with respect to the temperature every photon changes its energy in constant volume. With a known temperature known photons become other photons because distribution is always constant and it doesn't change. So, blackbody distribution and its shape which gives the spectra is always conserves itself. And it is simply comes as $\boldsymbol{x^3/(e^x - 1)}$.



As you can see both the total energy is proportional with T^4 and the power is proportional with T^4, and from this proportionality if we assume that ***if box is not supplied with heat*** we can say that with a constant surface area, radiation power and the internal energy changes ***exponentially with time*** because both of them dependent to the each other with T^4. If the total energy of the box changes ***exponentially*** with time than we can say that temperature's fourth power changes ***exponentially*** which means temperature is still changing exponentially, as you know, fourth power of exponential decrease dependency is still an exponential function. Another important thing in the Planck law is it always conserves its shape in every condition which is only dependent to the temperature. If you compare the spectra of the two temperatures with respect to the intensity and frequency of light you can see number of photons increase with T^3 and average photon energy increase with T which says simply ***every photon become another photon*** with changing temperature inside the box.

V=volume (=m^3)	$a=\frac{\pi^2\ {k_B}^4}{15\,\hbar^3 c^3}=\frac{8\pi^5\ {k_B}^4}{15\,h^3\,c^3}$
A=area (=m^2)	$\sigma=\frac{\pi^2\ {k_B}^4}{60\,\hbar^3\,c^2}=\frac{2\pi^5\ {k_B}^4}{15\,h^3\,c^2}$

Planck's law	$=\frac{\hbar\,\omega^3}{4\pi^3\,c^2}\frac{1}{(e^{\frac{\hbar\,\omega}{k_B\,T}}-1)}$	$=\frac{2\,h\,v^3}{c^2}\frac{1}{(e^{\frac{h\,v}{k_B\,T}}-1)}$
Stefan-Boltzmann law	$=\frac{\pi^2\ {k_B}^4\,T^4}{60\,\hbar^3\,c^2}A$	$=\frac{2\pi^5\ {k_B}^4\,T^4}{15\,h^3\,c^2}A$
Radiation constant	$U=\frac{\pi^2\ {k_B}^4\,T^4}{15\,\hbar^3\,c^3}V$	$U=\frac{8\pi^5\ {k_B}^4\,T^4}{15\,h^3\,c^3}V$



If we examine the ***Planck's law*** which gives the power of the light radiation coming from the constant surface area ***A***, at temperature ***T***, at every frequency, its shape is constant as we discusses before and if we look the Wien's law we can understand how number of photons changes with respect to the temperature, our graphs peak point goes with T^3 but the area under the curve which is the integral of Planck law or simply ***Stefan-Boltzmann law*** radiation power is proportional to T^4 and total energy is proportional to T^4 so, if we ask how energy decreases with time, answer is simply exponential decay function.$(U(t)=\ U_0e^{-x\,t}$ ***(t is time))***

$$\frac{d}{dt}(U)=Power=\sigma T^4=(-A\,\frac{\pi^2\,{k_B}^4 T^4}{60\,\hbar^3 c^2})=\frac{d}{dt}(\frac{\pi^2\,{k_B}^4\,T^4}{15\,\hbar^3\,c^3}V)$$

From this relation we can understand that temperature must decrease exponentially because its fourth power is proportional to energy. So we should write temperatures time dependency***;***

$$\frac{dT}{dt}=-\frac{A\,\sigma\,T}{4\,V\,a}=-\frac{A\,\sigma\,T}{4\,V\,\sigma\frac{4}{c}}=-\frac{A\,c\,T}{V\,16}$$ so, temperature function becomes $T=T_0\,e^{-\frac{c\,A\,t}{16\,V}}$

Than what will we do is simply use temperature in the Planck's law because we need total radiated light until our box temperature becomes ***0***.

The function of the ***total radiated intensity per frequency*** of the cooled black body box;

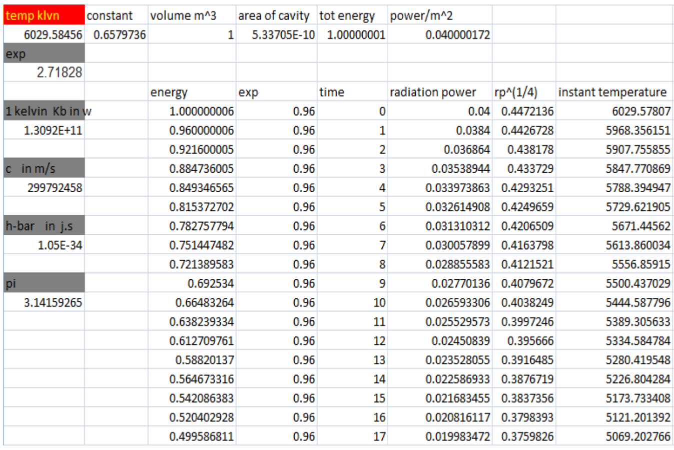
$$\int_{0}^{\infty} A\,\boldsymbol{\frac{\hbar\,\omega^{3}}{4\,\pi^{2}\,c^{2}}}\,\boldsymbol{\frac{dt}{e^{\left(\frac{\hbar\,\omega\,e^{\left(\frac{c\,A\,t}{16\,V}\right)}}{(k_B\,T_0\,)}\right)}-1}}$$

If we separate from the constants

$$A\,\boldsymbol{\frac{\hbar\,\omega^{3}}{4\,\pi^{2}\,c^{2}}}\quad\int_{0}^{\infty}\boldsymbol{\frac{dt}{e^{\left(\frac{\hbar\,\omega\,e^{\left(\frac{c\,A\,t}{16\,V}\right)}}{(k_B\,T_0\,)}\right)}-1}}$$

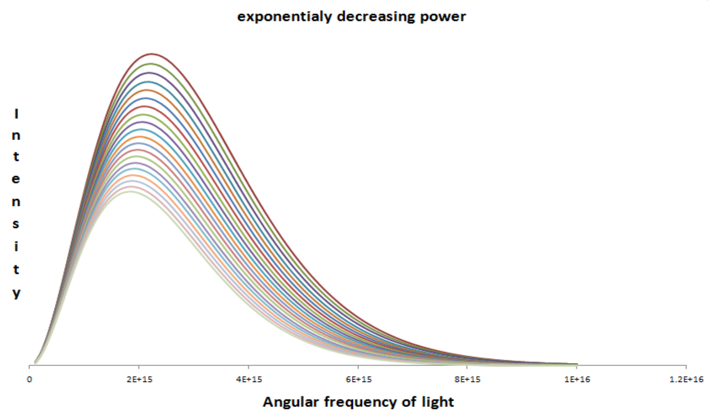
As we can see integral is only ***time dependent*** and we have lots of constant that are not dependent so, we can write it easily like $\int_0^\infty \frac{dx}{(e^{(\beta * e^{\gamma * x})})-1}$ but if we want to write in integral in most simple form it becomes $\int_0^\infty \frac{dx}{(e^{e^{x}})-1}$.

NUMERICAL ANALYSIS

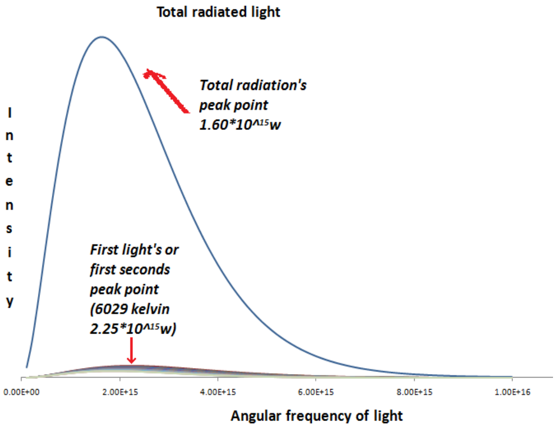


temp klvn	constant	volume m^3	area of cavity	tot energy	power/m^2		
6029.58456	0.6579736	1	5.33705E-10	1.00000001	0.040000172		
exp							
2.71828							
		energy	exp	time	radiation power	rp^(1/4)	instant temperature
1 kelvin Kb in w		1.000000006	0.96	0	0.04	0.4472136	6029.57807
1.3092E+11		0.960000006	0.96	1	0.0384	0.4426728	5968.356151
		0.921600005	0.96	2	0.036864	0.438178	5907.755855
c in m/s		0.884736005	0.96	3	0.03538944	0.433729	5847.770869
299792458		0.849346565	0.96	4	0.033973863	0.4293251	5788.394947
		0.815372702	0.96	5	0.032614908	0.4249659	5729.621905
h-bar in j.s		0.782757794	0.96	6	0.031310312	0.4206509	5671.44562
1.05E-34		0.751447482	0.96	7	0.030057899	0.4163798	5613.860034
		0.721389583	0.96	8	0.028855583	0.4121521	5556.85915
pi		0.692534	0.96	9	0.02770136	0.4079672	5500.437029
3.14159265		0.66483264	0.96	10	0.026593306	0.4038249	5444.587796
		0.638239334	0.96	11	0.025529573	0.3997246	5389.305633
		0.612709761	0.96	12	0.02450839	0.395666	5334.584784
		0.58820137	0.96	13	0.023528055	0.3916485	5280.419548
		0.564673316	0.96	14	0.022586933	0.3876719	5226.804284
		0.542086383	0.96	15	0.021683455	0.3837356	5173.733408
		0.520402928	0.96	16	0.020816117	0.3798393	5121.201392
		0.499586811	0.96	17	0.019983472	0.3759826	5069.202766

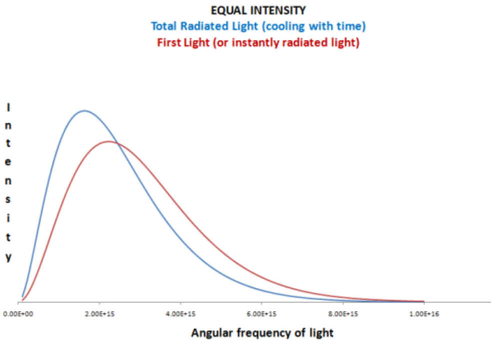
This integral has ***no exact solution*** at least it is not known yet, (*I have tried to solve it but I couldn't find a way and after lots of useless conversation I decided to solve it numerically, however I changed the way I go and* ***I calculated everything from the energy*** *not from the time*). In the end it is ***convergent***, and we can solve it by ***numerical analysis*** but, we should start from the energy calculation because it is too simple for us to understand. To start, we can take initial energy as ***1 joule*** in ***1*** m^3 with temperature of about ***6029 kelvin*** and we should have decreased it exponentially with four percent at every second (about every 17seconds energy becomes half gone), than we can found the temperature of the every second and we should calculate the time about ***1000 seconds*** when almost there is no energy. In this calculation our area is very small because as we can understand that cooling rate is dependent to the surface area and our ***area is smaller than needle***, about **5.33***$10^{-10}m^2$**.**



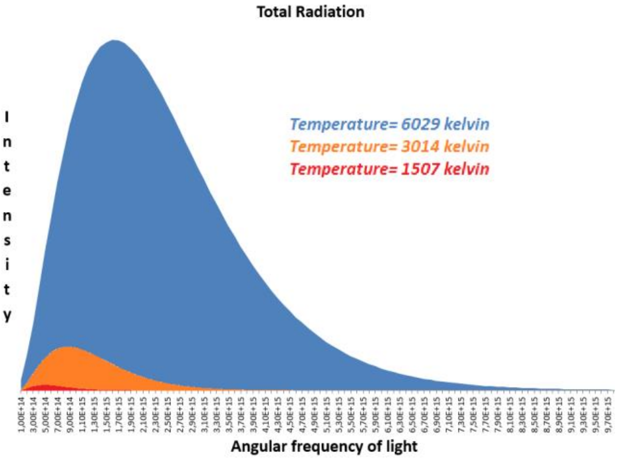
After finding the intensity of every second at every frequency we have total radiation. In the graph we can see the ***first 20 second*** radiation and it ***exponentially*** decreases. What will we do is to sum up all the data which is the logic of integration. After ***1000 seconds*** we can say that there is almost no energy left.



After we sum all the 1000 second data we can see a line in the upper part which shows the total radiated light in every angular frequency, and in the lower part of the graph we can see ***first 20 seconds*** radiation again (*It is not clearly seen because of 20 tight lines*). We can see the total radiations peak point is in the lower frequency range than the Wien's peak point of ***6029 kelvin*** temperature because with the time passes temperature decreases and peak points of every seconds moves to lower frequency or higher wavelength range. For comparison we can look at the graph in equal intensity which means thermodynamically two cases have same energy, but frequencies of the photons are different. In current theory all the heat is light as red line inside the box, but for us heat becomes the light after there is no heat inside the box as blue line.



But we have calculated everything only for one temperature which is 6029 kelvin so, we should change the temperature and we should try to understand how this graph changes with respect to the temperature. Good thing is graph's temperature proportionality is exactly the same with the Planck's law, most probably the constant coming from the integral have no effect on the Planck's distribution. The number of photon increases with T^3 and energy of every photon increases with T (or we can say average energy of photons increases with T) and absolutely the area under this graph gives us T^4 which gives the total internal energy that is heat. If we want to give an example; if we ***double the temperature*** again, height of the graph increases ***8 times*** and width of the graph increases ***2 times***.



We should be interested when we see the result because it was good to see an integral of exponential of exponential function becomes ***same type of increase in the graph*** with respect to the temperature and the good thing is we can easily understand that the solution must be proportional with the Planck's Law then we can easily say there must be a function to multiply the Planck's Law and it becomes our solution.

$$A \frac{\hbar\, \omega^3}{4\pi^3\, c^2} \int_0^\infty \frac{dt}{e^{\left(\frac{\hbar\, \omega\, e^{\left(\frac{c\, A\, t}{16\, V}\right)}}{(k_B\, T_0\,)}\right)} - 1}$$

But, we have ***Area A*** in the function however, our function must be independent from the ***A*** moreover, it must be dependent to the ***Volume V*** because it will give the height of the graph and it is a linear multiplier for our function. If we think that we need a function will give the total radiated light per wavelength it must be like

$$f\ \ V \frac{\hbar\, \omega^3}{4\pi^3\, c^3} \frac{1}{\left(e^{\frac{\hbar\, \omega}{k_B\, T}} - 1\right)}$$

Because, it has same type of increase and integral is independent from $\boldsymbol{\omega}$.

So, we should be annihilate the ***A*** in the integral and to write an energy function we need to multiply it with ''***V/c***'' because we need an energy function for a volume.

SERIES SOLUTION

Our integral is in the form of $I = \int_0^\infty \frac{dx}{(e^{(\beta\, e^{\gamma\, x})})-1}$

and we will assume $u = \beta * e^{\gamma * x}$

so, we have $du = \gamma\ \beta\ e^{\gamma * x}\ dx$ and it is equal to $du = \gamma\ u\ dx$

now we have the form of $I = \int_\beta^\infty \frac{\frac{1}{\gamma\, u} du}{e^u - 1} \rightarrow \gamma I = \int_\beta^\infty \frac{\frac{1}{u} du}{e^u - 1}$

$\rightarrow \gamma I = \int_\beta^\infty \frac{1}{e^u - 1} \frac{du}{u} \rightarrow \gamma I = \int_\beta^\infty \frac{e^{-u}}{1 - e^{-u}} \frac{du}{u}$

Note that

$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k ,$ *and* $\frac{x}{1-x} = \frac{e^{-u}}{1-e^{-u}} ,$

$$\frac{1-x^n}{1-x} = \sum_{k=0}^{n-1} x^k , \quad and\ if\ n \to n+1 \quad \frac{1-x^{n+1}}{1-x} = \sum_{k=0}^{n} x^k$$

$$\frac{1}{1-x} - \frac{x^{n+1}}{1-x} = 1 + \sum_{k=1}^{n} x^k \quad and \quad \frac{1}{1-x} - 1 = \frac{x^{n+1}}{1-x} + \sum_{k=1}^{n} x^k$$

$$\frac{1-(1-x)}{1-x} = \frac{x}{1-x} = \frac{x^{n+1}}{1-x} + \sum_{k=1}^{n} x^k \to \frac{e^{-u}}{1-\ e^{-u}} = \frac{e^{-(n+1)u}}{1-e^{-u}} + \sum_{k=1}^{n} e^{-ku}$$

$$\gamma I = \int_\beta^\infty \frac{e^{-u}}{1-\ e^{-u}} \frac{du}{u} = \sum_{k=1}^{n} \int_\beta^\infty \frac{e^{-ku}}{u} du + \int_\beta^\infty \frac{e^{-(n+1)u}}{u(1-e^{-u})} du$$

This integral is in the form of an **Exponential series integral + Residue**
Totally we have a form of

$$\gamma I = \int_\beta^\infty \frac{e^{-u}}{1-\ e^{-u}} \frac{du}{u} = \sum_{k=1}^{n} E_i(k\varphi) + R_n$$

Because of it is infinite series and it is not possible to convert it any kind of function or even empirical function we can't use it as a function, only thing we can do is we can use its table value in the book of ''Handbook of Mathematical Functions'' (*Abramowitz&Stegun*) and we can calculate it until the nth and we can calculate its error function which gives the limit of the sum of until the nth. So, if we want to convert it at least an empirical formula we can't use the series solution.

DISCUSSIONS ABOUT SOLUTION

As we discussed before we have ***same type of increase in the graph*** with respect to the temperature and we can easily understand that the solution must be proportional with the Planck’s Law then we can easily say that there must be a function to multiply the Planck’s Law and it becomes our solution.

$$\int_0^\infty \mathbf{A}\frac{\hbar\omega^3}{4\pi^2\mathbf{c}^2}\frac{\mathbf{dt}}{\mathbf{e}^{\frac{\hbar\omega}{\mathbf{k_BT}}}\mathbf{e}^{\frac{\mathbf{cAt}}{\mathbf{16V}}}-1}$$

But, we have ***Area A*** in the function however, our function must be independent from the ***A*** moreover, it must be dependent to the ***Volume V*** because it will give the height of the graph and it is a linear multiplier for our function and these constants comes from the γ inside the series solution. We should be annihilate the ***A*** in the integral and to write an energy function we need to multiply it with ‘’***V/c***’’ because we need an energy function for a volume. So, our final function must be increase with the same type proportionality of Planck’s Law which is $\boldsymbol{x^3/(e^x-1)}$. For this reason we can make an assumption with the Radiation Constant and the Planck Law to find an empirical formula to show the real function because even if it is not fits with the real results their dependencies with respect to Temperature is same and we can use it as an empirical function.

So, our function becomes like

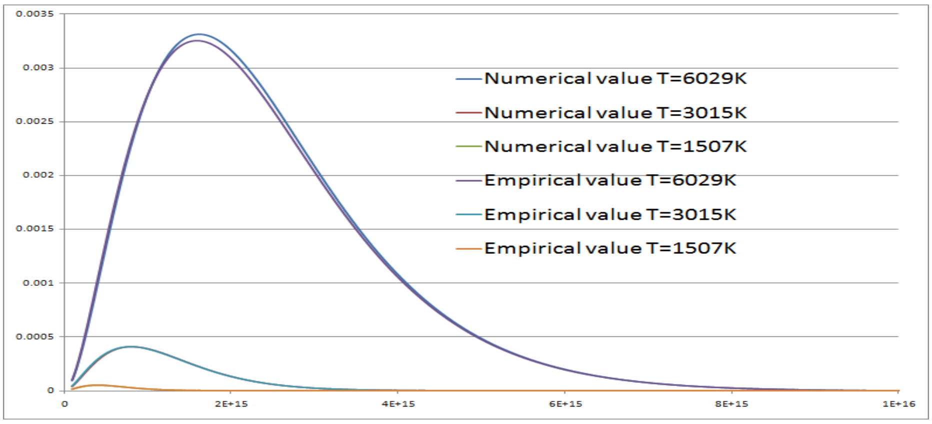
$$\mathbf{A}\frac{\hbar\omega^3}{4\pi^2\mathbf{c}^2}\frac{1}{\mathbf{e}^{\frac{\hbar\omega}{\mathbf{k_BT}}}-1}\frac{\mathbf{16V}}{\mathbf{cA}}f_1 \qquad \mathbf{V}\frac{4\hbar\omega^3}{\pi^2\mathbf{c}^3}\frac{1}{\mathbf{e}^{\frac{\hbar\omega}{\mathbf{k_BT}}}-1}f_1$$

And when we divide this function to the numerical values we can see an almost linear graph and we can write an empirical formula.

And it is comes as $\boldsymbol{f_1=\frac{1}{1+\frac{\hbar\omega}{k_BT}}}$ so in the end our empirical function becomes like

$$V\frac{4\hbar\omega^3}{\pi^2c^3}\frac{1}{e^{\frac{\hbar\omega}{k_BT}}-1}\frac{1}{\frac{\hbar\omega}{k_BT}+1}$$

We have to know that the function we found has no proof that it is the exact solution of the integral. However, it exactly fits for all the temperatures with the numerical values, so we can use it as an empirical formula at least. (*The error in the graph comes from the numeric part and it decreases with the increasing significant digits.*)



Prof. Cem Tezer'e

$$I = \int_0^\infty \frac{dx}{e^{\alpha e^{\beta x}} - 1} \qquad (\alpha, \beta > 0).$$

Letting $u = \alpha e^{\beta x}$ we have

$$\beta I = \int_\alpha^\infty \frac{e^{-u}}{1 - e^{-u}} \frac{du}{u}.$$

Note that

$$\frac{1 - x^n}{1 - x} = \sum_{k=0}^{n-1} x^k, \quad \frac{x}{1-x} = \frac{x^{n+1}}{1-x} + \sum_{k=1}^{n} x^k$$

$$\frac{e^{-u}}{1 - e^{-u}} = \frac{e^{-(n+1)u}}{1 - e^{-u}} + \sum_{k=1}^{n} e^{-ku}$$

Hence

$$\beta I = \sum_{k=1}^{n} \int_\alpha^\infty \frac{e^{-ku}}{u} du + R_n,$$

where

$$R_n = \int_\alpha^\infty \frac{e^{-(n+1)u}}{u(1 - e^{-u})} du.$$

We then have

$$\beta I = \sum_{k=1}^{n} E_1(k\alpha) + R_n$$

with

$$E_1(x) = \int_x^\infty \frac{e^{-u}}{u} du \qquad (x > 0)$$

denoting the exponential integral.

$$R_n = \int_a^{\infty} \frac{e^{-(n+1)u}}{u(1-e^{-u})}\, du$$

since $u(1-e^{-a}) \geq a(1-e^{-a})$ and

$$\int_a^{\infty} e^{-(n+1)u}\, du = \frac{e^{-(n+1)a}}{n+1}$$

we have

$$0 < R_n < \frac{1}{a(1-e^{-a})} \frac{e^{-(n+1)a}}{n+1}.$$

F. Ö.

22 / 09 / 2016

Note. The values of $E_1(a)$ are given in Table 5.1 Handbook of Mathematical Functions

REFERENCES

Black body radiation

*Planck, M. (1914) [1912]. The Theory of Heat Radiation. translated by Masius, M. P. Blakiston's Sons & Co.

Planck law

*Planck, M. (1914). The Theory of Heat Radiation. Masius, M. (transl.) (2nd ed.). P. Blakiston's Son & Co. OL 7154661M

Stefan-Boltzmann law

*Stefan, J. (1879), "Über die Beziehung zwischen der Wärmestrahlung und der Temperatur" [On the relationship between heat radiation and temperature] (PDF), Sitzungsberichte der mathematisch-naturwissenschaftlichen Classe der kaiserlichen Akademie der Wissenschaften (in German), Vienna, 79: 391–428

*Boltzmann, L. (1884), "Ableitung des Stefan'schen Gesetzes, betreffend die Abhängigkeit der Wärmestrahlung von der Temperatur aus der electromagnetischen Lichttheorie" [Derivation of Stefan's little law concerning the dependence of thermal radiation on the temperature of the electro-magnetic theory of light], Annalen der Physik und Chemie (in German), 258 (6): 291–294, Bibcode:1884AnP...258..291B, doi:10.1002/andp.18842580616