A new sufficient condition by Euler function for Riemann hypothesis

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Abstract: The aim of this paper is to show a new sufficient condition (NSC) by the Euler function for the Riemann hypothesis and its possibility. We build the NSC for any natural numbers ≥ 2 from well-known Robin theorem, and prove that the NSC holds for all odd and some even numbers while, the NSC holds for any even numbers under a certain condition, which would be called the condition (d).
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I. Introduction

Let N be the set of the natural numbers. The function $\varphi(n) = n \cdot \prod_{p|n} (1 - p^{-1})$ is called the Euler function of $n \in N$ ([1]), where $\varphi(1) = 1$ and $p|n$ denotes p is the prime divisor of n.
The function $\sigma(n) = \sum_{d|n} d$ is called divisor function of $n \in N$ ([1]), where $d|n$ denotes d is the divisor of n ([1]). Robin showed in his paper [4] (also see [2]).
Proposition 1. If the Riemann hypothesis (RH) is true, then

$$\sigma(n) \leq e^{\gamma} \cdot n \cdot \log\log n \tag{1.1}$$

holds for any $n \geq 5041$, where $\gamma = 0.577\cdots$ is the Euler constant ([1]).
Proposition 2. If the RH is false, then there exist constants $0 < \beta < 1/2$ and $c > 0$ such that

$$\sigma(n) \geq e^{\gamma} \cdot n \cdot \log\log n + \frac{c \cdot n \cdot \log\log n}{(\log n)^{\beta}} \tag{1.2}$$

holds for infinitely many $n \in N$.
From (1.1) and (1.2), one easily see that (1.1) is equivalent to the RH. So (1.1) is called the Robin criterion for the RH ([7, 8]). It is known that (1.1) holds for any odd numbers ≥ 11 and for many even numbers (see theorem 1.2 and theorem 1.4 of [8], theorem 3.2 and theorem 3.3 of [9]), and for any integers of the form $n = a^2 + b^2$ (see the corollary of the theorem 1 of [10]). Much papers have been attempted to the Robin criterion, but now new idea is required to prove it in full generality ([7]).
Another one of the sufficient conditions for the RH was given by Nicolas in [3].
Proposition 3. The RH is true iff for any $n \geq 2$

$$\frac{n}{\varphi(n)} \leq e^{\gamma} \cdot \log\log n + \frac{c_1}{\sqrt{\log n}} \tag{1.3}$$

holds, where $c_1 = 4.0628\cdots$ is determined constant ([3]).
It is known that (1.3) holds for any odd numbers ≥ 17 and for many even numbers (see also theorem 2.1 and theorem 3.1 of [8], theorem 2.2 of [9] and the corollary of theorem 1 of [10]).

As above seen, the proposition 1 and the proposition 3 are similar, but there is a essential difference here, that is, (1.1) is related to the divisor function and (1.3) is related to the Euler function. And (1.3) looks weaker than (1.1).
On the other hand, the formula

$$\sum_{p\le t} p^{-1} = \log\log t + b + E(t) \tag{1.4}$$

is called the Mertens' formula, where $t > 1$ is a real number, p is the prime number,

$$b = \gamma + \sum_{p} (\log(1-1/p) + 1/p) = 0.261497212\cdots$$

is the Mertens' constant and $\gamma = 0.577\cdots$ is the Euler constant ([1, 2]). As usual, we here will call $E(t)$ the error term of the Mertens' formula (1.4). By (3.17) and (3.20) of [5], for $t > 1$

$$-\frac{1}{\log^2 t} < E(t) < \frac{1}{\log^2 t}. \tag{1.5}$$

We recall the Chebyshev's function $\vartheta(t) = \sum_{p\le t} \log p$ ([1]). By the prime number theorem ([1]),

$$\vartheta(t) = t\cdot(1+\theta(t)) \tag{1.6}$$

holds for any real number $t > 1$, and by (3.15) and (3.16) of [5], for $t \ge 41$

$$-\frac{1}{\log t} < \theta(t) < \frac{1}{\log t}. \tag{1.7}$$

The function $\theta(t)$ is used as good tool with the function $E(t)$ in the study of the distribution of the prime numbers by the Euler function.
In this paper we build a new sufficient condition (NSC) by the Euler function for the RH from the proposition 2, and prove that the NSC holds for all odd and some even numbers while, the NSC holds for any even numbers under a certain condition (d), which would be called the condition (d). The validity of such condition (d) would be discussed in other opportunity in detail.

II. Main result of paper

From the proposition 2 we have
[Theorem 1] If there exists a constant $c_0 \ge 1$ such that

$$\frac{n}{\varphi(n)} \le e^{\gamma}\cdot\log\log(c_0\cdot n\cdot\rho(n)) \tag{2.1}$$

holds for any natural number $n \ge 2$, then the RH is true, where

$$\rho(n) = \exp(\sqrt{\log n}\cdot(\log\log n)^2).$$

This (2.1) is a NSC (new sufficient condition) for the RH. This (2.1) is clearly weaker than (1.1) and (1.3). From the proposition 1 and the proposition 2, we could see that (2.1) is the best possible one of the sufficient conditions for the RH by the divisor function or the Euler function. Also from the proposition 3, it is not difficult to see that the NSC is also a necessary condition for the RH. But our interest is to inquire whether the NSC holds without any condition or with what condition. In this connection, for $n \in N(n \ne 1)$ we define the function

$$\Phi_0(n) = \frac{\exp(\exp(e^{-\gamma}\cdot n/\varphi(n)))}{n\cdot\rho(n)}. \tag{2.2}$$

It is obvious that (2.1) is equivalent to that $\Phi_0(n) \le c_0$ holds for any $n \ge 2$. Then
[**Theorem 2**] We have $\Phi_0(n) \le 24$ for following integer $n \ge 2$.

(a) all odd numbers.

(b) all integers of $\omega(n) \le 9 \times 10^4$ ($\omega(n)$ is the number of distinct prime factors of $n \in N$ [9]).

(c) all integers of the form $n = a^k$ ($k \ge 2$; k, a is the natural number).

Moreover we provide
[**Theorem 3**] we have $\Phi_0(n) \le 24$ for any even number $n \ge 2$, if the condition

$$\Big(\mathfrak{B}(p) \cdot E(p) + \mathfrak{D}(p) \cdot \theta(p)\Big) \cdot \sqrt{p} \le 2 \tag{2.3}$$

holds for any prime number $p \ge 3$, where

$$\mathfrak{B}(p) := 1 + \frac{1}{2} \cdot \log p - \frac{2 \cdot \log p}{\log p + \log(1 + \theta(p))},$$

$$\mathfrak{D}(p) := \frac{1}{2} + \frac{2}{\log p + \log(1 + \theta(p))}.$$

We will call (2.3) the condition (d) below.

III. Proof of Theorem 1

It is clear that $\sigma(n) \cdot \varphi(n) \le n^2$ for any $n \ge 2$. If (2.1) holds, but the RH is false, then by (1.2),

$$e^\gamma \cdot \log\log n + \frac{c \cdot \log\log n}{(\log n)^\beta} \le \frac{\sigma(n)}{n} \le \frac{n}{\varphi(n)} \le e^\gamma \cdot \log\log(c_0 \cdot n \cdot \rho(n))$$

holds for infinitely many $n \in N$. On the other hand, since $\log(1 + t) \le t$ $(t > 0)$, we have

$$\log\log(c_0 \cdot n \cdot \rho(n)) = \log(\log c_0 + \log n + \sqrt{\log n} \cdot (\log\log n)^2) =$$

$$= \log\log n + \log\left(1 + \frac{\log c_0}{\log n} + \frac{(\log\log n)^2}{\sqrt{\log n}}\right) \le$$

$$\le \log\log n + \frac{\log c_0}{\log n} + \frac{(\log\log n)^2}{\sqrt{\log n}}$$

and

$$1 \le \frac{\mathrm{e}^\gamma \cdot c^{-1} \cdot \log c_0}{(\log n)^{1-\beta} \cdot \log\log n} + \frac{\mathrm{e}^\gamma \cdot c^{-1} \cdot \log\log n}{(\log n)^{1/2-\beta}} \to 0 \quad (n \to \infty),$$

but it is a contradiction.

IV. Reduction to the primorial number

Let $p_1 = 2$, $p_2 = 3$, $p_3 = 5, \cdots$ be the first consecutive primes. Then p_m $(m \in N)$ is m-th prime number. The number $(p_1 \cdots p_m)$ is called the primorial number ([3, 7]). Assume that $n = q_1^{\lambda_1} \cdots q_m^{\lambda_m}$ is the prime factorization of $n \in N$. Here $q_1, \cdots, q_m$ are distinct primes, $\lambda_1, \cdots, \lambda_m$ are nonnegative integers ≥ 1 and $\omega(n) = m$. Put $\Im_m := p_1 \cdots p_m$, then it is clear that $n \ge \Im_m$ and

$$\frac{n}{\varphi(n)} = \prod_{i=1}^{m}(1 - q_i^{-1})^{-1} \le \prod_{i=1}^{m}(1 - p_i^{-1})^{-1} = \frac{\Im_m}{\varphi(\Im_m)}$$

and so $\Phi_0(n) \le \Phi_0(\Im_m)$. This shows that the boundedness of the function $\Phi_0(n)$ for $n \in N$ ($n \neq 1$) is reduced to one for the primorial numbers. Now we put

$$C_m := \Phi_0(\Im_m) \quad (m \ge 1).$$

V. Proof of Theorem 2

Let $n \neq 1$ be an odd number and $\omega(n) = m$. Then it is clear that $n \ge 3 \cdot 5 \cdots p_{m+1} > \Im_m$ and

$$\frac{n}{\varphi(n)} \le \prod_{i=2}^{m+1} (1 - p_i^{-1})^{-1} < \prod_{i=1}^{m} (1 + p_i^{-1}).$$

By the Mertens' formula (1.4),

$$\sum_{i=1}^{m} \log(1 + p_i^{-1}) < \sum_{i=1}^{m} \frac{1}{p_i} = \log\log p_m + b + E(p_m).$$

and also by (1.6),

$$\log\log\log n > \log\log(\vartheta(p_m)) =$$
$$= \log\log p_m + \log\left(1 + \frac{1}{\log p_m} \cdot \log(1 + \theta(p_m))\right).$$

Here by (1.5) and (1.7) we see

$$E(p_m) - \log\left(1 + \frac{1}{\log p_m} \cdot \log(1 + \theta(p_m))\right) < \frac{5}{\log^2 p_m}.$$

If

$$\frac{5}{\log^2 p_m} < \gamma - b,$$

then

$$p_m > \exp\left(\sqrt{\frac{5}{\gamma - b}}\right) = 53.4934\cdots.$$

Therefore if $p_m \ge 54$, that is, $\omega(n) = m \ge 17$, then we have

$$\frac{n}{\varphi(n)} < e^{\gamma} \cdot \log\log n$$

and so $\Phi_0(n) < 1$. In the case of $1 \le \omega(n) = m \le 17$, it is confirmed by the proof of (b) that $\Phi_0(n) \le 24$. The proof of (b) is accomplished by MATLAB. If $1 \le m \le 4$, we see

$$C_1 = \frac{\exp(\exp(e^{-\gamma} \cdot 2))}{2 \cdot \exp(\sqrt{\log 2} \cdot (\log\log 2)^2)} = 9.6680\cdots,$$

$$C_2 = \frac{\exp(\exp(e^{-\gamma} \cdot 2 \cdot 3/2))}{(2 \cdot 3)\exp(\sqrt{\log(2 \cdot 3)} \cdot (\log\log(2 \cdot 3))^2)} = 23.1516\cdots,$$

$$C_3 = \frac{\exp(\exp(e^{-\gamma} \cdot 2 \cdot (3/2) \cdot (5/4)))}{(2 \cdot 3 \cdot 5)\exp(\sqrt{\log(2 \cdot 3 \cdot 5)} \cdot (\log\log(2 \cdot 3 \cdot 5))^2)} = 7.7386\cdots,$$

$$C_4 = \frac{\exp(\exp(e^{-\gamma} \cdot 2 \cdot (3/2) \cdot (5/4) \cdot (7/6)))}{(2 \cdot 3 \cdot 5 \cdot 7)\exp(\sqrt{\log(2 \cdot 3 \cdot 5 \cdot 7)} \cdot (\log\log(2 \cdot 3 \cdot 5 \cdot 7))^2)} = 0.8317\cdots.$$

If $5 \le m \le 93118$, then we see $C_m < 1$ from the table 1 and the table 2 below.
Therefore for any integers n of $1 \le \omega(n) = m \le 9 \times 10^4$ we have

$$\Phi_0(n) \le \Phi_0(\Im_m) = C_m \le 24.$$

On the other hand, by (3.30) of [5] it is known that

$$e^{-\gamma} \cdot \prod_{i=1}^{m} (1 - p_i^{-1})^{-1} < \log p_m + \frac{1}{\log p_m} \qquad (p_m \ge 2).$$

and by the theorem 10 of [5],

$$\vartheta(p_m) > 0.84 \cdot p_m \quad (p_m \ge 101).$$

If $\omega(n) = m \ge 93118$ and n is of the form a^k $(k \ge 2)$, then $\omega(n) = \omega(a)$, $n \ge (\Im_m)^k$ and $p_m \ge e^{14}$. Thus

$$\frac{n}{\varphi(n)} < e^{\gamma} \cdot \log\left(p_m \cdot \exp\left(\frac{1}{\log p_m}\right)\right) <$$
$$< e^{\gamma} \cdot \log(1.08 \cdot p_m) < e^{\gamma} \cdot \log(k \cdot \vartheta(p_m)) < e^{\gamma} \cdot \log\log n.$$

and so $\Phi_0(n) < 1$. Combining it with (b), the proof of (c) is given.
(***Remark***) (i) The result (a) is similar to one of the Theorem 2.1 of [8, 9], but the method of the proof is different each other. We here used the well-known estimates for the error terms $E(t)$ and $\theta(t)$ of the Mertens' formula and the Chebyshev's function.
(ii) The result (b) is a new one obtained from this paper. In the result (b), the limited value of $\omega(n) \le 9 \times 10^4$ for n is not essential. We could improve it to the possible value to calculate by MATLAB. But it is evident that the state, which explains that $\omega(n) = m$ tends to an infinite, could not give any guarantee by MATLAB. The theoretical assurance for it would be given by the condition (d) in the Theorem 3.
(iii) The result (c) includes one of the theorem 4.1 of [8] or the theorem 1 of [10]. In addition, by the method of the proof of (c) we could give the conclusion that $\Phi_0(n) \le 24$ holds for any integers $n = q_1^{\lambda_1} \cdots q_m^{\lambda_m}$ with

$$\min_{1 \le i \le m} \{\lambda_i\} \ge k \ge 2.$$

VI. Some estimates

Now we would ready to prove the theorem 3. The boundedness of the function $\Phi_0(\Im_m)$ for the general primorial numbers is not easily obtained as in the Theorem 2. It is needed some estimates and the condition (d) there.

6.1. Some symbols

Put $F_m := \Im_m / \varphi(\Im_m)$, then

$$\log(F_m) = -\sum_{i=1}^{m} (\log(1 - 1/p_i) + 1/p_i) + \sum_{i=1}^{m} 1/p_i =$$

$$= \log\log p_m + \gamma + E(p_m) + \varepsilon(p_m),$$

where

$$\varepsilon(p_m) = \sum_{p > p_m} (\log(1 - 1/p) + 1/p) = O(1/p_m).$$

From this we have

$$(e^{-\gamma} \cdot F_m) = \log p_m \cdot e_0, \quad \exp(e^{-\gamma} \cdot F_m) = p_m \cdot e_0',$$

where

$$e_0 = \exp(E(p_m) + \varepsilon(p_m)), \quad e_0' = \exp(\log p_m \cdot (e_0 - 1)).$$

Similarly, we easily have

$$(e^{-\gamma} \cdot F_{m-1}) = (\log p_{m-1}) \cdot e_1, \quad \exp(e^{-\gamma} \cdot F_{m-1}) = p_{m-1} \cdot e_1',$$

where

$$e_1 = \exp(E(p_{m-1}) + \varepsilon(p_{m-1})), \quad e_1' = \exp(\log p_{m-1} \cdot (e_1 - 1)).$$

On the other hand, we easily see

$$\log \Im_m = p_m \cdot \alpha_0, \quad \log \Im_{m-1} = p_{m-1} \cdot \alpha,$$

where

$$\alpha_0 = 1 + \theta(p_m), \quad \alpha = 1 + \theta(p_{m-1}).$$

Now put

$$N_0 = \sqrt{(p_m \cdot \alpha_0)} \cdot \log^2(p_m \cdot \alpha_0), \quad N_1 = \sqrt{(p_{m-1} \cdot \alpha)} \cdot \log^2(p_{m-1} \cdot \alpha).$$

6.2. An estimate of e_1 and e_1'

We put $p = p_{m-1}$, $p_0 = p_m$ below. For the theoretical calculation we assume $p \geq e^{14}$, because the case of $p \leq e^{14}$ was discussed in the Theorem 2. Since

$$(e^{-\gamma} \cdot F_{m-1}) = \prod_{i=1}^{m-1} (1 - p_i^{-1})^{-1} = (\log p) \cdot e_1 < \log p + \frac{1}{\log p} \qquad (p \geq 2)$$

by (3.30) of [5], we respectively have

$$e_1 < 1.0052 \quad (p \geq e^{14}), \qquad e_1' < 1.075 \quad (p \geq e^{14}),$$

$$(e_1 \cdot e_1') < 1.08 \quad (p \geq e^{14}).$$

6.3. An estimate of $(e_1 \cdot e_1')$

Since if $e_1 \leq 1$ then $e_1' \leq 1$, we have $(e_1 \cdot e_1') \leq 1$. On the other hand, Hence, since $\varepsilon(p) < 0$, if $e_1 > 1$, then

$$0 < r := E(p) + \varepsilon(p) < \frac{1}{\log^2 p} \leq 0.0052 \quad (p \geq e^{14})$$

and

$$e_1 = 1 + r + \sum_{n=2}^{\infty} \frac{r^n}{n!} \leq 1 + r + \frac{r^2}{2 \cdot (1 - r)} \leq 1 + r + 0.503 \cdot r^2,$$

$$e_1 \cdot e_1' = \exp(r + (\log p) \cdot (e_1 - 1)) \leq 1 + h + \frac{h^2}{2 \cdot (1 - h)},$$

where

$$h = (1 + \log p) \cdot r + 0.503 \cdot \log p \cdot r^2 \leq 0.1125 \quad (p \geq e^{14}).$$

Therefore we have

$$(e_1 \cdot e_1' - 1) \leq (1 + \log p) \cdot (E(p) + \varepsilon(p)) +$$
$$+ 0.6 \cdot (1 + \log p)^2 \cdot (E(p) + \varepsilon(p))^2 \quad (e_1 > 1, \ p \geq e^{14}).$$

6.4. An estimate of $V_0 := p_0 \cdot (e_0' - \alpha_0) - p \cdot (e_1' - \alpha)$

It is clear that $p_0 \cdot \alpha_0 - p \cdot \alpha = \log p_0$ and

$$E(p_0) - E(p) = \frac{1}{p_0} - \log\left(\frac{\log p_0}{\log p}\right),$$

$$\varepsilon(p_0) - \varepsilon(p) = -\log\left(1 - \frac{1}{p_0}\right) - \frac{1}{p_0}.$$

From this

$$\frac{e_0}{e_1} = \left(\frac{\log p}{\log p_0}\right) \cdot \left(1 + \frac{1}{p_0 - 1}\right), \qquad \frac{e_0'}{e_1'} = \frac{p}{p_0} \cdot \exp\left(\frac{\log p \cdot e_1}{p_0 - 1}\right).$$

Thus we have

$$V_0 = p \cdot e_1' \cdot \left(\frac{p_0 \cdot e_0'}{p \cdot e_1'} - 1\right) - \log p_0 = \log p_0 \cdot (\mu \cdot e_1' - 1),$$

where

$$\mu = \frac{p}{\log p_0} \cdot \left(\exp\left(\frac{\log p \cdot e_1}{p_0 - 1}\right) - 1\right).$$

Since

$$\mu \le e_1 + \frac{1}{2} \cdot \frac{\log p \cdot e_1}{p} \cdot \left(1 - \frac{\log p \cdot e_1}{p}\right)^{-1} \le$$

$$\le e_1 + 0.503 \cdot \frac{\log p}{p}, \quad (e_1 > 1,\ p \ge e^{14})$$

we have

$$\mu \cdot e_1' - 1 \le (e_1 \cdot e_1' - 1) + 0.55 \cdot \frac{\log p}{p} \quad (e_1 > 1,\ p \ge e^{14}).$$

6.5. An estimate of $G_0 := (\log p_0 \cdot R(\Im_{m-1}) - (N_0 - N_1))/N_0$

Here

$$R(\Im_{m-1}) := \frac{(\log\log \Im_{m-1})^2}{2 \cdot \sqrt{\log \Im_{m-1}}} \cdot \left(1 + \frac{4}{\log\log \Im_{m-1}}\right).$$

It is known that $p_{k+1}^2 \le (p_1 \cdots p_k)$ for $p_k \ge 7$ by 246p of [6] and hence

$$\frac{\log p_0}{\log \Im_{m-1}} < \frac{1}{2} \quad (p \ge e^{14}).$$

Since $\log(1+t) \ge (t - t^2/2) \quad (0 < t < 1/2)$, we have

$$N_0 - N_1 = (\sqrt{\log \Im_m} - \sqrt{\Im_{m-1}}) \cdot (\log\log \Im_m)^2 +$$

$$+ \sqrt{\log \Im_{m-1}} \cdot ((\log\log \Im_m)^2 - (\log\log \Im_{m-1})^2) \ge$$

$$\ge \frac{\log p_0}{2 \cdot \sqrt{\log \Im_m}} \cdot (\log\log \Im_{m-1})^2 +$$

$$+ 2 \cdot \sqrt{\log \Im_{m-1}} \cdot \log\log \Im_{m-1} \cdot \log\left(1 + \frac{\log p_0}{\log \Im_{m-1}}\right) \ge$$

$$\ge \frac{\log p_0}{2 \cdot \sqrt{\log \Im_m}} \cdot (\log\log \Im_{m-1})^2 +$$

$$+ \log p_0 \cdot \frac{2 \cdot \log\log \Im_{m-1}}{\sqrt{\log \Im_{m-1}}} \cdot \left(1 - \frac{\log p_0}{2 \cdot \log \Im_{m-1}}\right)$$

and

$$G_0 \cdot N_0 \leq \frac{\log p_0}{2} \cdot \left(\frac{1}{\sqrt{\log \Im_{m-1}}} - \frac{1}{\sqrt{\log \Im_m}} \right) \cdot (\log \log \Im_{m-1})^2 +$$

$$+ \frac{\log^2 p_0}{(\log \Im_{m-1})^{3/2}} \cdot \log \log \Im_{m-1} \leq$$

$$\leq \log^2 p_0 \cdot \frac{(\log \log \Im_{m-1})^2}{(\log \Im_{m-1})^{3/2}} \cdot \left(\frac{1}{4} + \frac{1}{\log \log \Im_{m-1}} \right).$$

And it is known that $p_{k+1}^2 \leq 2 \cdot p_k^2$ for $p_k \geq 7$ by 247p. of [6] and so

$$\log p_0 \leq (\log p) \cdot \left(1 + \frac{\log \sqrt{2}}{\log p} \right).$$

Since $p \geq e^{14}$, we have $\alpha \geq (1 - 1/14)$ and the function $(\log^3 t)/t$ is decreasing on the interval $(e^3, \ +\infty)$. Therefore we get

$$G_0 \leq \frac{\log^2 p_0}{(\log \Im_{m-1})^2} \cdot \left(\frac{1}{4} + \frac{1}{\log \log \Im_{m-1}} \right) \leq$$

$$\leq \frac{\log^3 p}{p \cdot \alpha^2} \cdot \left(1 + \frac{\log \sqrt{2}}{\log p} \right)^2 \cdot \left(\frac{1}{4} + \frac{1}{\log p + \log \alpha} \right) \cdot \frac{1}{p \cdot \log p} \leq$$

$$\leq \frac{0.01}{p \cdot \log p} \quad (p \geq e^{14}).$$

6.6. An estimate of $S(p') := \sum_{p \geq p'} 1/(p \cdot \log p)$

Put

$$s(t) := \sum_{p \leq t} p^{-1} = \log \log t + b + E(t).$$

Then by the Abel's identity [1], we have

$$S(p') = \int_{p'}^{+\infty} \frac{1}{\log t} \cdot ds(t) = \int_{p'}^{+\infty} \frac{1}{\log t} \cdot \left(\frac{dt}{t \cdot \log t} + dE(t) \right) \leq$$

$$\leq \frac{1}{\log p'} - \frac{E(p')}{\log p'} + \int_{p'}^{+\infty} \frac{1}{t \cdot \log^4 t} \cdot dt \leq$$

$$\leq \frac{1}{\log p'} + \frac{1}{\log^3 p'} - \frac{1}{3 \cdot \log^3 t} \Big|_{p'}^{+\infty} = \frac{1}{\log p'} + \frac{4}{3 \cdot \log^3 p'}$$

and

$$S(p') \geq \frac{1}{\log p'} - \frac{4}{3 \cdot \log^3 p'}.$$

If p' is a first prime $\geq e^{14}$, then $p' = 1202609$ and it is 93118-th prime. And we have

$$0.070 \leq S(p') \leq 0.072.$$

6.7. Lemma

Now we are ready for the proof of the following Lemma.

[Lemma] For any $m \geq 9 \times 10^4$ we have $C_m < 1$ under the condition (d).
Proof. Let

$$D_m := \frac{p_m \cdot (e_0' - \alpha_0)}{\sqrt{p_m \cdot \alpha_0} \cdot \log^2(p_m \cdot \alpha_0)} \quad (m \geq 1).$$

Then $C_m < 1$ is equivalent to $D_m < 1$. And for any $p_m \geq e^{14}$ we here have

$$D_m \leq a_m := 1 - 13 \cdot S(p_m).$$

We will prove $D_m \leq a_m$ for any $p_m \geq e^{14}$ by the mathematical induction with respect to m. If $p' = 1202609$ then we have

$$D_{93118} = 0.010 \cdots \leq 0.06 \leq 1 - 13 \cdot S(p') \leq 0.09 < 1.$$

Now assume $p \geq e^{14}$ and $D_{m-1} \leq a_{m-1}$. Then

$$D_m = \frac{1}{N_0} \cdot (p \cdot (e_1' - \alpha) + V_0) = D_{m-1} \cdot \frac{N_1}{N_0} + \frac{V_0}{N_0} \leq$$

$$\leq a_{m-1} \cdot \frac{N_1}{N_0} + \frac{1}{N_0} \cdot \log p_0 \cdot (\mu \cdot e_1' - 1) \leq a_{m-1} + b_{m-1},$$

where

$$b_{m-1} = \frac{1}{N_0} \cdot (\log p_0 \cdot (\mu \cdot e_1' - 1) - a_{m-1} \cdot (N_0 - N_1)).$$

By the assumption $D_{m-1} \leq a_{m-1}$, we get

$$e_1' \leq \alpha + a_{m-1} \cdot \frac{\sqrt{p \cdot \alpha} \cdot \log^2(p \cdot \alpha)}{p} = \alpha \cdot \left(1 + a_{m-1} \frac{\log^2(p\,\alpha)}{\sqrt{p\,\alpha}} \right)$$

and by taking logarithm of both sides

$$\log e_1' = (\log p) \cdot (e_1 - 1) \leq \theta(p) + a_{m-1} \cdot \frac{\log^2(p \cdot \alpha)}{\sqrt{p \cdot \alpha}}.$$

From this

$$e_1 \leq 1 + \frac{1}{\log p} \cdot \left(\theta(p) + a_{m-1} \cdot \frac{\log^2(p \cdot \alpha)}{\sqrt{p \cdot \alpha}} \right),$$

$$E(p) + \varepsilon(p) \leq \frac{1}{\log p} \left(\theta(p) + a_{m-1} \cdot \frac{\log^2(p \cdot \alpha)}{\sqrt{p \cdot \alpha}} \right).$$

Thus

$$\log p \cdot E(p) - \theta(p) \leq a_{m-1} \cdot \frac{\log^2(p \cdot \alpha)}{\sqrt{p \cdot \alpha}} - \log p \cdot \varepsilon(p)$$

and the both sides multiply by

$$\frac{p}{\sqrt{p} \cdot \log^2(p \cdot \alpha)},$$

then

$$d(p) := \frac{p \cdot \log p \cdot E(p) - p \cdot \theta(p)}{\sqrt{p} \cdot \log^2(p \cdot \alpha)} \leq \frac{a_{m-1}}{\sqrt{\alpha}} - \frac{p \cdot \log p \cdot \varepsilon(p)}{\sqrt{p} \cdot \log^2(p \cdot \alpha)}.$$

If the condition (d) holds, then by (2.3) we get

$$(1+\log p)\cdot E(p) \le a_{m-1}\cdot \frac{\log^2(p\cdot\alpha)}{2\cdot\sqrt{p\cdot\alpha}}\cdot\left(1+\frac{4}{\log(p\cdot\alpha)}\right)-$$

$$-(1+\log p)\cdot\varepsilon(p)+\frac{2}{\sqrt{p}},$$

because $\varepsilon(p)<0$ and

$$\frac{\log p}{2}\cdot\left(1+\frac{4}{\log(p\cdot\alpha)}\right)\le (1+\log p) \qquad (p\ge e^{14},\ \alpha\ge 1-1/14).$$

Thus we see

$$(1+\log p)\cdot(E(p)+\varepsilon(p))\le a_{m-1}\cdot\frac{\log^2(p\cdot\alpha)}{2\cdot\sqrt{p\cdot\alpha}}\cdot\left(1+\frac{4}{\log(p\cdot\alpha)}\right)+\frac{2}{\sqrt{p}}.$$

If $e_1>1$, then, since $0<a_{m-1}\le 1$ and $(1-1/14)\le\alpha\le(1+1/14)$, we also have

$$(1+\log p)^2\cdot(E(p)+\varepsilon(p))^2\le$$

$$\le\frac{\log^4(p\cdot\alpha)}{p\cdot\alpha}\cdot\left(\frac{1}{2}+\frac{2}{\log(p\cdot\alpha)}+\frac{2\cdot\sqrt{\alpha}}{\log^2(p\cdot\alpha)}\right)^2\le$$

$$\le 0.4287\cdot\frac{\log^4(p\cdot\alpha)}{p\cdot\alpha}\quad(p\ge e^{14})$$

and

$$\log p_0\cdot(\mu\cdot e_1'-1)-a_{m-1}\cdot(N_0-N_1)\le\log p_0\cdot(1+\log p)\cdot(E(p)+\varepsilon(p))-$$

$$-a_{m-1}\cdot(N_0-N_1)+0.55\cdot\frac{\log^2 p_0}{p}+0.6\cdot\log p_0\cdot(1+\log p)^2\cdot(E(p)+\varepsilon(p))^2\le$$

$$\le a_{m-1}\cdot G_0\cdot N_0+0.55\cdot\frac{\log^2 p_0}{p}+0.2572\cdot\log p_0\cdot\frac{\log^4(p\cdot\alpha)}{p\cdot\alpha}+\frac{2\cdot\log p_0}{\sqrt{p}}.$$

Finally, by the function $(\log^4 t)/\sqrt{t}$ is decreasing on the interval $(e^8,\ +\infty)$ we have

$$b_{m-1}\le a_{m-1}\cdot G_0+0.55\cdot\frac{\log p}{\sqrt{p\cdot\alpha}}\cdot\left(1+\frac{\log\sqrt{2}-\log\alpha}{\log p+\log\alpha}\right)^2\cdot\frac{1}{p\cdot\log p}+$$

$$+0.2572\cdot\frac{\log^4 p}{\sqrt{p}}\cdot\frac{1+\log\sqrt{2}/\log p}{\alpha^{3/2}}\cdot\frac{(1+\log\alpha/\log p)^2}{p\cdot\log p}+$$

$$+\frac{2}{\sqrt{\alpha}}\cdot\left(1+\frac{\log\sqrt{2}-\log\alpha}{\log p+\log\alpha}\right)^2\cdot\frac{1}{p\cdot\log p}\le$$

$$\le\frac{0.01}{p\cdot\log p}+\frac{0.01}{p\cdot\log p}+\frac{10.421}{p\cdot\log p}+\frac{2.203}{p\cdot\log p}\le\frac{13}{p\cdot\log p}\quad(p\ge e^{14}).$$

Next, if $e_1\le 1$ then we have

$$b_{m-1}\le 0.55\cdot\frac{\log^2 p_0}{p\cdot N_1}\le\frac{0.01}{p\cdot\log p}\quad(p\ge e^{14}).$$

VII. Proof of Theorem 3

It is obvious from the Theorem 2 and the Lemma. The Theorem 2 shows $\Phi_0(n) \le 24$ for any n of $1 \le \omega(n) = m \le e^{14}$ and the Lemma gives $\Phi_0(n) \le 1$ for any n of $\omega(n) = m \ge e^{14}$ under the condition (d) respectively.

VIII. Algorithm and Tables for Sequence $\{C_m\}$ and $\{\mathcal{R}_m\}$

Here

$$\mathcal{R}_m := \log(e^{-\gamma} \cdot \Im_m / \varphi(\Im_m)) - \log\log(\log \Im_m + \sqrt{\log \Im_m} \cdot (\log\log \Im_m)^2).$$

Then it is clear that $C_m < 1$ is equivalent to $\mathcal{R}_m < 0$. The table 1 shows the values of $C_m = \Phi_0(\Im_m)$ and $\mathcal{R}_m$ to $\omega(n) = m$ for $n \in N$. There are only values of C_m and $\mathcal{R}_m$ for $1 \le m \le 10$ here. But it is not difficult to verify them for $31 \le p_m \le e^{14}$. Note, if more informations, then it should be taken $\mathcal{R}_m < 0$, not $C_m < 1$, for $263 \le p_m \le e^{14}$, by reason of the limited values of MATLAB 6.5. The table 2 shows the values $\mathcal{R}_m$ for $93109 \le m \le 93118$. Of course, all the values in the table 1 and the table 2 are approximate.

The algorithm for $\mathcal{R}_m$ to $\omega(n) = m$ by MATLAB is as follows:

Function NSC-Index, clc, gamma=0.57721566490153286060; format long
$P = [2, 3, 5, 7, \cdots, 1202609]$; M=length(P);
for $m = 1 : M$; $p = P(1:m)$; $q = 1 - 1./p$; $F = -gamma + \log(prod(1./q))$;
$N1 = sum(\log(p.))$; $N2 = (N1)^{1/2}$; $N3 = (\log(N1))^2$; $N4 = N2 * N3$; $N5 = N1 + N4$;
m, $p(m)$, $C_m = \exp(\exp(\exp(F)))/\exp(N1)/\exp(N4)$, $\mathcal{R}_m = F - (\log(\log(N5)))$, end

Table 1

m	p_m	C_m	$\mathcal{R}_m$
1	2	9.66806133818849	$-$
2	3	23.15168798263150	0.73259862957209
3	5	7.73864609733096	0.14633620860732
4	7	0.83171792006862	-0.00636141995881
5	11	0.01114282713904	-0.09308687002330
6	13	$1.102119966548700e-004$	-0.12730939385590
7	17	$3.834259945131073e-007$	-0.15077316854133
8	19	$1.397561045763582e-009$	-0.15960912308179
9	23	$2.821898264763264e-012$	-0.16612788105591
10	29	$2.081541289212468e-015$	-0.17415284347098

Table 2

m	p_m	$\mathcal{R}_m$
93109	1202477	-0.01154791933871
93110	1202483	-0.01154786567870
93111	1202497	-0.01154781201949
93112	1202501	-0.01154775835370
93113	1202507	-0.01154770468282
93114	1202549	-0.01154765103339
93115	1202561	-0.01154759738330
93116	1202569	-0.01154754372957
93117	1202603	-0.01154749009141
93118	1202609	-0.01154743644815

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