An upper bound for error term of Mertens' formula

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Abstract: In this paper, it is obtained a new estimate for the error term $E(t)$ of Mertens' formula $\sum_{p \le t} p^{-1} = \log\log t + b + E(t)$, where $t > 1$ is a real number, p is the prime number and b is the well-known Mertens' constant. We, first, provide an upper bound, not a lower bound, of $E(p)$ for any prime number $p \ge 3$ and, next, give one in the form as $E(t) < \log t/\sqrt{t}$ for any real number $t \ge 3$. This is an essential improvement of already known results. Such estimate is very effective in the study of the distribution of the prime numbers.
***Keywords**: Mertens' formula; Chebyshev's function; Riemann Hypothesis.*

I. Introduction

The formula

$$\sum_{p \le t} p^{-1} = \log\log t + b + E(t) \tag{1.1}$$

is called the Mertens' formula, where $t > 1$ is a real number, p is the prime number,

$$b = \gamma + \sum_{p} (\log(1 - 1/p) + 1/p) = 0.261497212\cdots$$

is the Mertens' constant and $\gamma = 0.577\cdots$ is the Euler constant ([1, 2]). As usual, we here will call $E(t)$ the error term of the Mertens' formula (1.1). Much papers have been contributed in estimating the orders of the magnitude of $E(t)$ in the various approximations.
It is already well-known that

$$E(t) = O\left(\frac{1}{\log t}\right). \tag{1.2}$$

This (1.2) could be found in many books (for instance, see [1. 2]). A classic and epochal result appeared as the Theorem 23 on p. 65 of Ingham [3] (see also on p. 66 of [4]) in the form

$$\pi(t) = \sum_{p \le t} 1 = \int_2^t \frac{du}{\log u} + O(t \cdot \exp(-a \cdot \sqrt{\log t}))\ \ (a > 0), \tag{1.3}$$

where a is a positive absolute constant. And the improvement form of the Ingham's result was given in Vinogradov [6] (see also on p. 229 of [5]) as

$$\pi(t) = \int_2^t \frac{du}{\log u} + O(t \cdot \exp(-a \cdot (\log t)^{3/5})). \tag{1.4}$$

From (1.3) and (1.4) by the Abel's identity ([1]), it is easily given

$$E(t) = O(\exp(-a \cdot \sqrt{\log t})) \tag{1.5}$$

and more

$$E(t) = O(\exp(-a \cdot (\log t)^{3/5})). \tag{1.6}$$

It is obvious that the inverse process is also possible. Of course, these are not the best possible results. It is well-known that the Riemann Hypothesis (RH)([2]) is equivalent to that

$$\pi(t) = \int_2^t \frac{du}{\log u} + O(\sqrt{t} \cdot \log t). \tag{1.7}$$

holds (see the Equivalence 5.5 on p. 47 of [2]). Therefore the best one for $E(t)$ is as the form

$$E(t) = O\left(\frac{\log t}{\sqrt{t}}\right). \tag{1.8}$$

In deed, the (1.8) is another one of the forms equivalent to the RH.
Unlike above such estimates, Rosser and Schoenfeld showed, practically it is very useful, the widely applicable approximations in (3.17) and (3.20) of [4],

$$-\frac{1}{\log^2 t} < E(t) < \frac{1}{\log^2 t} \quad (t > 1). \tag{1.9}$$

We recall the Chebyshev's function $\vartheta(t) = \sum_{p \le t} \log p$ ([1]). In the work to obtain the estimate for $E(t)$, the function $\vartheta(t)$ is used as good tool. By the prime number theorem ([1]),

$$\vartheta(t) = t \cdot (1 + \theta(t)) \tag{1.10}$$

and by (3.15) and (3.16) of [4]

$$-\frac{1}{\log t} < \theta(t) < \frac{1}{\log t} \quad (t \ge 41). \tag{1.11}$$

In this paper we show a new estimate for $E(p)$ including $\theta(p)$ for any prime number $p \ge 3$. And using it, we give an upper bound, not a lower bound, for $E(t)$ in the form as $E(t) < \log t/\sqrt{t}$ for any real number $t \ge 3$. This is an essential improvement of already known results. Such estimate is very effective in the study of the RH and the distribution of the prime numbers by $E(t)$.

II. Main result of paper

We give the following theorem.
[Theorem 1] For any prime number $p \ge 3$ we have

$$\Big(\mathfrak{B}(p) \cdot E(p) + \mathfrak{D}(p) \cdot \theta(p)\Big) \cdot \sqrt{p} \le 2, \tag{2.1}$$

where

$$\mathfrak{B}(p) := 1 + \frac{1}{2} \cdot \log p - \frac{2 \cdot \log p}{\log p + \log(1 + \theta(p))},$$

$$\mathfrak{D}(p) := \frac{1}{2} + \frac{2}{\log p + \log(1 + \theta(p))}.$$

We will call (2.1) the condition (d) below.
It is not difficult to see that the condition (d) is equivalent to that

$$(1+\log p)\cdot E(p) \le d(p)\cdot \frac{\log^2(p\cdot\alpha)}{2\cdot\sqrt{p}}\cdot\left(1+\frac{4}{\log(p\cdot\alpha)}\right)+\frac{2}{\sqrt{p}} \tag{2.2}$$

holds for any prime number $p \ge 3$, where

$$\alpha := 1+\theta(p),$$

$$d(p) := \frac{p\cdot\log p\cdot E(p) - p\cdot\theta(p)}{\sqrt{p}\cdot\log^2(p\cdot\alpha)}.$$

For the convenient in the work, we take any prime $p \ge 3$ and introduce following functions.

$$f(t) = t\cdot\log t\cdot E(t) - t\cdot\theta(t),$$

$$g(t) = \sqrt{t}\cdot\log^2(t\cdot\alpha),$$

$$d(t) = \frac{f(t)}{g(t)} \quad (t\in[p,\ p+1]),$$

where $\alpha = 1+\theta(p)$ is a positive constant such that

$$(1-1/\log p) \le \alpha \le (1+1/\log p).$$

Then both $f(t)$ and $g(t)$ are continuously differentiable function on the interval $(p,\ p+1)$.
In fact, since the functions

$$\sum_{p\le t} p^{-1} - b, \quad \vartheta(t) = \sum_{p\le t}\log p$$

are constants on $(p,\ p+1)$, we have

$$E'(t) = \frac{-1}{t\cdot\log t}, \quad \theta'(t) = -\frac{1}{t} - \frac{\theta(t)}{t}, \tag{2.3}$$

where $E'(t)$ is the derivative of $E(t)$ and so on. Hence we obtain

$$f'(t) = (1+\log t)\cdot E(t). \tag{2.4}$$

Thus the function $d(t)$ is also continuously differentiable on $(p,\ p+1)$. Moreover, $d(t)$ has the right hand derivative at the point $t = p$. Put

$$d'(p) := \lim_{t\to p+0} d'(t).$$

Then we could rewrite Theorem 1 as
[Theorem 1'] For any prime number $p \ge 3$ we have

$$d'(p)\cdot g(p)\cdot\sqrt{p} \le 2. \tag{2.5}$$

Also it is clear that (2.5) is equivalent to (2.2).
From the Theorem 1 we obtain following important theorem.
[Theorem 2] For any real number $t \ge 3$ we have

$$\sum_{p\le t} p^{-1} < \log\log t + b + \frac{\log t}{\sqrt{t}}. \tag{2.6}$$

Rewriting (2.6), then for any real number $t \geq 3$,

$$E(t) < \frac{\log t}{\sqrt{t}} \tag{2.7}$$

holds. This (2.7) is a new estimate for $E(t)$. Unsatisfactorily, this (2.7) is to give only the upper bound of $E(t)$, however it also gives a possibility to get a lower bound for one. Here we accentuate that it is very useful not only (2.7) but also (2.2).

III. Some Preparations for Theorem 1

From the section III to the section VI we would handle the Theorem 1.
First, we make ready for the proof of the Theorem 1.

3.1. A Condition (d′)

If the Theorem 1 does not hold, then there exists a prime number $p \geq 3$ such that

$$d'(p) \cdot \ g(p) \cdot \sqrt{p} > 2.$$

We fix such prime p. Then from the table 1 and the table 2 we see $p \geq e^{14}$, because $H_m \leq 0$ for any $3 \leq p_m \leq e^{14}$ (see (6.1) below). Here (2.2) is equivalent to $H_m \leq 0$ for p_m there.
Now we define the function

$$G(t) := d'(t) \cdot g(t) \cdot \sqrt{t}, \quad t \in [p, \ p+1].$$

Then

$$G'(t) = \frac{1}{\sqrt{t}} \cdot \left(\partial_0(t) - 1 - \frac{1}{\log t} + \left(1 + \frac{2}{\log(t \cdot \alpha)}\right) \cdot D_1(t) \right),$$

where

$$\partial_0(t) := E(t) + \frac{f(t)}{4 \cdot t} \cdot \left(1 - \frac{8}{\log^2(t \cdot \alpha)}\right) -$$

$$-f'(t) \cdot \left(1 + \frac{4}{\log(t \cdot \alpha)}\right) + \frac{f(t)}{2 \cdot t} \cdot \left(1 + \frac{4}{\log(t \cdot \alpha)}\right)^2$$

and

$$D_1(t) := d'(t) \cdot g(t) = f'(t) - d(t) \cdot g'(t).$$

Hence $G'(t) < 0$ is equivalent to

$$\partial_0(t) + \left(1 + \frac{2}{\log(t \cdot \alpha)}\right) \cdot D_1(t) < 1 + \frac{1}{\log t}.$$

Since $t \geq e^{14}$ and $\alpha \geq (1 - 1/14)$ by (1.11), we get

$$\log(t \cdot \alpha) = \log t + \log \alpha \geq 13.925 > 0$$

and so

$$|\partial_0(t)| + \left(1 + \frac{2}{\log(t \cdot \alpha)}\right) \cdot |D_1(t)| < \frac{1}{\log^2 t} + \frac{1}{2 \cdot \log t} +$$

$$\left(1 + \frac{4}{\log(t \cdot \alpha)}\right) \cdot \left(\frac{1}{\log t} + \frac{1}{\log^2 t}\right) + \left(1 + \frac{4}{\log(t \cdot \alpha)}\right)^2 \cdot \frac{1}{\log t} +$$

$$+\left(1 + \frac{2}{\log(t \cdot \alpha)}\right) \cdot \left(\frac{1}{\log t} + \frac{1}{\log^2 t} + \left(1 + \frac{4}{\log(t \cdot \alpha)}\right) \cdot \frac{1}{\log t}\right) < 0.46 \ \ (t \geq e^{14}). \tag{3.1}$$

This shows that the function $G(t)$ is decreasing on the interval $[p,\ p+1]$. Thus there exists a point t_1 such that $p < t_1 < p+1$ and

$$G(p+1) = G(p) + G(p+1) - G(p) =$$

$$= G(p) + G'(t_1) \cdot (p+1-p) > 2 - \frac{1.6}{\sqrt{p}}.$$

For the convenient discussion, we put $x_1 = p,\ x_2 = p+1$. Then, since $G(t) \geq G(p+1)$, for any $t \in (x_1,\ x_2)$ we have

$$d'(t) \cdot\ g(t) \cdot \sqrt{t} > 2 \cdot (1 - 1/\sqrt{t}). \tag{3.2}$$

We will call (3.2) the condition (d'). For the proof of the Theorem 1, we must obtain a contradiction from the condition (d').

3.2. Proof of $d''(t) < 0$

For any $t \in (x_1,\ x_2)$ we here have $d''(t) < 0$. In fact, since

$$d''(t) = \frac{1}{t \cdot g(t)} \cdot \left(\partial_0(t) - 1 - \frac{1}{\log t} \right),$$

we easily see that $d''(t) < 0$ is equivalent to $\partial_0(t) < 1 + 1/\log t$, and

$$|\partial_0(t)| \leq 0.2577 < 1 \quad (t \geq e^{14}).$$

3.3. Function $F(t)$ and $F'(t)$

Put

$$F(t) := (d_2 - d(t)) \cdot g'(t) - (g(t) - g_1) \cdot d'(t), \qquad t \in (x_1,\ x_2).$$

Then it is clear

$$\int_{x_1}^{x_2} F(t) dt = 0, \tag{3.3}$$

where

$$g_1 := g(x_1) = \lim_{t \to x_1 + 0} g(t), \quad d_2 := d(x_2) = \lim_{t \to x_2 - 0} d(t).$$

Hence there exists a point ξ_0 such that $x_1 < \xi_0 < x_2$ and

$$\int_{x_1}^{x_2} F(t) dt = F(\xi_0) \cdot (x_2 - x_1) = 0$$

and so

$$(d_2 - d(\xi_0)) \cdot g'(\xi_0) = (g(\xi_0) - g_1) \cdot d'(\xi_0). \tag{3.4}$$

We here have $F'(t) < 0$ for any $t \in (x_1,\ x_2)$ under the condition (d'). In fact, since $d'(t) > 0$ from the condition (d'), for $F'(t) < 0$ it is sufficient to show

$$(g(t) - g_1) \cdot (-d''(t)) < 2 \cdot d'(t) \cdot g'(t).$$

And there exists a point t_1 such that $x_1 < t_1 < t$ and

$$g(t) - g(x_1) = g'(t_1) \cdot (t - x_1) \leq g'(t_1) \leq$$

$$\leq g'(t)\cdot\left(1-\frac{g''(x_1)}{g'(x_2)}\right)\leq 1.01\cdot g'(t) \quad (t\geq e^{14}).$$

Hence

$$(g(t)-g_1)\cdot(-d''(t))\leq 1.01\cdot\frac{g'(t)}{t\cdot g(t)}\cdot\left(1+\frac{1}{\log t}-\partial_0(t)\right)\leq$$

$$\leq\frac{2\cdot(1-1/\sqrt{t})\cdot g'(t)}{\sqrt{t}\cdot g(t)}\leq 2\cdot d'(t)\cdot g'(t) \quad (t\geq e^{14}).$$

Moreover, we note that $F''(t)>0$ holds for any $t\in(x_1,\ x_2)$.

3.4. An estimate of the point ξ_0

From

$$\int_{x_1}^{x_2}F(t)dt=\int_{x_1}^{\xi_0}F(t)dt+\int_{\xi_0}^{x_2}F(t)dt=0,$$

there exist $\lambda_1,\ \lambda_2$ such that $x_1<\lambda_1<\xi_0<\lambda_2<x_2$ and

$$F(\lambda_1)\cdot(\xi_0-x_1)+F(\lambda_2)\cdot(x_2-\xi_0)=0. \tag{3.5}$$

Then, since $F'(t)<0 \quad (t\in(x_1,\ x_2))$, we have

$$F(\lambda_1)>F(\xi_0)=0>F(\lambda_2).$$

Put $x_0:=x_1+x_2-\xi_0$. Then from (3.5) we have

$$F(\lambda_1)\cdot(x_2-x_0)+F(\lambda_2)\cdot(x_0-x_1)=0. \tag{3.6}$$

This (3.6) shows that the line passing the points $(x_1,\ F(\lambda_1))$ and $(x_2,\ F(\lambda_2))$ passes the point $(x_0,\ 0)$. On the other hand, by the mean value theorem, there exist the points η_1 and η_2 such that $x_1<\eta_1<\xi_0<\eta_2<x_2$ and

$$d(x_2)-d(\xi_0)=d'(\eta_2)\cdot(x_2-\xi_0),$$

$$g(\xi_0)-g(x_1)=g'(\eta_1)\cdot(\xi_0-x_1).$$

Since the function $g'(t)$ is decreasing on $(x_1,\ x_2)$ and from the condition (d'), we have

$$g'(\xi_0)\leq g'(\eta_1),\quad d'(t)>0 \quad (t\in(x_1,\ x_2)).$$

Here if $x_0\leq\xi_0$, then $x_2-\xi_0\leq\xi_0-x_1$ and by (3.4) we get

$$d'(\xi_0)=d'(\eta_2)\cdot\frac{x_2-\xi_0}{\xi_0-x_1}\cdot\frac{g'(\xi_0)}{g'(\eta_1)}\leq d'(\eta_2),$$

but it is a contradiction to $d''(t)<0$. Thus we have $x_0-\xi_0>0$ under the condition (d').

3.5. An estimate of $\varepsilon_0:=x_0-\xi_0$

Since $F(\xi_0)=0$, we have

$$F(x_1)=(d_2-d_1)\cdot g'(x_1)=F(x_1)-F(\xi_0)=-F'(\beta_0)\cdot(\xi_0-x_1),$$

where $x_1<\beta_0<\xi_0$. Also since $\xi_0-x_1=(1-\varepsilon_0)/2$ and

$$F'(t)=(d_2-d(t))\cdot g''(t)-(g(t)-g_1)\cdot d''(t)-2\cdot d'(t)\cdot g'(t), \tag{3.7}$$

we have

$$d'(\beta_0) \cdot g'(\beta_0) \cdot \varepsilon_0 = T_1 + T_2,$$

where

$$T_1 = d'(\beta_0) \cdot g'(\beta_0) - (d_2 - d_1) \cdot g'(x_1),$$

$$T_2 = -\Big((d_2 - d(\beta_0)) \cdot g''(\beta_0) - (g(\beta_0) - g_1) \cdot d''(\beta_0) \Big) \cdot (\xi_0 - x_1).$$

By the condition (d'), we get

$$d'(\beta_0) \cdot g'(\beta_0) \cdot \varepsilon_0 \geq \frac{2 \cdot (1 - 1/\sqrt{\beta_0}) \cdot g'(\beta_0)}{g(\beta_0) \cdot \sqrt{\beta_0}} \cdot \varepsilon_0 \geq \frac{\varepsilon_0}{x_2 \cdot \sqrt{x_2}}$$

and

$$\begin{aligned} T_1 &= d'(\beta_0) \cdot g'(\beta_0) - (d_2 - d_1) \cdot g'(x_1) = \\ &= d'(\beta_0) \cdot g'(\beta_0) - d'(\beta_1) \cdot g'(x_1) = \\ &= d''(\beta_1') \cdot g'(\beta_0) \cdot (\beta_0 - \beta_1) + d'(\beta_1) \cdot g''(\beta_0') \cdot (\beta_0 - x_1) \leq \\ &\leq a_2 + b_2 \leq \frac{0.9005}{x_1^2} \quad (x_1 \geq e^{14}), \end{aligned}$$

where $x_1 < \beta_1 < x_2, \quad \beta_0 < \beta_1' < \beta_1, \quad x_1 < \beta_0' < \beta_0$ and

$$a_2 := |d'(t_1) \cdot g''(t_2)|, \quad b_2 := |d''(t_1) \cdot g'(t_2)| \quad (\textit{see section } 4.4 \textit{ below}).$$

We also have

$$\begin{aligned} T_2 &= -\Big((d_2 - d(\beta_0)) \cdot g''(\beta_0) - (g(\beta_0) - g_1) \cdot d''(\beta_0) \Big) \cdot (\xi_0 - x_1) = \\ &= -\Big(d'(\beta_2) \cdot g''(\beta_0) \cdot (x_2 - \beta_0) - g'(\beta_2') \cdot d''(\beta_0) \cdot (\beta_0 - x_1) \Big) \cdot (\xi_0 - x_1) \leq \\ &\leq (a_2 + b_2)/2 \leq \frac{0.4503}{x_2^2} \quad (x_2 \geq e^{14}), \end{aligned}$$

where $\beta_0 < \beta_2 < x_2, \quad x_1 < \beta_2' < \beta_0$. Thus we have

$$0 < \varepsilon_0 \leq \frac{1.3508}{\sqrt{x_2}} \leq 0.0015 \quad (x_2 \geq e^{14}). \tag{3.8}$$

3.6. An estimate of $\delta_0 := \lambda_0 - x_0$

Here a point λ_0 is determined as follows. If the line passing the points $(\lambda_1, F(\lambda_1)$ and $(\lambda_2, F(\lambda_2)$ intersects the line $y = 0$ at the point λ_0, then we obtain

$$F(\lambda_1) \cdot (\lambda_2 - \lambda_0) + F(\lambda_2) \cdot (\lambda_0 - \lambda_1) = 0. \tag{3.9}$$

Since the function $F(t)$ is decreasing and convex on $(x_1,\, x_2)$ under the condition (d'), it is clear $\xi_0 < \lambda_0$. And the equation of the line passing $(\xi_0,\, F(\lambda_1))$ and $(\lambda_0,\, 0)$ is

$$F(\lambda_1) \cdot x + (\lambda_0 - \xi_0) \cdot y - F(\lambda_1) \cdot \lambda_0 = 0 \tag{3.10}$$

and one passing $(\xi_0,\, -F(\lambda_2))$ and $(x_0,\, 0)$ is

$$-F(\lambda_2) \cdot x + (x_0 - \xi_0) \cdot y + F(\lambda_2) \cdot x_0 = 0. \tag{3.11}$$

We put

$$\triangle_0 := F(\lambda_1)\cdot(x_0-\xi_0)+F(\lambda_2)\cdot(\lambda_0-\xi_0)\neq 0,$$
$$\triangle_1 := F(\lambda_1)\cdot F(\lambda_2)\cdot\delta_0.$$

Then y-coordinate of the cross point of above two lines (3.10) and (3.11) is $\triangle_1/\triangle_0$. Here if $\delta_0>0$ then $\triangle_1/\triangle_0<0$ and $\triangle_1<0$, if $\delta_0<0$ then $\triangle_1/\triangle_0>0$ and $\triangle_1>0$. Hence we always have $\triangle_0>0$ and

$$F(\lambda_1)\cdot(x_0-\xi_0)+F(\lambda_2)\cdot(\lambda_0-\xi_0)>0. \tag{3.12}$$

From this

$$-F(\lambda_2)\cdot\delta_0<(F(\lambda_1)+F(\lambda_2))\cdot(x_0-\xi_0).$$

And since

$$x_2-x_0=\frac{1-\varepsilon_0}{2},$$
$$x_0-x_1=\frac{1+\varepsilon_0}{2},$$

by (3.6) we have

$$\frac{F(\lambda_1)+F(\lambda_2)}{-F(\lambda_2)}=\frac{x_0-x_1}{x_2-x_0}-1=\frac{2\cdot\varepsilon_0}{1-\varepsilon_0}$$

and so

$$\delta_0<2\cdot\varepsilon_0^2/(1-\varepsilon_0).$$

Similarly, for the lines passing the points $(\xi_0, F(\lambda_1))$, $(x_0,0)$ and the points $(\xi_0,-F(\lambda_2))$, $(\lambda_0,0)$, we have

$$F(\lambda_1)\cdot(\lambda_0-\xi_0)+F(\lambda_2)\cdot(x_0-\xi_0)>0$$

and from this

$$\delta_0>(-1)\cdot 2\cdot\varepsilon_0^2/(1+\varepsilon_0).$$

Therefore we have

$$(-1)\cdot\frac{2\cdot\varepsilon_0^2}{1+\varepsilon_0}<\delta_0<\frac{2\cdot\varepsilon_0^2}{1-\varepsilon_0}. \tag{3.13}$$

3.7. An estimate of $\delta_1 := \lambda_1-\lambda_1'$

Here $\lambda_1' := x_1+\xi_0-\lambda_1$. By the same method as in δ_0, for the lines passing $(x_1,\ F(\lambda_1))$, $(\lambda_1,\ 0)$ and $(x_1,\ -F(\lambda_2))$, $(\lambda_1',\ 0)$ we have

$$F(\lambda_1)\cdot(\lambda_1'-x_1)+F(\lambda_2)\cdot(\lambda_1-x_1)>0.$$

From this, since

$$\frac{F(\lambda_1)+F(\lambda_2)}{F(\lambda_1)}=\frac{2\cdot\varepsilon_0}{1+\varepsilon_0},$$
$$\lambda_1-x_1=\frac{1}{4}\cdot(1-\varepsilon_0)+\frac{1}{2}\cdot\delta_1,$$

we have $\delta_1<\varepsilon_0/2$. Also for the lines passing the points $(x_1,F(\lambda_1))$, $(\lambda_1',0)$ and $(x_1,-F(\lambda_2))$, $(\lambda_1,0)$ we have

$$F(\lambda_1)\cdot(\lambda_1-x_1)+F(\lambda_2)\cdot(\lambda_1'-x_1)>0.$$

and so $\delta_1 > (-\varepsilon_0)/2$. Therefore we get

$$(-1)\cdot\frac{\varepsilon_0}{2} < \delta_1 < \frac{\varepsilon_0}{2}. \tag{3.14}$$

3.8. An estimate of $\omega_0 := (\lambda_1+\lambda_2) - 2\cdot\xi_0$

Let $\lambda_0' := \lambda_1 + \lambda_2 - \lambda_0$. Then from (3.5) and (3.9) we get

$$-\frac{F(\lambda_1)}{F(\lambda_2)} = \frac{x_2-\xi_0}{\xi_0-x_1} = \frac{\lambda_0-\lambda_1}{\lambda_2-\lambda_0} = \frac{\lambda_2-\lambda_0'}{\lambda_0'-\lambda_1}$$

and

$$\lambda_0 = \lambda_1\cdot(\xi_0-x_1) + \lambda_2\cdot(x_2-\xi_0),$$
$$\lambda_0' = \lambda_1\cdot(x_2-\xi_0) + \lambda_2\cdot(\xi_0-x_1).$$

Hence

$$\lambda_0 - \lambda_0' = (\lambda_2-\lambda_1)\cdot\varepsilon_0$$

Put $\omega_1 := (\lambda_1+\lambda_2) - (x_1+x_2)$. Then, since

$$\lambda_1+\lambda_2 = \lambda_0+\lambda_0' = 2\cdot\lambda_0 - (\lambda_0-\lambda_0'),$$

$$x_1+x_2 = x_0+\xi_0 = 2\cdot x_0 - (x_0-\xi_0),$$

we have

$$\omega_1 = 2\cdot\delta_0 + (1-(\lambda_2-\lambda_1))\cdot\varepsilon_0.$$

Thus

$$\omega_0 = \omega_1 + \varepsilon_0 = 2\cdot\delta_0 + 2\cdot\varepsilon_0 - (\lambda_2-\lambda_1)\cdot\varepsilon_0. \tag{3.15}$$

3.9. An estimate of $\Lambda_0 := (\lambda_2-\lambda_1)$

Since

$$\xi_0 - \lambda_1 = \frac{1}{4}\cdot(1-\varepsilon_0) - \frac{1}{2}\cdot\delta_1,$$

we have

$$(\lambda_2-\lambda_1) = 2\cdot(\lambda_0-\lambda_1) - (\lambda_0-\lambda_0') =$$
$$= 2\cdot\delta_0 + 2\cdot\varepsilon_0 + \frac{1}{2}\cdot(1-\varepsilon_0) - \delta_1 - (\lambda_2-\lambda_1)\cdot\varepsilon_0,$$

and

$$(\lambda_2-\lambda_1) = \frac{1}{2} + \frac{\varepsilon_0}{1+\varepsilon_0} - \frac{\delta_1}{1+\varepsilon_0} + \frac{2\cdot\delta_0}{1+\varepsilon_0}$$

and hence

$$\frac{1}{2} < (\lambda_2-\lambda_1) < \frac{1}{2} + 2\cdot\varepsilon_0. \tag{3.16}$$

3.10. An estimate of $\delta_2 := \lambda_1 - \eta_1$

By the same method as above, for the lines passing the points $(x_1,\ F(\lambda_1))$, $(\lambda_1,\ 0)$ and $(x_1,\ -F(\lambda_2))$, $(\eta_1,\ 0)$ we have

$$F(\lambda_1)\cdot(\eta_1-x_1) + F(\lambda_2)\cdot(\lambda_1-x_1) > 0.$$

From this

$$\delta_2 < \frac{F(\lambda_1)+F(\lambda_2)}{F(\lambda_1)} \cdot (\lambda_1 - x_1) =$$

$$= \frac{2\cdot\varepsilon_0}{1+\varepsilon_0} \cdot \left(\frac{1}{4}\cdot(1-\varepsilon_0) + \frac{1}{2}\cdot\delta_1\right) < \frac{1}{2}\cdot\frac{\varepsilon_0}{1+\varepsilon_0}.$$

Similarly, for the lines passing the points $(x_1,\, F(\lambda_1))$, $(\eta_1,\, 0)$ and $(x_1,\, -F(\lambda_2))$, $(\lambda_1,\, 0)$ we have

$$F(\lambda_1)\cdot(\lambda_1 - x_1) + F(\lambda_2)\cdot(\eta_1 - x_1) > 0.$$

From this

$$\delta_2 > \frac{F(\lambda_1)+F(\lambda_2)}{F(\lambda_2)} \cdot (\lambda_1 - x_1) > -\frac{1}{2}\cdot\frac{\varepsilon_0}{1-\varepsilon_0}.$$

Consequently, we have

$$(-1)\cdot\frac{\varepsilon_0}{2\cdot(1-\varepsilon_0)} < \delta_2 < \frac{\varepsilon_0}{2\cdot(1+\varepsilon_0)}. \tag{3.17}$$

3.11. An estimate of $Q_0 := (\eta_2 + \eta_1 - 2\cdot\xi_0)\cdot(\xi_0 - x_1)$

First, we will find the lower bound of Q_0.
If the line passing $(\eta_1,\, F(\eta_1))$, $(\eta_2,\, F(\eta_2))$ intersects $y = 0$ at η_0, then

$$F(\eta_1)\cdot(\eta_2 - \eta_0) + F(\eta_2)\cdot(\eta_0 - \eta_1) = 0$$

and from this

$$F(\lambda_1)\cdot(\eta_2 - \eta_0) + F(\lambda_2)\cdot(\eta_0 - \eta_1) + W_0 = 0,$$

where

$$W_0 = (F(\eta_1) - F(\lambda_1))\cdot(\eta_2 - \eta_0) + (F(\eta_2) - F(\lambda_2))\cdot(\eta_0 - \eta_1).$$

From (4.10) we have

$$\eta_0 = \eta_1 + (\eta_2 - \eta_1)\cdot(x_2 - \xi_0) + W_1,$$

where

$$W_1 = \frac{x_2 - \xi_0}{F(\lambda_1)}\cdot W_0.$$

Since $F(t)$ is convex on $(x_1,\, x_2)$ and $\eta_1 < \xi_0 < \eta_2$, we have $\eta_0 > \xi_0$ and

$$\eta_1 + (\eta_2 - \eta_1)\cdot(x_2 - \xi_0) + W_1 > \xi_0.$$

From this

$$Q_0 > -(\eta_2 - \xi_0)\cdot\varepsilon_0 - W_1.$$

Here if $W_1 < 0$ then

$$Q_0 > -\varepsilon_0$$

and if $W_1 > 0$ then we put $\eta_0' := \eta_1 + \eta_2 - \eta_0$. Then

$$F(\lambda_1)\cdot(\eta_0' - \eta_1) + F(\lambda_2)\cdot(\eta_2 - \eta_0') + W_0 = 0$$

and by same way as above

$$\eta_0' = \eta_1 + (\eta_2 - \eta_1)\cdot(\xi_0 - x_1) - W_1.$$

Thus we have

$$\eta_0 - \eta_0' = (\eta_2 - \eta_1) \cdot \varepsilon_0 + 2 \cdot W_1$$

and so $(\eta_0 - \eta_0') > 0$. Similarly, for the lines passing the points $(\lambda_1,\ F(\lambda_1))$, $(\eta_0,\ 0)$ and $(\lambda_1,\ -F(\lambda_2))$, $(\eta_0',\ 0)$ we have

$$F(\lambda_1) \cdot (\eta_0' - \lambda_1) + F(\lambda_2) \cdot (\eta_0 - \lambda_1) > 0.$$

On the other hand, since $\eta_2 > \eta_0$ and

$$(\xi_0 - \lambda_1) < \frac{1}{4},$$

we have

$$(\eta_0 - \eta_0') < \frac{F(\lambda_1) + F(\lambda_2)}{F(\lambda_1)} \cdot (\eta_0 - \lambda_1) <$$

$$< \frac{2 \cdot \varepsilon_0}{1 + \varepsilon_0} \cdot ((\eta_0 - \xi_0) + (\xi_0 - \lambda_1)) <$$

$$< 2 \cdot \varepsilon_0 \cdot (\eta_2 - \xi_0) + \frac{\varepsilon_0}{2}.$$

Hence

$$2 \cdot W_1 = (\eta_0 - \eta_0') - (\eta_2 - \eta_1) \cdot \varepsilon_0 <$$

$$< 2 \cdot \varepsilon_0 \cdot (\eta_2 - \xi_0) + \frac{\varepsilon_0}{2} - (\eta_2 - \xi_0) \cdot \varepsilon_0 - (\xi_0 - \eta_1) \cdot \varepsilon_0 \leq$$

$$\leq \varepsilon_0 \cdot (\eta_2 - \xi_0) + \frac{\varepsilon_0}{2} - (\xi_0 - \lambda_1) \cdot \varepsilon_0 - \delta_2 \cdot \varepsilon_0 =$$

$$= \varepsilon_0 \cdot (\eta_2 - \xi_0) + \frac{\varepsilon_0}{2} - \varepsilon_0 \cdot \left(\frac{1}{4} \cdot (1 - \varepsilon_0) - \frac{1}{2} \cdot \delta_1 \right) - \delta_2 \cdot \varepsilon_0 <$$

$$< \varepsilon_0 \cdot (\eta_2 - \xi_0) + \frac{\varepsilon_0}{4} + 2 \cdot \varepsilon_0^2$$

and, since $x_2 > \eta_2$ and $x_2 - \xi_0 = (1 + \varepsilon_0)/2$,

$$Q_0 > -(\eta_2 - \xi_0) \cdot \varepsilon_0 - W_1 >$$

$$> -(\eta_2 - \xi_0) \cdot \varepsilon_0 - \frac{\varepsilon_0}{2} \cdot (\eta_2 - \xi_0) - \frac{\varepsilon_0}{8} - \varepsilon_0^2 >$$

$$> -\frac{3}{2} \cdot \varepsilon_0 \cdot (x_2 - \xi_0) - \frac{\varepsilon_0}{8} - \varepsilon_0^2 =$$

$$= -\frac{3}{4} \cdot \varepsilon_0 \cdot (1 + \varepsilon_0) - \frac{\varepsilon_0}{8} - \varepsilon_0^2 > -\varepsilon_0.$$

Therefore, generally, we have

$$Q_0 = (\eta_2 + \eta_1 - 2 \cdot \xi_0) \cdot (\xi_0 - x_1) > -\varepsilon_0. \tag{3.18}$$

Next, we will find the upper bound of Q_0. It is easy to see that

$$(\eta_2 + \eta_1 - 2 \cdot \xi_0) = (\eta_2 - \xi_0) - (\xi_0 - \eta_1) <$$

$$< (x_2 - \xi_0) - (\xi_0 - \eta_1) =$$

$$= \frac{1}{2} \cdot (1 + \varepsilon_0) - \left(\frac{1}{4} \cdot (1 - \varepsilon_0) - \frac{1}{2} \cdot \delta_1 + \delta_2 \right) <$$

$$< \frac{1}{4} + 2 \cdot \varepsilon_0$$

and

$$(2 \cdot \xi_0 - \eta_2 - \eta_1) = (\xi_0 - \eta_1) - (\eta_2 - \xi_0) <$$

$$< (\xi_0 - \eta_1) < \frac{1}{4} + 2 \cdot \varepsilon_0.$$

Therefore, since

$$|(\eta_2 + \eta_1 - 2 \cdot \xi_0)| \cdot |\xi_0 - x_1| <$$

$$< \left(\frac{1}{4} + 2 \cdot \varepsilon_0\right) \cdot \frac{1}{2} \cdot (1 - \varepsilon_0) < \frac{1}{8} + \varepsilon_0,$$

we have

$$|Q_0| < \frac{1}{8} + \varepsilon_0. \tag{3.19}$$

3.12. An estimate of $H_0 := (\xi_0 - \eta_1) \cdot (\xi_0 - x_1)$

Since

$$(\xi_0 - \eta_1) = \frac{1}{4} \cdot (1 - \varepsilon_0) - \frac{1}{2} \cdot \delta_1 + \delta_2,$$

$$(\xi_0 - x_1) = \frac{1}{2} \cdot (1 - \varepsilon_0),$$

we easily have

$$\frac{1}{8} - \varepsilon_0 < H_0 < \frac{1}{8} + \varepsilon_0. \tag{3.20}$$

IV. New equality and inequality

In this section we make a new equality from (3.5) and (3.6), and derive a new inequality from the estimates for the various points discussed in the section III.

4.1. A new equality

Now we add (3.5) and (3.6), then we have

$$F(\lambda_1) + F(\lambda_2) = (F(\lambda_1) - F(\lambda_2)) \cdot \varepsilon_0$$

and both sides multiply by $d'(\eta_2) \cdot g'(\xi_0)$, then

$$(F(\lambda_1) + F(\lambda_2)) \cdot d'(\eta_2) \cdot g'(\xi_0) =$$

$$= (F(\lambda_1) - F(\lambda_2)) \cdot d'(\eta_2) \cdot g'(\xi_0) \cdot \varepsilon_0. \tag{4.1}$$

On the other hand, since $F(\xi_0) = 0$, we get

$$F(\lambda_1) + F(\lambda_2) = (F(\lambda_2) - F(\xi_0)) - (F(\xi_0) - F(\lambda_1)) =$$

$$= F'(\alpha_2) \cdot (\lambda_2 - \xi_0) - F'(\alpha_1) \cdot (\xi_0 - \lambda_1) =$$

$$= (F'(\alpha_2) - F'(\alpha_1)) \cdot (\xi_0 - \lambda_1) + F'(\alpha_2) \cdot ((\lambda_2 - \xi_0) - (\xi_0 - \lambda_1)) =$$

$$= F''(\tau_0) \cdot (\alpha_2 - \alpha_1) \cdot (\xi_0 - \lambda_1) + F'(\alpha_2) \cdot (\lambda_1 + \lambda_2 - 2 \cdot \xi_0)$$

and

$$F(\lambda_1) - F(\lambda_2) = (F(\lambda_1) - F(\xi_0)) - (F(\lambda_2) - F(\xi_0)) =$$
$$= -F'(\alpha_1) \cdot (\xi_0 - \lambda_1) - F'(\alpha_2) \cdot (\lambda_2 - \xi_0) =$$
$$= (F'(\alpha_2) - F'(\alpha_1)) \cdot (\xi_0 - \lambda_1) - F'(\alpha_2) \cdot ((\lambda_2 - \xi_0) + (\xi_0 - \lambda_1)) =$$
$$= F''(\tau_0) \cdot (\alpha_2 - \alpha_1) \cdot (\xi_0 - \lambda_1) - F'(\alpha_2) \cdot (\lambda_2 - \lambda_1),$$

where $\lambda_1 < \alpha_1 < \xi_0 < \alpha_2 < \lambda_2$ and $\alpha_1 < \tau_0 < \alpha_2$.
From (3.4), there exist the points η_1 and η_2 such that $x_1 < \eta_1 < \xi_0 < \eta_2 < x_2$ and

$$(d_2 - d(\xi_0)) \cdot g'(\xi_0) - (g(\xi_0) - g_1) \cdot d'(\xi_0) =$$
$$= d'(\eta_2) \cdot g'(\xi_0) \cdot (x_2 - \xi_0) - g'(\eta_1) \cdot d'(\xi_0) \cdot (\xi_0 - x_1) =$$
$$= \Big(d'(\eta_2) \cdot g'(\xi_0) - g'(\eta_1) \cdot d'(\xi_0) \Big) \cdot (\xi_0 - x_1) +$$
$$+ d'(\eta_2) \cdot g'(\xi_0) \cdot (x_0 - \xi_0) = 0. \tag{4.2}$$

Here also there exist μ_1, μ_2 such that $\eta_1 < \mu_1 < \xi_0 < \mu_2 < \eta_2$ and

$$d'(\eta_2) \cdot g'(\xi_0) - g'(\eta_1) \cdot d'(\xi_0) =$$
$$= (d'(\eta_2) - d'(\xi_0)) \cdot g'(\xi_0) + (g'(\xi_0) - g'(\eta_1)) \cdot d'(\xi_0) =$$
$$= d''(\mu_2) \cdot g'(\xi_0)) \cdot (\eta_2 - \xi_0) + g''(\mu_1) \cdot d'(\xi_0) \cdot (\xi_0 - \eta_1) =$$
$$= \Big(d''(\mu_2) \cdot g'(\xi_0) + g''(\mu_1) \cdot d'(\xi_0) \Big) \cdot (\xi_0 - \eta_1) +$$
$$+ d''(\mu_2) \cdot g'(\xi_0) \cdot (\eta_2 + \eta_1 - 2 \cdot \xi_0). \tag{4.3}$$

We put

$$A_0 := d'(\eta_2) \cdot g'(\xi_0),$$
$$A_1 := (-d''(\mu_2)) \cdot g'(\xi_0) + (-g''(\mu_1)) \cdot d'(\xi_0),$$
$$A_2 := (-d''(\mu_2)) \cdot g'(\xi_0),$$
$$B_0 := d'(\alpha_2) \cdot g'(\alpha_2),$$
$$B_1 := (d_2 - d(\alpha_2)) \cdot g''(\alpha_2) - (g(\alpha_2) - g_1) \cdot d''(\alpha_2),$$
$$U_0 := d''(\tau_0) \cdot g'(\tau_0) + d'(\tau_0) \cdot g''(\tau_0),$$
$$U_1 := (d_2 - d(\tau_0)) \cdot g'''(\tau_0) - (g(\tau_0) - g_1) \cdot d'''(\tau_0).$$

Then since

$$F''(t) = (d_2 - d(t)) \cdot g'''(t) - (g(t) - g_1) \cdot d'''(t) -$$
$$-3 \cdot (d''(t) \cdot g'(t) + d'(t) \cdot g''(t)), \tag{4.4}$$

from (3.7) we have

$$F'(\alpha_2) = B_1 - 2 \cdot B_0,$$
$$F''(\tau_0) = U_1 - 3 \cdot U_0,$$

and from (4.2) and (4.3) we get

$$\mathfrak{M}_\mathfrak{o} := A_0 \cdot \varepsilon_0 = A_1 \cdot (\xi_0 - \eta_1) \cdot (\xi_0 - x_1) +$$

$$+A_2\cdot(\eta_2+\eta_1-2\cdot\xi_0)\cdot(\xi_0-x_1)=O(1/t^2). \tag{4.5}$$

Thus the left side of (4.1) is

$$L_0:=(F(\lambda_1)+F(\lambda_2))\cdot d'(\eta_2)\cdot g'(\xi_0)=L_1+L_2,$$

where

$$L_1=F''(\tau_0)\cdot d'(\eta_2)\cdot g'(\xi_0)\cdot(\alpha_2-\alpha_1)\cdot(\xi_0-\lambda_1)=L_{11}-L_{12},$$
$$L_2=F'(\alpha_2)\cdot d'(\eta_2)\cdot g'(\xi_0)\cdot(\lambda_1+\lambda_2-2\cdot\xi_0)=L_{21}-L_{22}.$$

and

$$L_{11}=U_1\cdot A_0\cdot(\alpha_2-\alpha_1)\cdot(\xi_0-\lambda_1)=O(1/t^4),$$
$$L_{12}=3\cdot U_0\cdot A_0\cdot(\alpha_2-\alpha_1)\cdot(\xi_0-\lambda_1)=O(1/t^3).$$

and

$$L_{21}=2\cdot A_0\cdot B_1\cdot\delta_0-4\cdot A_0\cdot B_0\cdot\delta_0+$$
$$+B_1\cdot(2-(\lambda_2-\lambda_1))\cdot\mathfrak{M}_\mathfrak{o}=O(1/t^4),$$
$$L_{22}=2\cdot(2-(\lambda_2-\lambda_1))\cdot B_0\cdot\mathfrak{M}_\mathfrak{o}=O(1/t^3).$$

Similarly, the right side of (4.1) is

$$R_0:=(F(\lambda_1)-F(\lambda_2))\cdot\mathfrak{M}_\mathfrak{o}=R_1-R_2,$$

where

$$R_1=F''(\tau_0)\cdot\mathfrak{M}_\mathfrak{o}\cdot(\alpha_2-\alpha_1)\cdot(\xi_0-\lambda_1)=$$
$$=(U_1-3\cdot U_0)\cdot\mathfrak{M}_\mathfrak{o}\cdot(\alpha_2-\alpha_1)\cdot(\xi_0-\lambda_1)=O(1/t^4).$$

And

$$R_2:=F'(\alpha_2)\cdot\mathfrak{M}_\mathfrak{o}\cdot(\lambda_2-\lambda_1)=R_{21}-R_{22},$$

where

$$R_{21}=B_1\cdot\mathfrak{M}_\mathfrak{o}\cdot(\lambda_2-\lambda_1)=O(1/t^4),$$
$$R_{22}=2\cdot\ B_0\cdot\mathfrak{M}_\mathfrak{o}\cdot(\lambda_2-\lambda_1)=O(1/t^3).$$

Thus (4.1) is equivalent to $L_0=R_0$, that is, we have a new equality

$$(L_{11}+L_{21})-(L_{12}+L_{22})=R_1-(R_{21}-R_{22}). \tag{4.6}$$

4.2. A new inequality

Put

$$K_1:=(L_{12}+L_{22})+R_{22},$$
$$K_2:=(L_{11}+L_{21})-R_1+R_{21},$$

then, from the equality (4.6), we have $K_1=K_2$. And by $\mathfrak{M}_\mathfrak{o}$ of (4.5) we have

$$K_1=\mathfrak{A}_\mathfrak{o}+K_{11}+K_{12},$$

where

$$\mathfrak{A}_\mathfrak{o}=A_1\cdot B_0\cdot(\xi_0-\eta_1)\cdot(\xi_0-x_1),$$
$$K_{11}=3\cdot U_0\cdot A_0\cdot(\alpha_2-\alpha_1)\cdot(\xi_0-\lambda_1)+$$
$$+3\cdot A_1\cdot B_0\cdot(\xi_0-\eta_1)\cdot(\xi_0-x_1),$$

$$K_{12} = 4 \cdot A_2 \cdot B_0 \cdot (\eta_2 + \eta_1 - 2 \cdot \xi_0) \cdot (\xi_0 - x_1).$$

By the condition (d') we have

$$\left| \frac{d'(\alpha_2) \cdot g'(\alpha_2)}{d'(\eta_2) \cdot g'(\xi_0)} \right| \leq 1 + x_2 \cdot \sqrt{x_2} \cdot (a_2 + b_2) \leq$$

$$\leq 1 + \frac{0.9005}{\sqrt{x_1}} \quad (x_1 \geq e^{14})$$

and

$$\mathfrak{M}_1 := B_0 \cdot \varepsilon_0 = \frac{d'(\alpha_2) \cdot g'(\alpha_2)}{d'(\eta_2) \cdot g'(\xi_0)} \cdot \mathfrak{M}_0 = O(1/t^2).$$

And also we see $A_2 > 0$, $\mathfrak{M}_1 > 0$. Thus from (3.18) we get

$$K_{12} \geq K'_{12} := -4 \cdot A_2 \cdot \mathfrak{M}_1 = O(1/t^4).$$

It is clear that $d'(t) > 0$ by the condition (d') and $g'(t) > 0$, $g''(t) < 0$, $d''(t) < 0$ for any $t \in (x_1, \ x_2)$, so we have $A_0 > 0$, $A_1 > 0$, $B_0 > 0$, $U_0 < 0$. On the other hand,

$$(\alpha_2 - \alpha_1) \cdot (\xi_0 - \lambda_1) < (\lambda_2 - \lambda_1) \cdot (\xi_0 - \lambda_1) <$$

$$< \Lambda_0 \cdot \frac{1}{4} < \frac{1}{8} + \varepsilon_0$$

and hence by (3.20),

$$K_{11} \geq K'_{11} + K''_{11},$$

where

$$K'_{11} = \frac{3}{8} \cdot (U_0 \cdot A_0 + A_1 \cdot B_0) = O(1/t^4) \quad (\textit{see below}),$$

$$K''_{11} = 3 \cdot \varepsilon_0 \cdot (U_0 \cdot A_0 - A_1 \cdot B_0) = O(1/t^4).$$

We also have from (3.20)

$$\mathfrak{A}_0 \geq \Omega_0 - \mathfrak{A}_1,$$

where

$$\Omega_0 = \frac{1}{8} \cdot A_1 \cdot B_0,$$

$$\mathfrak{A}_1 = A_1 \cdot \mathfrak{M}_1 = O(1/t^4).$$

From this we have a new inequality

$$\Omega_0 \leq \Omega_1 := (L_{11} + L_{21}) + (R_{21} - R_1) - (K'_{12} + K'_{11} + K''_{11}) + \mathfrak{A}_1. \tag{4.7}$$

Now we intend to obtain the estimates for the lower bound of Ω_0 and the upper bound of Ω_1 respectively.

4.3. Lower bound of $\Omega_0 := (A_1 \cdot B_0)/8$

Using the condition (d'), we get

$$B_0 = d'(\alpha_2) \cdot g'(\alpha_2) \geq \frac{1}{x_2 \cdot \sqrt{x_2}}$$

and

$$(-g''(\mu_1)) \cdot d'(\xi_0) \geq \frac{0.4789}{x_2^2 \cdot \sqrt{x_2}} \quad (x_2 \geq e^{14})$$

and

$$(-d''(\mu_2))\cdot g'(\xi_0)=\frac{g'(\xi_0)}{\mu_2\cdot g(\mu_2)}\cdot\left(1+\frac{1}{\log\mu_2}-\partial_0(\mu_2)\right)\geq$$

$$\geq\frac{g'(x_2)}{x_2\cdot g(x_2)}\cdot(1-0.2577)\geq\frac{0.3711}{x_2^2}\quad(x_2\geq e^{14}).$$

Thus we have

$$\Omega_0\geq\frac{1}{8}\cdot\left(\frac{0.4789}{x_2^4}+\frac{0.3711}{x_2^3\cdot\sqrt{x_2}}\right)\geq\frac{0.0462}{x_1^3\cdot\sqrt{x_1}}\quad(x_1\geq e^{14}).$$

Put

$$\Omega_0':=\frac{0.0462}{x_1^3\cdot\sqrt{x_1}}\quad(x_1\geq e^{14}).\tag{4.8}$$

4.4. Some estimates

For any $t\in(x_1,\ x_2)$ it is easy to see that

$$g'(t)=\frac{\log^2(t\cdot\alpha)}{2\cdot\sqrt{t}}\cdot\left(1+\frac{4}{\log(t\cdot\alpha)}\right),$$

$$g''(t)=-\frac{\log^2(t\cdot\alpha)}{4\cdot t\cdot\sqrt{t}}\cdot\left(1-\frac{8}{\log^2(t\cdot\alpha)}\right),$$

$$g'''(t)=\frac{\log^2(t\cdot\alpha)}{t^2\cdot\sqrt{t}}\cdot\left(\frac{3}{8}-\frac{1}{2\cdot\log(t\cdot\alpha)}-\frac{3}{\log^2(t\cdot\alpha)}\right)$$

and

$$d'(t)\cdot g(t)=f'(t)-d(t)\cdot g'(t),$$

$$d''(t)\cdot g(t)=f''(t)-d(t)\cdot g''(t)-2\cdot d'(t)\cdot g'(t),$$

$$d'''(t)\cdot g(t)=f'''(t)-d(t)\cdot g'''(t)-$$

$$-3\cdot\left(d''(t)\cdot g'(t)+d'(t)\cdot g''(t)\right).$$

And it is also clear that

$$f'(t)=(1+\log t)\cdot E(t),$$

$$f''(t)=\frac{1}{t}\cdot\left(E(t)-1-\frac{1}{\log t}\right),$$

$$f'''(t)=\frac{1}{t^2}\cdot\left(1+\frac{1}{\log^2 t}-E(t)\right).$$

From (1.9) and (1.11), we respectively have

$$D_0(t):=|\log t\cdot E(t)-\theta(t)|\leq\frac{2}{\log t}\quad(t\geq e^{14}),$$

$$G_1(t):=\left|\frac{g'(t)}{g(t)}\right|=\frac{1}{2\cdot t}\cdot\left|1+\frac{4}{\log(t\cdot\alpha)}\right|\leq\frac{0.6437}{x_1}\quad(x_1\geq e^{14}),$$

$$D_1(t):=|d'(t)\cdot g(t)|\leq|(1+\log t)\cdot E(t)|+t\cdot D_0(t)\cdot G_1(t)\leq$$

$$\leq 0.1685\quad(t\geq e^{14}),$$

$$G_2(t) := \left|\frac{g''(t)}{g(t)}\right| = \frac{1}{4\cdot t^2}\cdot\left|1 - \frac{8}{\log^2(t\cdot\alpha)}\right| \le \frac{0.26}{x_1^2} \quad (x_1 \ge e^{14}),$$

$$D_2(t) := |d''(t)\cdot g(t)| \le |f''(t)| + t\cdot D_0(t)\cdot G_2(t) + 2\cdot D_1(t)\cdot G_1(t) \le$$
$$\le \frac{1.3307}{x_1} \quad (x_1 \ge e^{14}),$$

$$G_3(t) := \left|\frac{g'''(t)}{g(t)}\right| = \frac{1}{t^3}\cdot\left|\frac{3}{8} - \frac{1}{2\cdot\log(t\cdot\alpha)} - \frac{3}{\log^2(t\cdot\alpha)}\right| \le$$
$$\le \frac{0.3751}{x_1^3} \quad (x_1 \ge e^{14})$$

$$D_3(t) := |d'''(t)\cdot g(t)| \le |f'''(t)| + t\cdot D_0(t)\cdot G_3(t) +$$
$$+3\cdot(D_2(t)\cdot G_1(t) + D_1(t)\cdot G_2(t)) \le \frac{3.7650}{x_1^2} \quad (x_1 \ge e^{14}).$$

From this for any $t,\ t_1 \in (x_1,\ x_2)$ we have

$$r_0 := \left|\frac{g(t_1)}{g(t)}\right| \le 1 + \frac{g'(t_0)}{g(t)} \le 1.000001 \quad (t < t_0 < t_1),$$

$$a_1 := |d'(t)\cdot g'(t_1)| = r_0\cdot D_1(t)\cdot G_1(t_1) \le \frac{0.1085}{x_1} \quad (x_1 \ge e^{14}),$$

$$a_2 := |d'(t)\cdot g''(t_1)| = r_0\cdot D_1(t)\cdot G_2(t_1) \le \frac{0.0439}{x_1^2} \quad (x_1 \ge e^{14}),$$

$$a_3 := |d'(t)\cdot g'''(t_1)| = r_0\cdot D_1(t)\cdot G_3(t_1) \le \frac{0.0633}{x_1^3} \quad (x_1 \ge e^{14}),$$

$$b_2 := |d''(t)\cdot g'(t_1)| = r_0\cdot D_2(t)\cdot G_1(t_1) \le \frac{0.8566}{x_1^2} \quad (x_1 \ge e^{14}),$$

$$b_3 := |d'''(t)\cdot g'(t_1)| = r_0\cdot D_3(t)\cdot G_1(t_1) \le \frac{2.4236}{x_1^3} \quad (x_1 \ge e^{14}),$$

$$b_1 := |d''(t)\cdot g''(t_1)| = r_0\cdot D_2(t)\cdot G_2(t_1) \le \frac{0.3460}{x_1^3} \quad (x_1 \ge e^{14}).$$

By (4.5), we get

$$|\mathfrak{M}_\mathfrak{o}| \le (|A_1| + |A_2|)\cdot|Q_0| \le (a_2 + 2\cdot b_2)\cdot|Q_0| \le$$
$$\le \frac{0.2223}{x_1^2} \quad (x_1 \ge e^{14}),$$

$$|\mathfrak{M}_\mathfrak{1}| \le |\mathfrak{M}_\mathfrak{o}|\cdot\left(1 + \frac{0.9005}{\sqrt{x_1}}\right) \le \frac{0.2225}{x_1^2} \quad (x_1 \ge e^{14}).$$

4.5. Upper bound of Ω_1

The upper bound of Ω_1 is obtained as follows. First we will show the upper bound of

$$K'_{11} = \frac{3}{8}\cdot(U_0\cdot A_0 + A_1\cdot B_0).$$

Since

$$U_0\cdot A_0 + A_1\cdot B_0 = (U_0 + A_1)\cdot A_0 - (A_0 - B_0)\cdot A_1,$$

we have

$$U_0 + A_1 = \Big(d''(\tau_0) \cdot g'(\tau_0) - d''(\mu_2) \cdot g'(\xi_0) \Big) +$$

$$+ \Big(d'(\tau_0) \cdot g''(\tau_0) - g''(\mu_1) \cdot d'(\xi_0) \Big) =$$

$$= \Big(d''(\tau_0) \cdot (g'(\tau_0) - g'(\xi_0)) + (d''(\tau_0) - d''(\mu_2)) \cdot g'(\xi_0) \Big) +$$

$$+ \Big((d'(\tau_0) - d'(\xi_0)) \cdot g''(\tau_0) + (g''(\tau_0) - g''(\mu_1)) \cdot d'(\xi_0) \Big) \leq$$

$$\leq (b_1 + b_3) + (b_1 + a_3)$$

and

$$A_0 - B_0 = d'(\eta_2) \cdot g'(\xi_0) - d'(\alpha_2) \cdot g'(\alpha_2) =$$

$$= \Big((d'(\eta_2) - d'(\alpha_2)) \cdot g'(\xi_0) \Big) +$$

$$+ \Big((g'(\xi_0) - g'(\alpha_2)) \cdot d'(\alpha_2) \Big) < a_2 + b_2$$

and more

$$A_0 = d'(\eta_2) \cdot g'(\xi_0) < a_1,$$

$$A_1 = (-g''(\mu_1)) \cdot d'(\xi_0) + (-d''(\mu_2)) \cdot g'(\xi_0) < a_2 + b_2.$$

Thus we have

$$|K'_{11}| \leq \frac{3}{8} \cdot \Big((a_2 + b_2)^2 + a_1 \cdot (a_3 + b_3) + 2 \cdot a_1 \cdot b_1 \Big) \leq$$

$$\leq \frac{0.4334}{x_1^4} \quad (x_1 \geq e^{14}).$$

Next, we would show the upper bound of

$$L_{21} = 2 \cdot A_0 \cdot B_1 \cdot \delta_0 - 4 \cdot A_0 \cdot B_0 \cdot \delta_0 +$$

$$+ B_1 \cdot (2 - (\lambda_2 - \lambda_1)) \cdot \mathfrak{M}_{\mathfrak{o}} = O(1/t^4).$$

Since

$$A_0 \cdot \varepsilon_0 = \mathfrak{M}_{\mathfrak{o}}, \quad B_0 \cdot \varepsilon_0 = \mathfrak{M}_{\mathfrak{1}}$$

and

$$B_1 = (d_2 - d(\alpha_2)) \cdot g''(\alpha_2) - (g(\alpha_2) - g_1) \cdot d''(\alpha_2) < a_2 + b_2$$

and

$$\frac{1}{2} < \Lambda_0 = (\lambda_2 - \lambda_1) < \frac{1}{2} + 2 \cdot \varepsilon_0 < 0.5034,$$

we have

$$|L_{21}| \leq \frac{4 \cdot \varepsilon_0}{1 - \varepsilon_0} \cdot (a_2 + b_2) \cdot |\mathfrak{M}_{\mathfrak{o}}| + \frac{8}{1 - \varepsilon_0} \cdot |\mathfrak{M}_{\mathfrak{o}}| \cdot |\mathfrak{M}_{\mathfrak{1}}| +$$

$$+ \frac{3}{2} \cdot (a_2 + b_2) \cdot |\mathfrak{M}_{\mathfrak{o}}| \leq \frac{0.6978}{x_1^4} \quad (x_1 \geq e^{14}).$$

And since

$$\mathcal{V}_0 := (\alpha_2 - \alpha_1) \cdot (\xi_0 - \lambda_1) < \frac{1}{8} + \varepsilon_0 < 0.1267$$

and

$$U_1 - 3 \cdot U_0 \le (a_3 + b_3) + 3 \cdot (a_2 + b_2),$$

we also have

$$|R_1| \le \Big((a_3 + b_3) + 3 \cdot (a_2 + b_2)\Big) \cdot |\mathfrak{M}_\mathfrak{o}| \cdot \mathcal{V}_0 \le$$
$$\le \frac{0.1462}{x_1^4} \quad (x_1 \ge e^{14}).$$

Similarly, we have respectively

$$|K''_{11}| \le 3 \cdot (a_2 + b_2) \cdot (|\mathfrak{M}_\mathfrak{o}| + |\mathfrak{M}_\mathfrak{1}|) \le \frac{1.2017}{x_1^4} \quad (x_1 \ge e^{14}),$$

$$|L_{11}| \le (a_3 + b_3) \cdot a_1 \cdot \mathcal{V}_0 \le \frac{0.0342}{x_1^4} \quad (x_1 \ge e^{14}),$$

$$|R_{21}| \le (a_2 + b_2) \cdot |\mathfrak{M}_\mathfrak{o}| \cdot (\lambda_2 - \lambda_1) \le \frac{0.1008}{x_1^4} \quad (x_1 \ge e^{14}),$$

$$|\mathfrak{A}_\mathfrak{1}| \le (a_2 + b_2) \cdot |\mathfrak{M}_\mathfrak{1}| \le \frac{0.2004}{x_1^4} \quad (x_1 \ge e^{14}),$$

$$|K'_{12}| \le 4 \cdot b_2 \cdot |\mathfrak{M}_\mathfrak{1}| \le \frac{0.7625}{x_1^4} \quad (x_1 \ge e^{14}).$$

Consequently we obtain

$$\Omega_1 \le |K'_{11}| + |L_{21}| + |R_1| + |K''_{11}| + |L_{11}| + |R_{21}| + |\mathfrak{A}_\mathfrak{1}| + |K'_{12}| \le$$
$$\le \frac{0.4334}{x_1^4} + \frac{0.6978}{x_1^4} + \frac{0.1462}{x_1^4} + \frac{1.2017}{x_1^4} +$$
$$+ \frac{0.0342}{x_1^4} + \frac{0.1008}{x_1^4} + \frac{0.2004}{x_1^4} + \frac{0.7625}{x_1^4} \le$$
$$\le \frac{3.5770}{x_1^4} \quad (x_1 \ge e^{14}). \tag{4.9}$$

V. Proof of Theorem 1

We take arbitrarily the prime number $p \ge 3$. If $3 \le p \le e^{14}$, then we could confirm that the condition (d) holds from the table 1 and the table 2. Hence if the Theorem 1 does not hold, then there exists a prime number $p \ge e^{14}$ such that the condition (d') holds. For such prime p, we have $\Omega'_0 \le \Omega_1$ and, finally, from (4.8) and (4.9) we get

$$1 \le \frac{3.5770}{0.0462 \cdot \sqrt{x_1}} \le 0.08 \quad (x_1 \ge e^{14}),$$

but it is a contradiction. This shows that the condition (d') is not valid. Consequently, the Theorem 1 holds for any prime number $p \ge 3$.

VI. Algorithm and Tables for Sequence $\{H_m\}$

Here

$$H_m := (1 + \log p_m) \cdot E(p_m) -$$
$$- d(p_m) \cdot \frac{\log^2(p_m \cdot \alpha)}{2 \cdot \sqrt{p_m}} \cdot \left(1 + \frac{4}{\log(p_m \cdot \alpha)}\right) - \frac{2}{\sqrt{p_m}}, \tag{6.1}$$

where

$$\alpha := 1 + \theta(p_m),$$

$$d(p_m) := \frac{p_m \cdot (\log p_m \cdot E(p_m) - \theta(p_m))}{\sqrt{p_m} \cdot \log^2(p_m \cdot \alpha)}.$$

The table 1 and 2 show the values of H_m for $2 \le p_m \le 29$ and $93109 \le p_m \le 93118$. Note that (1.11) holds for any p_m $(3 \le p_m \le 41)$, and the condition (d) holds if and only if $H_m \le 0$ for any $m \ge 2$. It is easy to verify that $H_m \le 0$ for any $29 \le p_m \le 93109$.

The algorithm for H_m by MATLAB is as follows:
Function EMF-Index, clc, b=0.261497212847643; format long,
$P = [2, 3, 5, 7, \cdots, 1202609]$; M=length(P);
for $m = 1 : M$; $p = P(1 : m)$; $E = sum(1./p) - b - \log(\log(p(m)))$; $E1 = (1 + \log(p(m))) * E$;
$V1 = sum(\log(p.))$; $Q = (V1/p(m)) - 1$; $R = (p(m))^{1/2}$; $V = \log(V1)$; $g = R * V^2$;
$f = p(m) * (\log(p(m)) * E - Q)$; $d = f/g$; $B = (V^2) * (1 + 4/V)/2/R$;
m, $p(m)$, $H_m = E1 - d * B - 2/R$, end.

Table 1

m	p_m	H_m
1	2	4.92781518770647
2	3	−3.79708871931795
3	5	−1.82084025624172
4	7	−1.24240415973621
5	11	−1.05892911097784
6	13	−0.82377421885520
7	17	−0.75298886049588
8	19	−0.60562813217931
9	23	−0.56797602737022
10	29	−0.59342397038654

Table 2

m	p_m	H_m
93109	1202477	−0.00169503567169
93110	1202483	−0.00168790073361
93111	1202497	−0.00168788503420
93112	1202501	−0.00167897043350
93113	1202507	−0.00167183566691
93114	1202549	−0.00169673633799
93115	1202561	−0.00169494084824
93116	1202569	−0.00168958602998
93117	1202603	−0.00170736660136
93118	1202609	−0.00170023228562

VII. Some Preparations for Theorem 2

From the section VII to the section IX we would handle the Theorem 2.
As in the section III, we make ready for the proof of the Theorem 2.

7.1. Some symbols

Let $p_1 = 2$, $p_2 = 3$, $p_3 = 5, \cdots$ be the first consecutive primes. Then p_m $(m \in N)$ is m-th prime number. We arbitrary choose the prime number $p_m \ge 3$ and fix it. We put $p_0 = p_m,\ p = p_{m-1}$ below. For the theoretical calculation we assume $p \ge e^{14}$. The discussion for $p \le e^{14}$ is supported by MATLAB. Put

$$e_0 := \exp(E(p_0)), \qquad e_0' := \exp(\log p_0 \cdot (e_0 - 1)),$$

$$e_1 := \exp(E(p)), \qquad e_1' := \exp(\log p \cdot (e_1 - 1)),$$

$$\alpha_0 := 1 + \theta(p_0), \qquad \alpha := 1 + \theta(p),$$

$$N_0 := \sqrt{p_0 \cdot \alpha_0} \cdot \log^2(p_0 \cdot \alpha_0), \qquad N_1 := \sqrt{p \cdot \alpha} \cdot \log^2(p \cdot \alpha).$$

7.2. An estimate of e_1 , e_1' and $(e_1 \cdot e_1')$
From (1.9) we respectively have

$$e_1 < \exp(1/\log^2 p) < 1.0052 \quad (p \geq e^{14}), \qquad e_1' < 1.075 \quad (p \geq e^{14}),$$

$$(e_1 \cdot e_1') < 1.08 \quad (p \geq e^{14}).$$

Since if $e_1 \leq 1$ then $e_1' \leq 1$, we have $(e_1 \cdot e_1') \leq 1$. If $e_1 > 1$, then

$$0 < r := E(p) < \frac{1}{\log^2 p} \leq 0.0052 \quad (p \geq e^{14})$$

and

$$e_1 = 1 + r + \sum_{n=2}^{\infty} \frac{r^n}{n!} \leq 1 + r + \frac{r^2}{2 \cdot (1-r)} \leq 1 + r + 0.503 \cdot r^2,$$

$$e_1 \cdot e_1' = \exp(r + (\log p) \cdot (e_1 - 1)) \leq 1 + h + \frac{h^2}{2 \cdot (1-h)},$$

where

$$h = (1 + \log p) \cdot r + 0.503 \cdot \log p \cdot r^2 \leq 0.1125 \quad (p \geq e^{14}).$$

Therefore we have

$$(e_1 \cdot e_1' - 1) \leq (1 + \log p) \cdot E(p) + 0.6 \cdot (1 + \log p)^2 \cdot E(p)^2 \quad (e_1 > 1,\ p \geq e^{14}).$$

7.3. An estimate of $V_0 := p_0 \cdot (e_0' - \alpha_0) - p \cdot (e_1' - \alpha)$
By (1.10) it is clear that

$$p_0 \cdot \alpha_0 - p \cdot \alpha = \vartheta(p_0) - \vartheta(p) = \log p_0$$

and, since

$$E(t) = \sum_{p \leq t} p^{-1} - \log\log t - b,$$

we have

$$E(p_0) - E(p) = \frac{1}{p_0} - \log\left(\frac{\log p_0}{\log p}\right).$$

From this

$$\frac{e_0}{e_1} = \left(\frac{\log p}{\log p_0}\right) \cdot \exp(p_0^{-1}),$$

$$\frac{e_0'}{e_1'} = \frac{p}{p_0} \cdot \exp\left(\log p \cdot e_1 \cdot \left(\exp(p_0^{-1}) - 1\right)\right).$$

Thus we have

$$V_0 = p \cdot e_1' \cdot \left(\frac{p_0 \cdot e_0'}{p \cdot e_1'} - 1\right) - \log p_0 = \log p_0 \cdot (\mu \cdot e_1' - 1),$$

where

$$\mu = \frac{p}{\log p_0} \cdot \left(\exp\left(\log p \cdot e_1 \cdot \left(\exp(p_0^{-1}) - 1\right)\right) - 1\right).$$

Since

$$\exp(t) < 1 + t + \frac{1}{2} \cdot \frac{t^2}{1-t} \quad (0 < t < 1),$$

we have

$$\mu \le e_1 + 0.505 \cdot \frac{\log p}{p} \quad (e_1 > 1,\ \ p \ge e^{14})$$

and hence

$$\mu \cdot e_1' - 1 \le (e_1 \cdot e_1' - 1) + 0.55 \cdot \frac{\log p}{p} \quad (e_1 > 1,\ p \ge e^{14}).$$

7.4. An estimate of $G_0 := (\log p_0 \cdot R(p \cdot \alpha) - (N_0 - N_1))/N_0$

Here

$$R(p \cdot \alpha) := \frac{\log^2(p \cdot \alpha)}{2 \cdot \sqrt{p \cdot \alpha}} \cdot \left(1 + \frac{4}{\log(p \cdot \alpha)}\right).$$

It is known that $p_{k+1}^2 \le (p_1 \cdots p_k)$ for $p_k \ge 7$ on p. 246 of [6] and hence

$$\frac{\log p_0}{\log(p \cdot \alpha)} < \frac{1}{2} \quad (p \ge e^{14}).$$

Since $\log(1 + t) \ge (t - t^2/2)$ for any t $(0 < t < 1/2)$, we have

$$N_0 - N_1 = (\sqrt{p_0 \cdot \alpha_0} - \sqrt{p \cdot \alpha}) \cdot \log^2(p_0 \cdot \alpha_0) +$$

$$+\sqrt{p \cdot \alpha} \cdot (\log^2(p_0 \cdot \alpha_0) - \log^2(p \cdot \alpha)) \ge$$

$$\ge \frac{\log p_0}{2 \cdot \sqrt{p_0 \cdot \alpha_0}} \cdot \log^2(p \cdot \alpha) +$$

$$+2 \cdot \sqrt{p \cdot \alpha} \cdot \log(p \cdot \alpha) \cdot \log\left(1 + \frac{\log p_0}{p \cdot \alpha}\right) \ge$$

$$\ge \frac{\log p_0}{2 \cdot \sqrt{p_0 \cdot \alpha_0}} \cdot \log^2(p \cdot \alpha) +$$

$$+ \log p_0 \cdot \frac{2 \cdot \log(p \cdot \alpha)}{\sqrt{p \cdot \alpha}} \cdot \left(1 - \frac{\log p_0}{2 \cdot (p \cdot \alpha)}\right)$$

and

$$G_0 \cdot N_0 \le \frac{\log p_0}{2} \cdot \left(\frac{1}{\sqrt{p \cdot \alpha}} - \frac{1}{\sqrt{p_0 \cdot \alpha_0}}\right) \cdot \log^2(p \cdot \alpha) +$$

$$+ \frac{\log^2 p_0}{(p \cdot \alpha)^{3/2}} \cdot \log(p \cdot \alpha) \le$$

$$\le \log^2 p_0 \cdot \frac{\log^2(p \cdot \alpha)}{(p \cdot \alpha)^{3/2}} \cdot \left(\frac{1}{4} + \frac{1}{\log(p \cdot \alpha)}\right).$$

And it is known that $p_{k+1}^2 \le 2 \cdot p_k^2$ for $p_k \ge 7$ on p. 247 of [6] and so

$$\log p_0 \le (\log p) \cdot \left(1 + \frac{\log \sqrt{2}}{\log p}\right).$$

Since $p \ge e^{14}$, we have $\alpha \ge (1 - 1/14)$ and the function $(\log^3 t)/t$ is decreasing on the interval $(e^3,\ +\infty)$. Therefore we get

$$G_0 \le \frac{\log^2 p_0}{(p \cdot \alpha)^2} \cdot \left(\frac{1}{4} + \frac{1}{\log(p \cdot \alpha)}\right) \le$$

$$\leq \frac{\log^3 p}{p \cdot \alpha^2} \cdot \left(1 + \frac{\log \sqrt{2}}{\log p}\right)^2 \cdot \left(\frac{1}{4} + \frac{1}{\log p + \log \alpha}\right) \cdot \frac{1}{p \cdot \log p} \leq$$

$$\leq \frac{0.01}{p \cdot \log p} \quad (p \geq e^{14}).$$

7.5. An estimate of $S(p') := \sum_{p \geq p'} 1/(p \cdot \log p)$

Put

$$s(t) := \sum_{p \leq t} p^{-1} = \log\log t + b + E(t).$$

Then by the Abel's identity [1], we have

$$S(p') = \int_{p'}^{+\infty} \frac{1}{\log t} \cdot ds(t) = \int_{p'}^{+\infty} \frac{1}{\log t} \cdot \left(\frac{dt}{t \cdot \log t} + dE(t)\right) \leq$$

$$\leq \frac{1}{\log p'} - \frac{E(p')}{\log p'} + \int_{p'}^{+\infty} \frac{dt}{t \cdot \log^4 t} \leq$$

$$\leq \frac{1}{\log p'} + \frac{1}{\log^3 p'} - \frac{1}{3 \cdot \log^3 t}\Big|_{p'}^{+\infty} =$$

$$= \frac{1}{\log p'} + \frac{4}{3 \cdot \log^3 p'}$$

and

$$S(p') \geq \frac{1}{\log p'} - \frac{4}{3 \cdot \log^3 p'}.$$

If p' is a first prime $\geq e^{14}$, then $p' = 1202609$ and it is 93118-th prime. And we have

$$0.070 \leq S(p') \leq 0.072.$$

7.6. Lemma

For any $m \geq 1$ we put

$$D_m := \frac{p_m \cdot (e_0' - \alpha_0)}{\sqrt{p_m \cdot \alpha_0} \cdot \log^2(p_m \cdot \alpha_0)}.$$

Then we are ready for the proof of the following lemma.

[**Lemma**] For any $m \geq 5$ we have $D_m < 1$.

Proof. First, for any $11 \leq p_m \leq e^{14}$ we see $D_m < 1$ by MATLAB (see the table 5 and the table 6 below). Next, we will prove that

$$D_m \leq a_m := 1 - 13 \cdot S(p_m)$$

holds for any $p_m \geq e^{14}$ by the mathematical induction with respect to m.

If $p' = 1202609$ then we have

$$D_{93118} = 0.0103 \cdots \leq 1 - 13 \cdot S(p') \leq 0.1 < 1.$$

Now assume $p \geq e^{14}$ and $D_{m-1} \leq a_{m-1}$. Then

$$D_m = \frac{1}{N_0} \cdot (p \cdot (e_1' - \alpha) + V_0) = D_{m-1} \cdot \frac{N_1}{N_0} + \frac{V_0}{N_0} \leq$$

$$\leq a_{m-1} \cdot \frac{N_1}{N_0} + \frac{1}{N_0} \cdot \log p_0 \cdot (\mu \cdot e_1' - 1) \leq a_{m-1} + b_{m-1},$$

where

$$b_{m-1} = \frac{1}{N_0} \cdot (\log p_0 \cdot (\mu \cdot e_1' - 1) - a_{m-1} \cdot (N_0 - N_1)).$$

By the assumption $D_{m-1} \leq a_{m-1}$, we get

$$e_1' \leq \alpha + a_{m-1} \cdot \frac{\sqrt{p \cdot \alpha} \cdot \log^2(p \cdot \alpha)}{p} = \alpha \cdot \left(1 + a_{m-1} \, \frac{\log^2(p\,\alpha)}{\sqrt{p\,\alpha}}\right)$$

and by taking logarithm of both sides

$$\log e_1' = (\log p) \cdot (e_1 - 1) \leq \theta(p) + a_{m-1} \cdot \frac{\log^2(p \cdot \alpha)}{\sqrt{p \cdot \alpha}}.$$

From this

$$e_1 \leq 1 + \frac{1}{\log p} \cdot \left(\theta(p) + a_{m-1} \cdot \frac{\log^2(p \cdot \alpha)}{\sqrt{p \cdot \alpha}}\right),$$

$$E(p) \leq \frac{1}{\log p} \left(\theta(p) + a_{m-1} \cdot \frac{\log^2(p \cdot \alpha)}{\sqrt{p \cdot \alpha}}\right).$$

Thus

$$\log p \cdot E(p) - \theta(p) \leq a_{m-1} \cdot \frac{\log^2(p \cdot \alpha)}{\sqrt{p \cdot \alpha}}$$

and the both sides multiply by

$$\frac{p}{\sqrt{p} \cdot \log^2(p \cdot \alpha)},$$

then

$$d(p) := \frac{p \cdot \log p \cdot E(p) - p \cdot \theta(p)}{\sqrt{p} \cdot \log^2(p \cdot \alpha)} \leq \frac{a_{m-1}}{\sqrt{\alpha}}.$$

From the Theorem 1 we get

$$(1 + \log p) \cdot E(p) \leq a_{m-1} \cdot \frac{\log^2(p \cdot \alpha)}{2 \cdot \sqrt{p \cdot \alpha}} \cdot \left(1 + \frac{4}{\log(p \cdot \alpha)}\right) + \frac{2}{\sqrt{p}}.$$

If $e_1 > 1$, then, since $0 < a_{m-1} \leq 1$ and

$$(1 - 1/14) \leq \alpha \leq (1 + 1/14),$$

we also have

$$(1 + \log p)^2 \cdot E(p)^2 \leq \frac{\log^4(p \cdot \alpha)}{p \cdot \alpha} \cdot \left(\frac{1}{2} + \frac{2}{\log(p \cdot \alpha)} + \frac{2 \cdot \sqrt{\alpha}}{\log^2(p \cdot \alpha)}\right)^2 \leq$$

$$\leq 0.4287 \cdot \frac{\log^4(p \cdot \alpha)}{p \cdot \alpha} \quad (p \geq e^{14})$$

and

$$\log p_0 \cdot (\mu \cdot e_1' - 1) - a_{m-1} \cdot (N_0 - N_1) \leq \log p_0 \cdot (1 + \log p) \cdot E(p) -$$

$$- a_{m-1} \cdot (N_0 - N_1) + 0.55 \cdot \frac{\log^2 p_0}{p} + 0.6 \cdot \log p_0 \cdot (1 + \log p)^2 \cdot E(p)^2 \leq$$

$$\leq a_{m-1}\cdot G_0\cdot N_0+0.55\cdot\frac{\log^2 p_0}{p}+0.2572\cdot\log p_0\cdot\frac{\log^4(p\cdot\alpha)}{p\cdot\alpha}+\frac{2\cdot\log p_0}{\sqrt{p}}.$$

Finally, by the function $(\log^4 t)/\sqrt{t}$ is decreasing on the interval $(e^8,\ +\infty)$ we have

$$b_{m-1}\leq a_{m-1}\cdot G_0+0.55\cdot\frac{\log p}{\sqrt{p\cdot\alpha}}\cdot\left(1+\frac{\log\sqrt{2}-\log\alpha}{\log p+\log\alpha}\right)^2\cdot\frac{1}{p\cdot\log p}+$$

$$+0.2572\cdot\frac{\log^4 p}{\sqrt{p}}\cdot\frac{1+\log\sqrt{2}/\log p}{\alpha^{3/2}}\cdot\frac{(1+\log\alpha/\log p)^2}{p\cdot\log p}+$$

$$+\frac{2}{\sqrt{\alpha}}\cdot\left(1+\frac{\log\sqrt{2}-\log\alpha}{\log p+\log\alpha}\right)^2\cdot\frac{1}{p\cdot\log p}\leq$$

$$\leq\frac{0.01}{p\cdot\log p}+\frac{0.01}{p\cdot\log p}+\frac{10.421}{p\cdot\log p}+\frac{2.203}{p\cdot\log p}\leq$$

$$\leq\frac{13}{p\cdot\log p}\quad(p\geq e^{14}).$$

Next, if $e_1\leq 1$ then we have

$$b_{m-1}\leq 0.55\cdot\frac{\log^2 p_0}{p\cdot N_1}\leq\frac{0.01}{p\cdot\log p}\quad(p\geq e^{14}).$$

VIII. Proof of Theorem 2

First, we would show that (2.6) holds for any prime number $p\geq 3$.
Since $\exp(t)>1+t\ \ (-1<t<1)$, for any prime number $p\geq 3$ we have

$$p\cdot\log p\cdot E(p)-p\cdot\theta(p)=$$

$$=p\cdot(1+\log p\cdot E(p))-p\cdot(1+\theta(p))\leq$$

$$\leq p\cdot e_1'-p\cdot\alpha.$$

From this, by the Lemma, we have $d(p_m)\leq D_m$ for any $m\geq 6$. Hence $d(p)<1$ for any $p\geq 11$ and by (2.1) of the Theorem 1

$$E(p)\leq\frac{1}{1+\log p}\cdot\left(\frac{\log^2(p\cdot\alpha)}{2\cdot\sqrt{p}}\cdot\left(1+\frac{4}{\log(p\cdot\alpha)}\right)+\frac{2}{\sqrt{p}}\right).$$

Here if $p\geq 71$ then

$$\frac{1}{\log p\cdot(1+\log p)}\cdot\left(\frac{\log^2(p\cdot\alpha)}{2}+2\cdot\log(p\cdot\alpha)+2\right)<0.935\quad(p\geq 71),$$

and if $3\leq p\leq 71$ then we get (2.6) (see the Table 3 and the Table 4 below). Thus

$$E(p)<\frac{\log p}{\sqrt{p}}.\quad(p\geq 3).$$

holds for any prime number $p\geq 3$. Next, for any real number $t\geq 3$, there exists a prime number $p\geq 3$ such that $p\leq t<p_0$. Put

$$Z(t):=E(t)-\frac{\log t}{\sqrt{t}}\quad(t\in[p,\ p_0)),$$

then $Z'(t) < 0 \ \ (t \in (p,\ p_0))$ and so $Z(t) \leq Z(p) < 0$. This shows that (2.6) holds for any real number $t \geq 3$.

IX. Algorithm and Tables for Sequence $\{Z_m\}$ and $\{D_m\}$

The table 3 and the table 4 show the values of

$$Z_m := E(p_m) - \frac{\log p_m}{\sqrt{p_m}}$$

for $1 \leq m \leq 20$ to m. Note that (2.6) holds for $m \geq 1$ if and only if $Z_m < 0$. And the table 5 shows the values of D_m. There are only values of D_m for $1 \leq m \leq 10$ here. But it is not difficult to verify them for $31 \leq p_m \leq e^{14}$. The table 6 shows the values D_m for $93109 \leq m \leq 93118$. Of course, all the values in the tables are approximate.

The algorithm for Z_m and D_m to m by matlab is as follows:

Function EMF-Index, clc, b=0.261497212847643; format long

$P = [2, 3, 5, 7, \cdots, 1202609]$; M=length(P);

for $m = 1 : M$; $p = P(1 : m)$; $E(p(m)) = sum(1./p) - b - \log(\log(p(m)))$;

$A(p(m)) = \log(p(m))/\sqrt{p(m)}$; $\vartheta(p(m)) = sum(\log(p.))$; $\alpha_0 = \vartheta(p(m))/p(m)$; $R = (p(m))^{1/2}$;

$V = \log(\vartheta(p(m)))$; $g = R * V^2$; $e_0' = \exp(\log(p(m)) * (\exp(E(p(m))) - 1))$,

$m,\ p_m,\ Z_m = E(p(m)) - A(p(m)),\quad D_m = p(m) * (e_0' - \alpha_0)/g$, end

Table 3

m	p_m	Z_m
1	2	0.11488663599975
2	3	−0.15649580772857
3	5	−0.42381139039502
4	7	−0.48652145124644
5	11	−0.59198165652987
6	13	−0.57080225951896
7	17	−0.58721773344997
8	19	−0.56143838923050
9	23	−0.55912374510489
10	29	−0.56745957678030

Table 4

m	p_m	Z_m
11	31	−0.54628472181999
12	37	−0.54636660696827
13	41	−0.53633985985008
14	43	−0.51944301813516
15	47	−0.50956392869224
16	53	−0.50518392944789
17	59	−0.50037701659240
18	61	−0.48761926443285
19	67	−0.48260095592014
20	71	−0.47441536685574

Table 5

m	p_m	D_m
1	2	25.62071141247196
2	3	8.97923715714347
3	5	1.91868003953127
4	7	1.04417674546040
5	11	0.65533994162650
6	13	0.50557929260089
7	17	0.40546150815241
8	19	0.35633549506425
9	23	0.31785034607111
10	29	0.27811621292050

Table 6

m	p_m	D_m
93109	1202477	0.01038794622881
93110	1202483	0.01038795465981
93111	1202497	0.01038796309106
93112	1202501	0.01038797210397
93113	1202507	0.01038798158228
93114	1202549	0.01038798943309
93115	1202561	0.01038799740043
93116	1202569	0.01038800571686
93117	1202603	0.01038801287098
93118	1202609	0.01038802049040

References

[1] Tom M. Apostol, Introduction to Analytic Number Theory, Springer-Verlag, New York, Heidelberg Berlin, 1976.
[2] P. Borwein, S. Choi, B. Rooney, A. Weirathmueller, The Riemann Hypothesis, Springer, Berlin Heidelberg, 2007.
[3] A. E. Ingham, The distribution of prime numbers, Cambridge Tract No.30, Cambridge University Press, 1932.
[4] J. B. Rosser, L. Schoenfeld, Approximate formulas for some functions of prime numbers, Illinois J. Math. 6 (1962), 64-94.
[5] J. Sandor, D. S. Mitrinovic, B. Crstici, Handbook of Number theory 1, Springer, 2006.
[6] I. M. Vinogradov, Novaya ocenka funkcii $\zeta(1+it)$, Izv. Akad. Nauk SSSR, Ser. Mat., vol. 22(1958), pp. 161-164.

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