The Number s: Formulas and Fractals

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abstract

Some formulas and fractals related with the number s .

1. Introduction: The Number s .

$$s = \frac{2+\sqrt{3}-\sqrt{4\sqrt{3}-1}}{2} \quad (1)$$

$$s = 0.6486303127934829527722790480187 9... \quad (2)$$

$$s = \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+...}}} = [0:1,1,1,5,2,39,2,1,1,1,4,20,1,7,1,1,4,4,1,4,2,1,...] \quad (3)$$

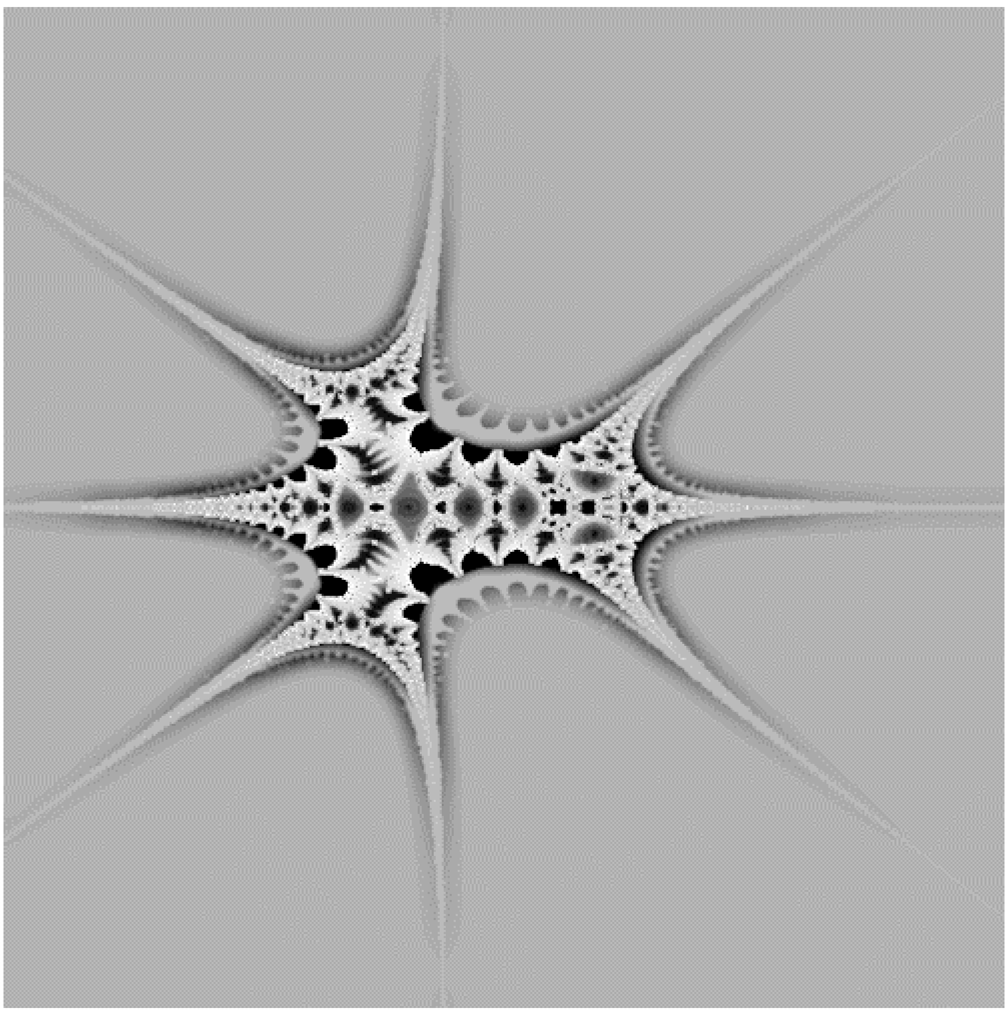

Figure 1.

2. The Number $1/s$.

$$\frac{1}{s}=\frac{2+\sqrt{3}+\sqrt{4\sqrt{3}-1}}{4} \tag{4}$$

$$\frac{1}{s}=1.54171024738769717037758364674353... \tag{5}$$

$$\frac{1}{s}=1+\frac{\sqrt{3}}{4}+\frac{1}{4}\sum_{n=0}^{\infty}\binom{2n}{n}\frac{(-1)^n\,2^{-2n}}{n+1}\left(\sqrt{3}-\frac{5}{4}\right)^{n+1} \tag{6}$$

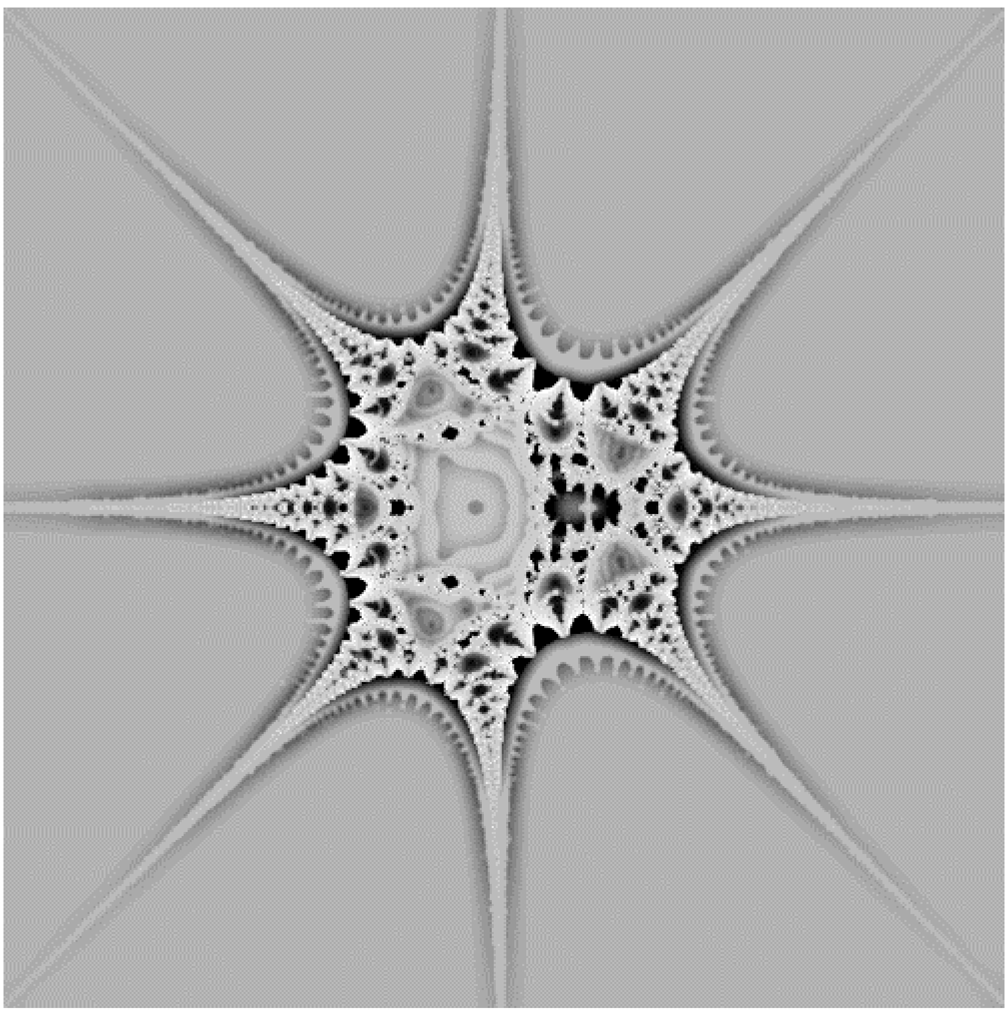

Figure 2.

3. Continued Fraction

$$s = \cfrac{2}{2+\sqrt{3} - \cfrac{2}{2+\sqrt{3} - \cfrac{2}{2+\sqrt{3} - \ldots}}} \tag{7}$$

$$x_{n+1} = \frac{2}{2+\sqrt{3}-x_n} \quad , x_1 = 0 \Rightarrow x_n \to s \tag{8}$$

$$x_n = \left\{ 0, 4-2\sqrt{3}, \frac{4+6\sqrt{3}}{23}, \frac{28}{13} - \frac{34\sqrt{3}}{39}, \frac{12+146\sqrt{3}}{409}, \frac{1612-526\sqrt{3}}{1081}, \ldots \right\} \tag{9}$$

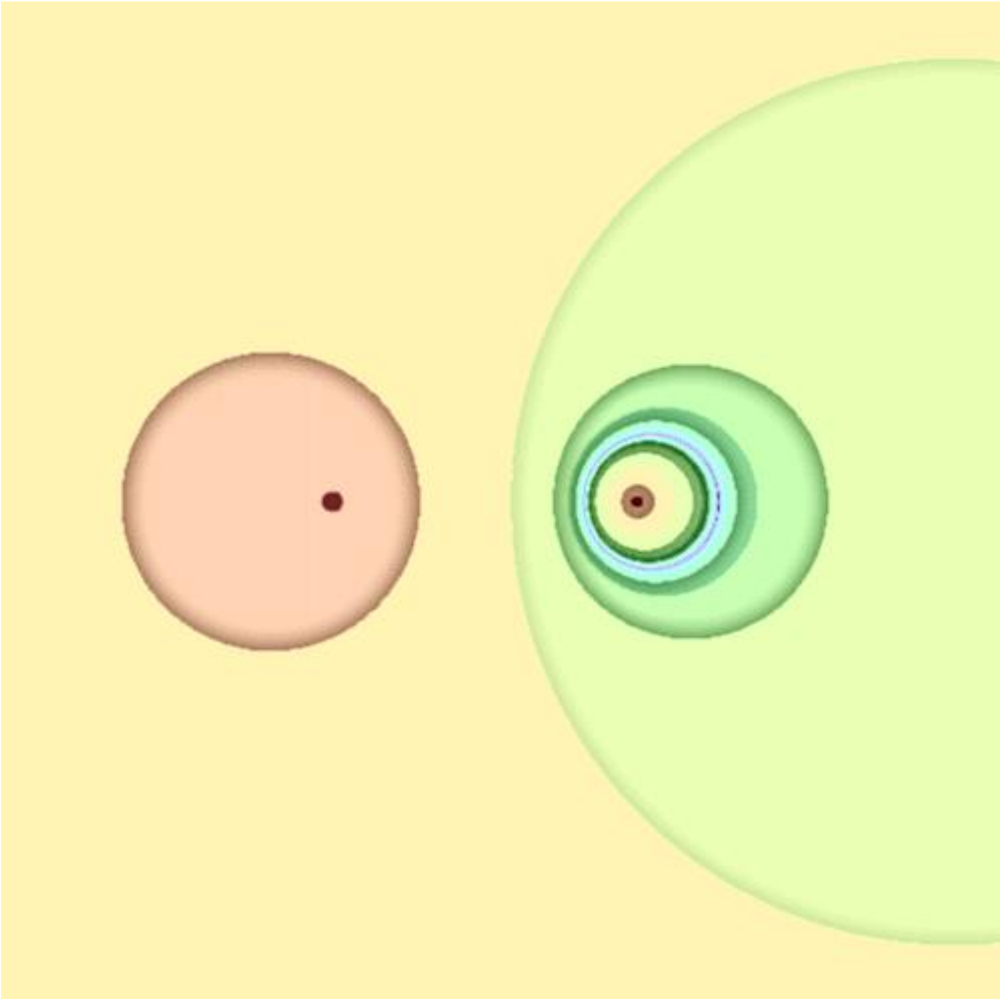

Figure 3.

4. Sequence

$$x_{n+1} = 4 - 2\sqrt{3} + \left(2 - \sqrt{3}\right) x_n^2 \quad , x_1 = 0 \Rightarrow x_n \to s \tag{10}$$

$$x_n = \left\{0, 4 - 2\sqrt{3}, 108 - 62\sqrt{3}, 86572 - 49982\sqrt{3}, \ldots\right\} \tag{11}$$

$$s = 4 - 2\sqrt{3} + \left(2 - \sqrt{3}\right)\left(4 - 2\sqrt{3} + \left(2 - \sqrt{3}\right)\left(4 - 2\sqrt{3} + \ldots\right)^2\right)^2 \tag{12}$$

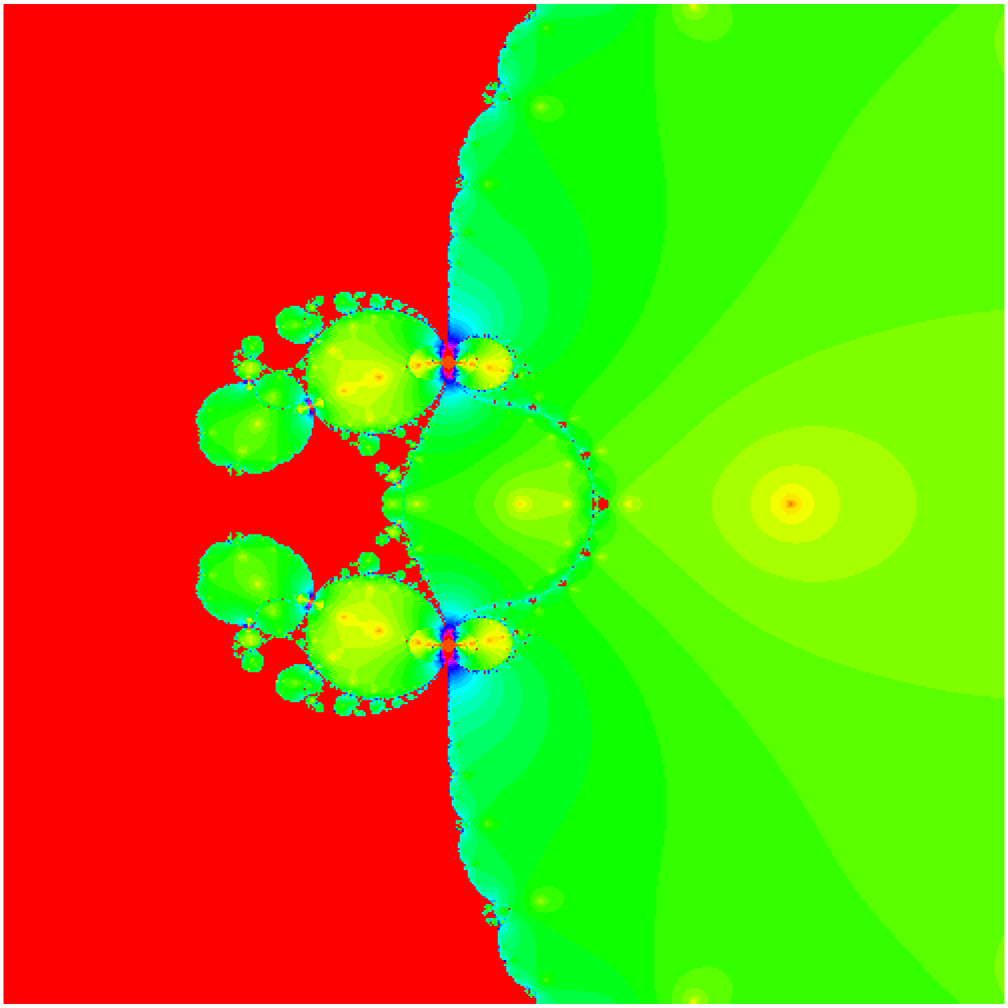

Figure 4.

5. Linear Recurrence

$$x_{n+4} = 4x_{n+3} - 5x_{n+2} + 8x_{n+1} - 4x_n \quad , n \in \mathbb{N} \tag{13}$$

$$x_1 = 1, x_2 = 4, x_3 = 11, x_4 = 32 \tag{14}$$

$$\lim_{n\to\infty} \frac{2x_n}{x_{n+1}} = s \tag{15}$$

$$x_n = \{1, 4, 11, 32, 101, 316, 971, 2984, 9205, 28404, 87579, 270000, ...\} \tag{16}$$

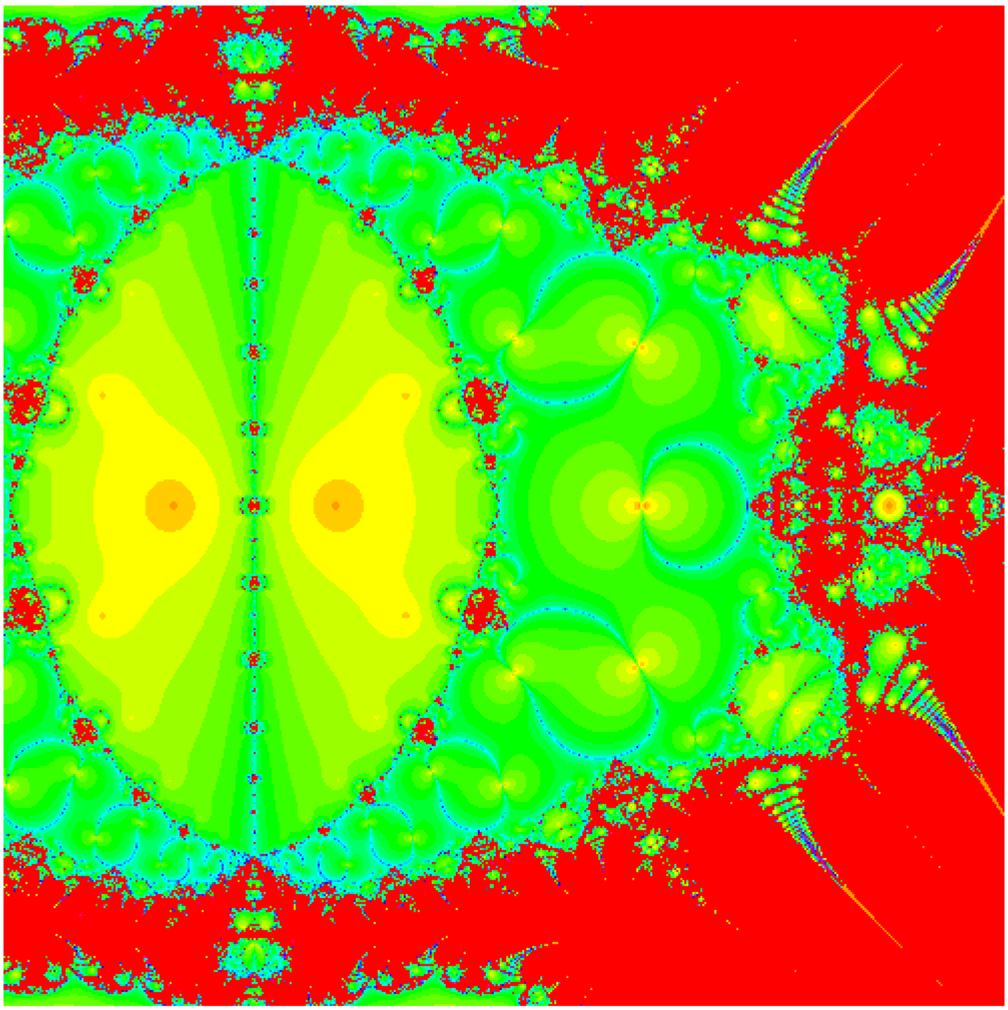

Figure 5.

6. Radicals for s .

$$s=1+\frac{\sqrt{3}}{2}-\frac{1}{2}\sqrt[4]{47-2\sqrt{47-2\sqrt{47-2\sqrt{47-\ldots}}}} \tag{17}$$

$$s=1+\frac{\sqrt{3}}{2}-\frac{1}{2}\sqrt{-2+\sqrt{47+2\sqrt{47+2\sqrt{47+\ldots}}}} \tag{18}$$

$$s=1+\frac{\sqrt{3}}{2}-\frac{1/2}{\sqrt[4]{\frac{1}{47}+\frac{2}{47}\sqrt{\frac{1}{47}+\frac{2}{47}\sqrt{\frac{1}{47}+\ldots}}}} \tag{19}$$

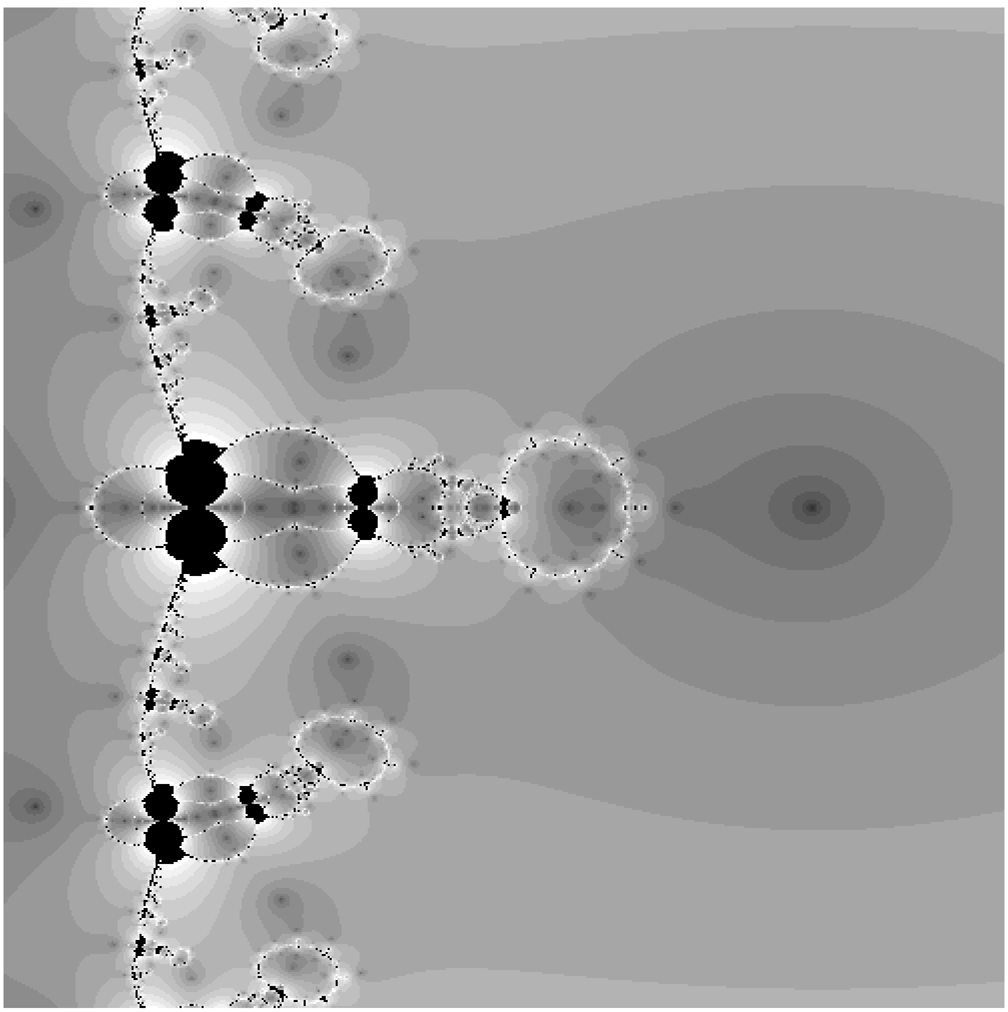

Figure 6.

7. Radicals for $1/s$.

$$\frac{1}{s}=1+\frac{1}{2}\sqrt{2\sqrt{3}-2-\left(2-\sqrt{3}\right)\sqrt{2\sqrt{3}-2-\left(2-\sqrt{3}\right)\sqrt{2\sqrt{3}-2-...}}} \tag{20}$$

$$\frac{1}{s}=\frac{1}{2}+\frac{1}{4}\sqrt{2+2\sqrt{9+12\sqrt{2+2\sqrt{9+12\sqrt{2+2\sqrt{9+...}}}}}} \tag{21}$$

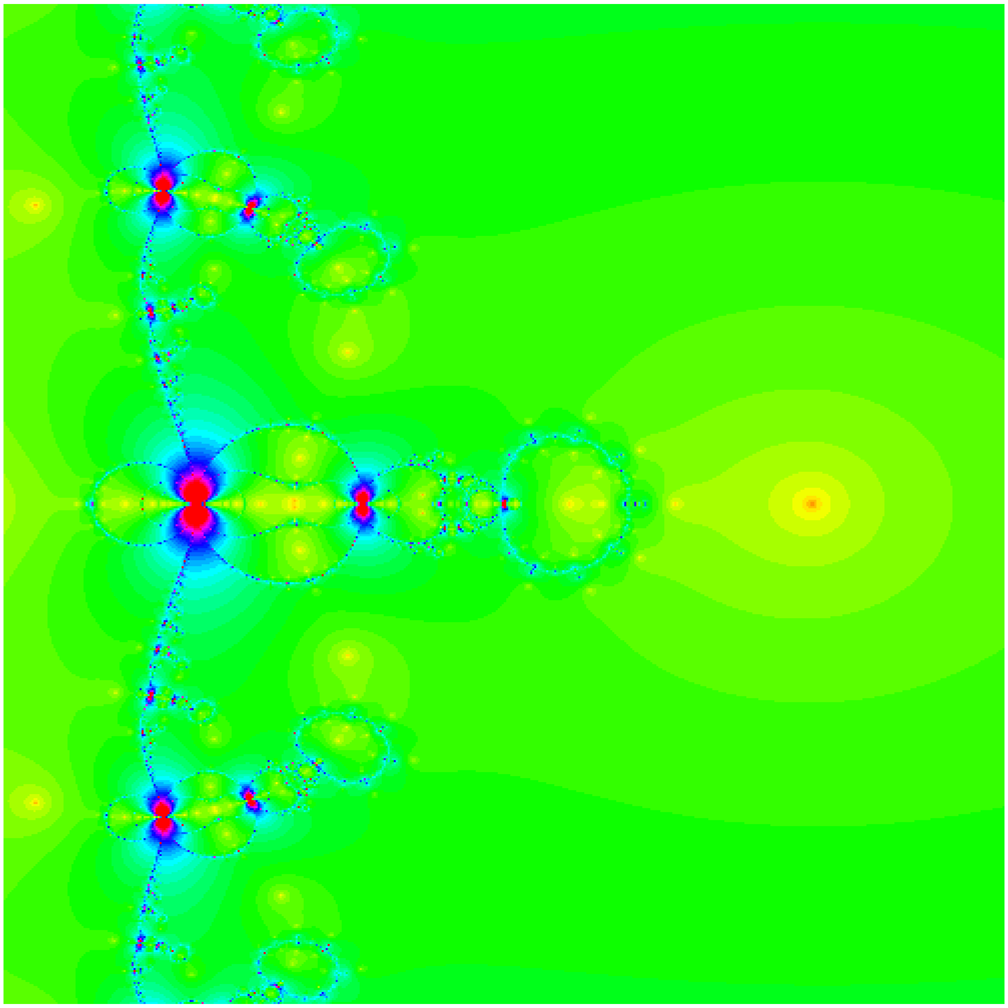

Figure 7.

8. Recurrences

$$x_{n+1} = \frac{4+5x_n^2-4x_n^3+x_n^4}{8} \quad , x_1 = 0 \Rightarrow x_n \to s \tag{22}$$

$$x_{n+1} = \frac{3x_n^4-8x_n^3+5x_n^2-4}{4x_n^3-12x_n^2+10x_n-8} \quad , x_1 = 0 \Rightarrow x_n \to s \tag{23}$$

$$x_{n+1} = \frac{4-2x_n+5x_n^2-4x_n^3+x_n^4}{6} \quad , x_1 = 0 \Rightarrow x_n \to s \tag{24}$$

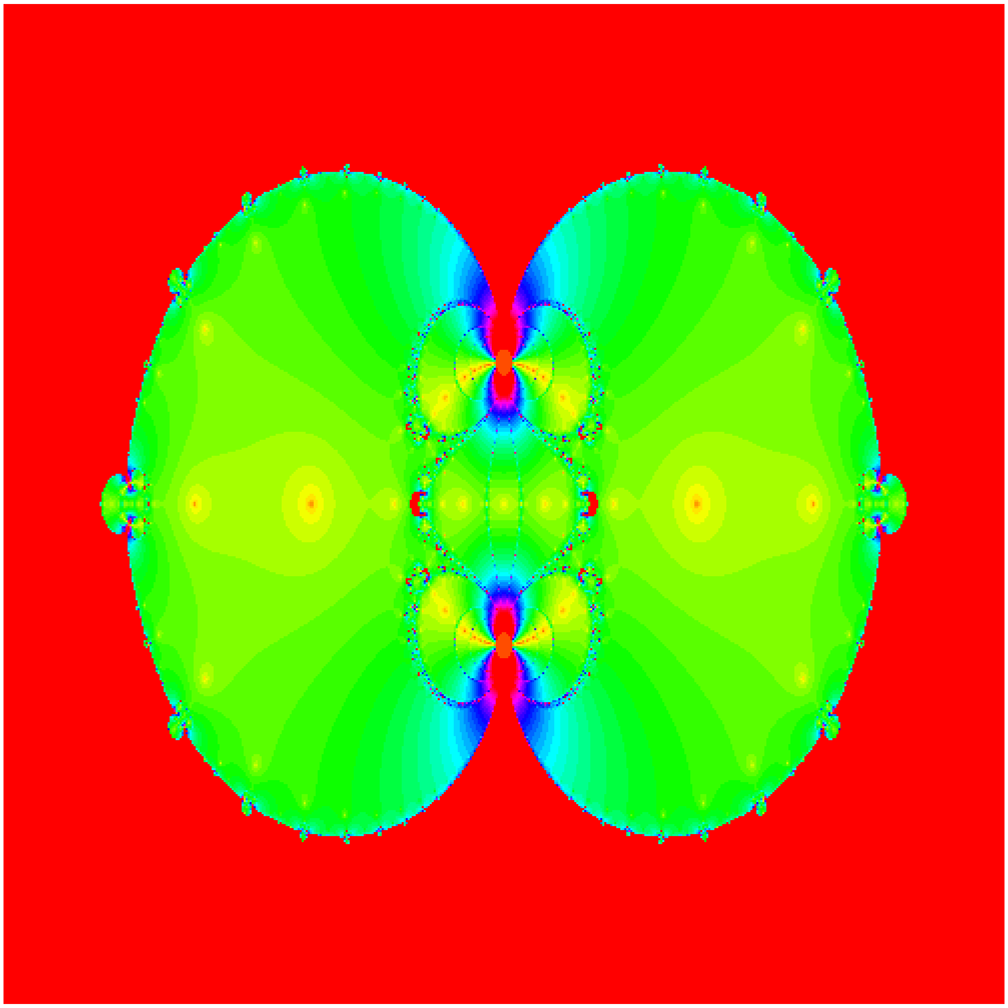

Figure 8.

9. Equations

$$s^4 - 4s^3 + 5s^2 - 8s + 4 = 0 \tag{25}$$

$$s^5 - 11s^3 + 12s^2 - 28s + 16 = 0 \tag{26}$$

$$s^6 - 32s^3 + 27s^2 - 72s + 44 = 0 \tag{27}$$

$$s^7 - 101s^3 + 88s^2 - 212s + 128 = 0 \tag{28}$$

$$s^8 - 316s^3 + 293s^2 - 680s + 404 = 0 \tag{29}$$

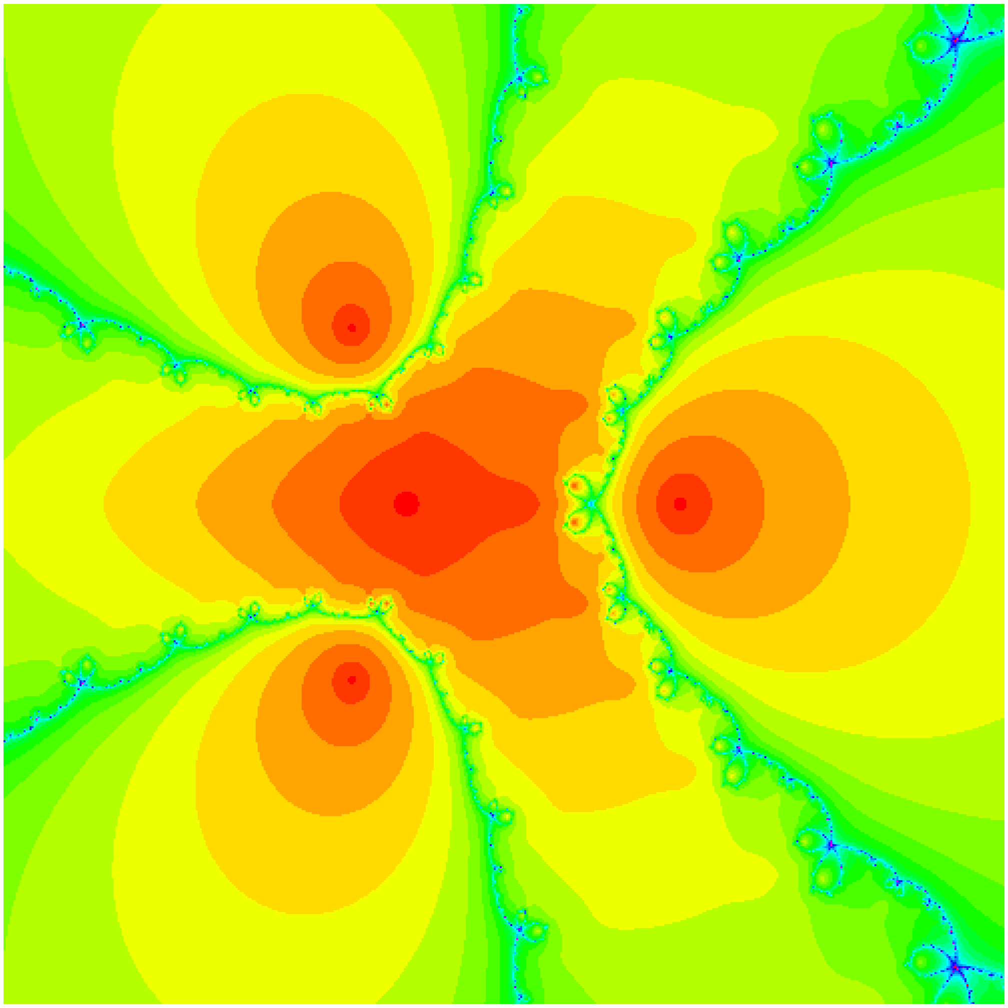

Figure 9.

10. Number Pi

$$\pi = 12\sum_{n=0}^{\infty}\frac{(-1)^n\left(1-2^{-2n-1}\right)s^{2n+1}}{2n+1} = 3\sum_{n=0}^{\infty}\frac{c_n\,2^{-2n}s^{2n+1}}{2n+1} \tag{30}$$

$$c_{n+2} = -5c_{n+1} - 4c_n \quad , c_0 = 2, c_1 = -14 \tag{31}$$

$$\pi = 6\sum_{n=0}^{\infty}(-2)^{-n}\,s^{2n+1}\sum_{k=0}^{n}\binom{n+k}{n-k}\frac{2^{-k}}{2k+1} \tag{32}$$

$$\pi = 6\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n+k}{n-k}\frac{(-1)^n}{2k+1}s^{2k+1}(1-s)^{2n-2k} \tag{33}$$

$$\pi = 3\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n+k}{n-k}\frac{(-4)^{-k}}{2k+1}s^{n+k+1}\left(1-\frac{s}{2}\right)^{n-k} \tag{34}$$

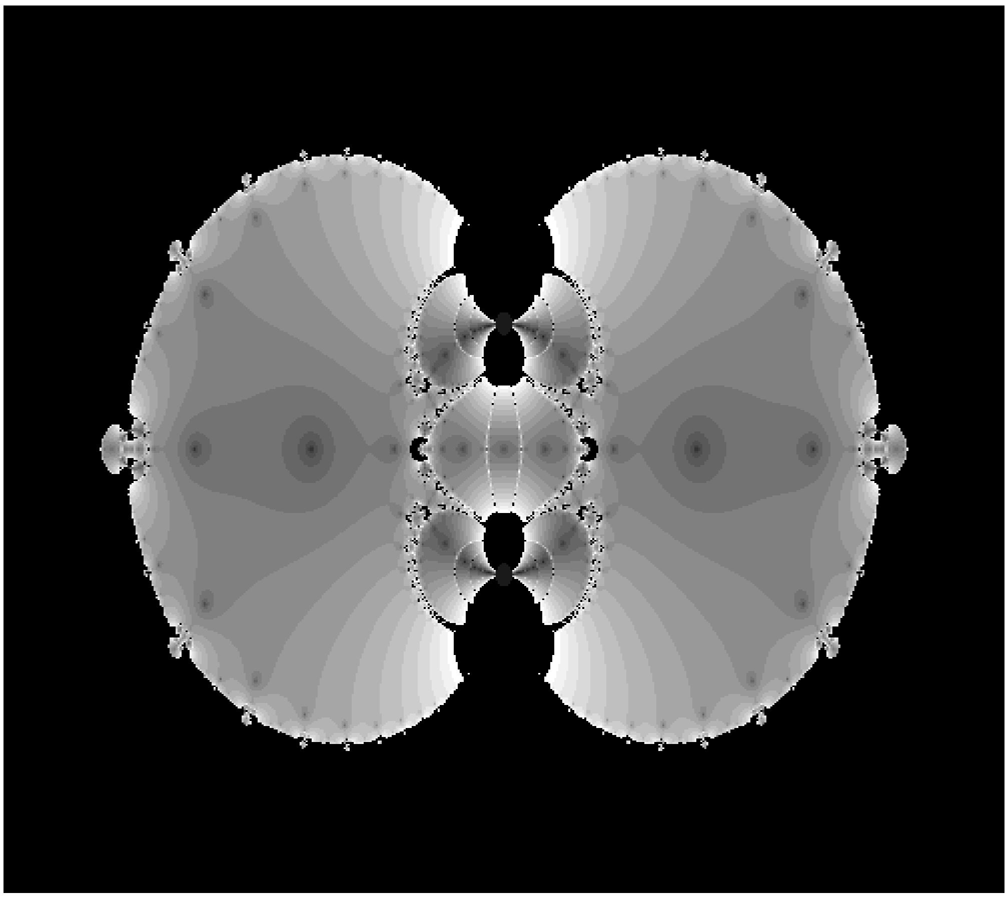

Figure 10.

11. Double Integrals

$$\frac{\pi s}{12}-1+\ln\left(\frac{4+s^2}{1+s^2}\right)=\int_0^1\int_0^1\frac{(1-x)\cos\left(s\ln(xy)\right)}{\left(\ln(xy)\right)^2}dxdy \tag{35}$$

$$-\frac{\pi}{6}+\frac{s}{2}\ln\left(\frac{4+s^2}{1+s^2}\right)=\int_0^1\int_0^1\frac{(1-x)\sin\left(s\ln(xy)\right)}{\left(\ln(xy)\right)^2}dxdy \tag{36}$$

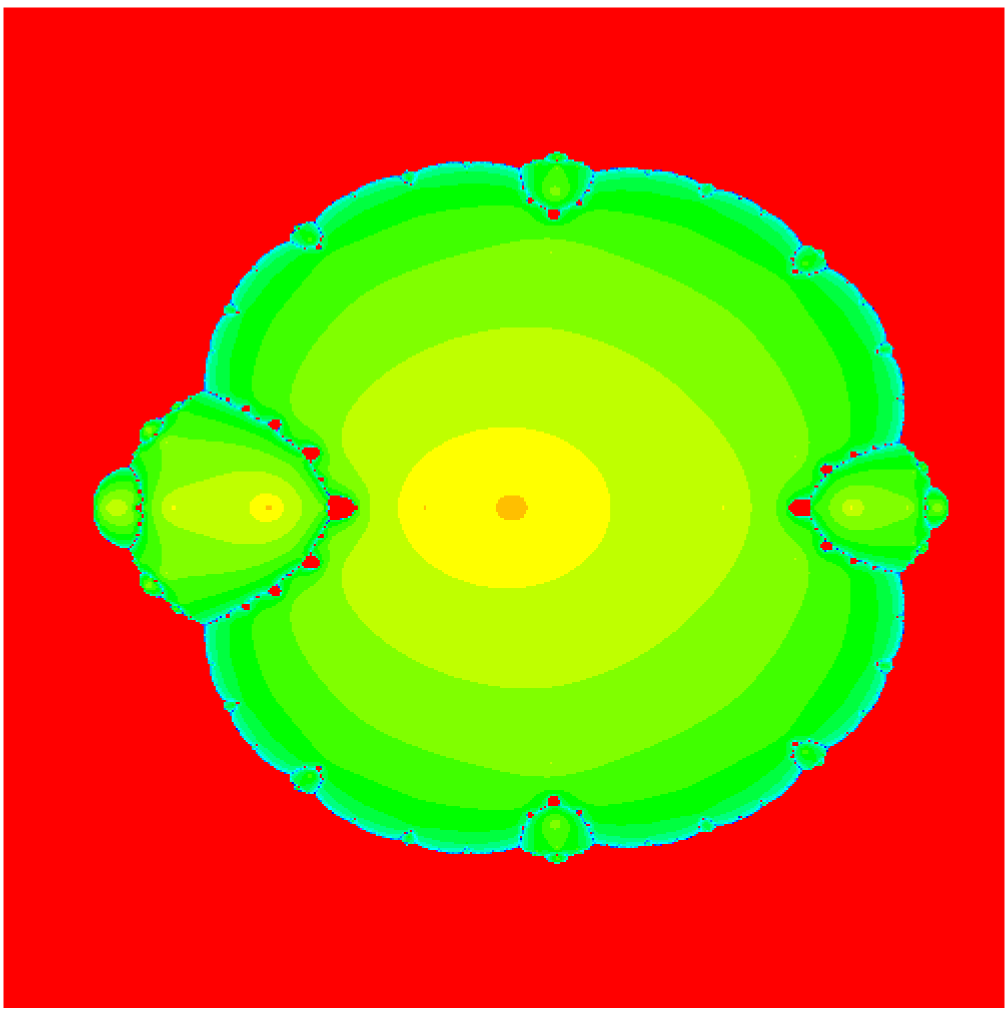

Figure 11.

12. Hypergeometric Formula

$$\pi = 8sF\left(\left\{1,\frac{1}{2}\right\},\left\{\frac{3}{2}\right\},-s^2\right)-\frac{4}{3}s^3F\left(\left\{1,\frac{3}{2}\right\},\left\{\frac{5}{2}\right\},-s^2\right)$$
$$-2sF\left(\left\{1,\frac{1}{2}\right\},\left\{\frac{3}{2}\right\},-\frac{s^2}{4}\right)+\frac{1}{3}s^3F\left(\left\{1,\frac{3}{2}\right\},\left\{\frac{5}{2}\right\},-\frac{s^2}{4}\right) \quad (37)$$

Remark: $F\left(\{a,b\},\{c\},x\right)$ is the hypergeometric function.

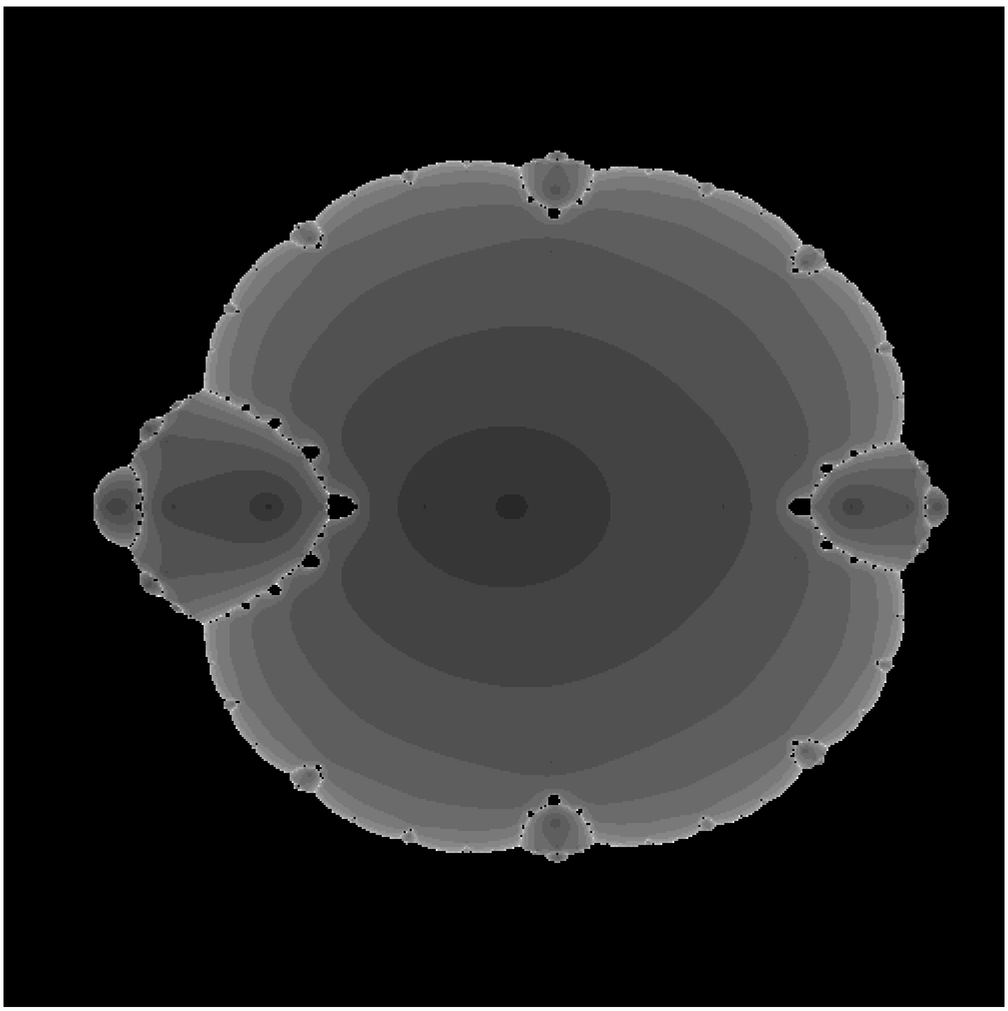

Figure 12.

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