Further research of single valued neutrosophic rough set model

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Abstract—Neutrosophic sets (NSs), as a new mathematical tool for dealing with problems involving incomplete, indeterminant and inconsistent knowledge, were proposed by Smarandache. By simplifying NSs, Wang et al. proposed the concept of single valued neutrosophic sets (SVNSs) and studied some properties of SVNSs. In this paper, we mainly investigate the topological structures of single valued neutrosophic rough sets which is constructed by combining SVNSs and rough sets. Firstly, we introduce the concept of single valued neutrosophic topological spaces. Then, we discuss the relationships between single valued neutrosophic approximation spaces and single valued neutrosophic topological spaces. Concretely, a reflexive and transitive single valued neutrosophic relation can induce a single valued neutrosophic topological space such that its single valued neutrosophic interior and closure operators are the lower and upper approximation operators induced by this single valued neutrosophic relation, respectively. Conversely, a single valued neutrosophic interior (closure, respectively) operator derived from a single valued neutrosophic topological space is just the single valued neutrosophic lower (upper, respectively) approximation operator derived from a single valued neutrosophic approximation space under some conditions. Finally, we show there exists a one-to-one correspondence between the set of all reflexive and transitive single valued neutrosophic relations and the set of all single valued neutrosophic rough topologies.

Keywords—Neutrosophic sets; Single valued neutrosophic sets; Single valued neutrosophic rough sets; Single valued neutrosophic topological spaces

1 Introduction

Rough set theory, initiated by Pawlak [15, 16], is an effective mathematical tool for the study of intelligent systems characterized by insufficient and incomplete information.

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Pawlak rough set model is established based on an equivalence relation. In the real application, the equivalence relation is a very stringent condition which limits the applications of rough sets in real world. For this reason, by replacing the equivalence relation with covering, similarity relation, tolerance relation, fuzzy relation, etc, different kinds of generalizations of Pawlak rough set model were proposed [3, 5, 9, 11, 19, 20, 23, 30, 31, 32, 33]. Rough set theory has been successfully applied to many fields, such as machine learning, knowledge acquisition, and decision analysis, etc.

Smarandache [21, 22] introduced the concept of NSs which consists of three membership functions (truth membership function, indeterminacy membership function and falsity membership function), where every function value is a real standard or non-standard subset of the nonstandard unit interval $]0^-, 1^+[$. The NS generalizes the concepts of the classical set, fuzzy set [37], interval-valued fuzzy set [24], intuitionistic fuzzy set [1] and interval-valued intuitionistic fuzzy set [2]. Wang et al. [25] proposed SVNSs by simplifying NSs. SVNSs can also be looked as an extension of intuitionistic fuzzy sets [1], in which three membership functions are unrelated and their function values belong to the unit closed interval. Many researchers have studied the theory and applications of SVNSs. Ye [34, 35] proposed decision making based on correlation coefficients and weighted correlation coefficient of SVNSs, and gave the application of proposed methods. Majumdar and Samant [13] studied distance, similarity and entropy of SVNSs from a theoretical aspect. Yang et al. [29] proposed SVNRs and studied some kinds of kernels and closures of SVNRs. Broumi and Smarandance [6] proposed single valued neutrosophic information systems based on rough set theory to exploit simultaneously the advantages of SVNSs and rough sets. They studied rough approximation of a SVNS in the single valued neutrosophic information systems and investigated the knowledge reduction and extension of the single valued neutrosophic information systems. Yang et al. [28] proposed single valued neutrosophic rough sets by combining SVNSs and rough sets, and explored a general framework of the study of single valued neutrosophic rough sets.

Topological structures and properties [10] of rough sets are important research issues for the study of rough sets. Many researchers have addressed the issues [4, 7, 8, 12, 14, 17, 18, 26, 27]. Wiweger [26] and Chuchro [7, 8] established the relationships between crisp rough sets and crisp topological spaces. Boixader et al. [4] investigated the connection between fuzzy rough sets and fuzzy topological spaces. Qin et al. [17, 18]

discussed topological structures of fuzzy rough sets. Wu and Zhou [27] generalized the results to IF rough sets and established relationships between IF rough approximations and IF topologies. Ma and Hu [14] studied topological and lattice structures of L-fuzzy rough sets determined by lower and upper sets. Li and Cui [12] studied similarity of fuzzy relations which is based on fuzzy topologies induced by fuzzy rough approximation operators. Zhang et al. [38] discussed the topological structures of interval-valued hesitant fuzzy rough set and its application. Along this line, in the present paper, we shall study topological structures of single valued neutrosophic rough sets and establish the relationships between the single valued neutrosophic approximation spaces and single valued neutrosophic topological spaces.

The rest of this paper is organized as follows. In the next section, we recall some basic notions on Pawlak rough sets, NSs, SVNSs and single valued neutrosophic rough sets. In Section 3, we give notions of single valued neutrosophic topology and its interior operation and closure operation. Some related properties are also studied. Section 4 investigates the relationships between single valued neutrosophic approximation spaces and single valued neutrosophic topology spaces. The last section summarizes the conclusion.

2 Preliminaries

In this section, we recall some basic notions and results which will be used in the paper.

2.1. *Pawlak rough sets*

Definition 2.1([15, 16]). Let U be a nonempty finite universe and R be an equivalence relation in U. (U, R) is called a Pawlak approximation space. $\forall X \subseteq U$, the lower and upper approximations of X, denoted by $\underline{R}(X)$ and $\overline{R}(X)$, are defined as follows, respectively:

$$\underline{R}(X) = \{x \in U \mid [x]_R \subseteq X\},$$
$$\overline{R}(X) = \{x \in U \mid [x]_R \cap X \neq \emptyset\},$$

where $[x]_R = \{y \in U \mid (x, y) \in R\}$. $\underline{R}$ and $\overline{R}$ are called as lower and upper approximation operators, respectively. The pair $(\underline{R}(X), \overline{R}(X))$ is called a Pawlak rough set.

2.2. NSs *and* SVNSs

Definition 2.2([21]). Let U be a space of points (objects), with a generic element in U denoted by x. A NS $\widetilde{A}$ in U is characterized by three membership functions, a truth-membership function $T_{\widetilde{A}}$, an indeterminacy membership function $I_{\widetilde{A}}$ and a falsity-membership function $F_{\widetilde{A}}$, where $\forall x \in U$, $T_{\widetilde{A}}(x)$, $I_{\widetilde{A}}(x)$ and $F_{\widetilde{A}}(x)$ are real standard or non-standard subsets of $]0^-, 1^+[$.

There is no restriction on the sum of $T_{\widetilde{A}}(x)$, $I_{\widetilde{A}}(x)$ and $F_{\widetilde{A}}(x)$, thus $0^- \leq \sup\, T_{\widetilde{A}}(x) + \sup\, I_{\widetilde{A}}(x) + \sup\, F_{\widetilde{A}}(x) \leq 3^+$.

Definition 2.3([25]). Let U be a space of points (objects), with a generic element in U denoted by x. A SVNS $\widetilde{A}$ in U is characterized by three membership functions, a truth-membership function $T_{\widetilde{A}}$, an indeterminacy membership function $I_{\widetilde{A}}$, and a falsity-membership function $F_{\widetilde{A}}$, where $\forall x \in U$, $T_{\widetilde{A}}(x), I_{\widetilde{A}}(x), F_{\widetilde{A}}(x) \in [0,1]$.

The SVNS $\widetilde{A}$ can be denoted by $\widetilde{A} = \{\langle x, T_{\widetilde{A}}(x), I_{\widetilde{A}}(x), F_{\widetilde{A}}(x)\rangle \mid x \in U\}$ or $\widetilde{A} = (T_{\widetilde{A}}, I_{\widetilde{A}}, F_{\widetilde{A}})$. $\forall x \in U$, $\widetilde{A}(x) = (T_{\widetilde{A}}(x), I_{\widetilde{A}}(x), F_{\widetilde{A}}(x))$, and $(T_{\widetilde{A}}(x), I_{\widetilde{A}}(x), F_{\widetilde{A}}(x))$ is called a single valued neutrosophic number.

In this paper, SVNS(U) will denote the family of all SVNSs in U. Let $\widetilde{A}$ be a SVNS in U. If $\forall x \in U$, $T_{\widetilde{A}}(x) = 0$ and $I_{\widetilde{A}}(x) = F_{\widetilde{A}}(x) = 1$, then $\widetilde{A}$ is called an empty SVNS, denoted by $\widetilde{\emptyset}$. If $\forall x \in U$, $T_{\widetilde{A}}(x) = 1$, and $I_{\widetilde{A}}(x) = F_{\widetilde{A}}(x) = 0$, then $\widetilde{A}$ is called a full SVNS, denoted by $\widetilde{U}$. $\forall \alpha_1, \alpha_2, \alpha_3 \in [0,1]$, $\alpha_1\widehat{, \alpha_2,}\alpha_3$ denotes a constant SVNS satisfying, $T_{\alpha_1\widehat{, \alpha_2,}\alpha_3}(x) = \alpha_1, I_{\alpha_1\widehat{, \alpha_2,}\alpha_3}(x) = \alpha_2, F_{\alpha_1\widehat{, \alpha_2,}\alpha_3}(x) = \alpha_3$.

For any $y \in U$, a single valued neutrosophic singleton set 1_y and its complement $1_{U-\{y\}}$ are, defined as: $\forall x \in U$,

$$T_{1_y}(x) = \begin{cases} 1, & x = y \\ 0, & x \neq y \end{cases}, \qquad I_{1_y}(x) = F_{1_y}(x) = \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases};$$

$$T_{1_{U-\{y\}}}(x) = \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases}, \quad I_{1_{U-\{y\}}}(x) = F_{1_{U-\{y\}}}(x) = \begin{cases} 1, & x = y \\ 0, & x \neq y \end{cases}.$$

Definition 2.4([36]). Let $\widetilde{A}$ and $\widetilde{B}$ be two SVNSs in U. If for any $x \in U$, $T_{\widetilde{A}}(x) \leq T_{\widetilde{B}}(x)$, $I_{\widetilde{A}}(x) \geq I_{\widetilde{B}}(x)$ and $F_{\widetilde{A}}(x) \geq F_{\widetilde{B}}(x)$, then we called $\widetilde{A}$ is contained in $\widetilde{B}$, i.e. $\widetilde{A} \Subset \widetilde{B}$. If $\widetilde{A} \Subset \widetilde{B}$ and $\widetilde{B} \Subset \widetilde{A}$, then we called $\widetilde{A}$ is equal to $\widetilde{B}$, denoted by $\widetilde{A} = \widetilde{B}$.

Definition 2.5([25]). Let $\widetilde{A}$ be a SVNS in U. The complement of $\widetilde{A}$ is denoted by $\widetilde{A}^c$, where $\forall x \in U$, $T_{\widetilde{A}^c}(x) = F_{\widetilde{A}}(x)$, $I_{\widetilde{A}^c}(x) = 1 - I_{\widetilde{A}}(x)$, and $F_{\widetilde{A}^c}(x) = T_{\widetilde{A}}(x)$.

Definition 2.6([29]). Let $\widetilde{A}$ and $\widetilde{B}$ be two SVNSs in U.

(1) The union of $\widetilde{A}$ and $\widetilde{B}$ is a SVNS $\widetilde{C}$, denoted by $\widetilde{C} = \widetilde{A} \Cup \widetilde{B}$, where $\forall x \in U$,

$T_{\widetilde{C}}(x) = T_{\widetilde{A}}(x) \vee T_{\widetilde{B}}(x)$, $I_{\widetilde{C}}(x) = I_{\widetilde{A}}(x) \wedge I_{\widetilde{B}}(x)$ and $F_{\widetilde{C}}(x) = F_{\widetilde{A}}(x) \wedge F_{\widetilde{B}}(x)$;

(2) The intersection of $\widetilde{A}$ and $\widetilde{B}$ is a SVNS $\widetilde{D}$, denoted by $\widetilde{D} = \widetilde{A} \Cap \widetilde{B}$, where $\forall x \in U$,

$T_{\widetilde{D}}(x) = T_{\widetilde{A}}(x) \wedge T_{\widetilde{B}}(x)$, $I_{\widetilde{D}}(x) = I_{\widetilde{A}}(x) \vee I_{\widetilde{B}}(x)$ and $F_{\widetilde{D}}(x) = F_{\widetilde{A}}(x) \vee F_{\widetilde{B}}(x)$,

where " $\vee$ " and " $\wedge$ " denote maximum and minimum, respectively.

It is easy to verify that the union and intersection of SVNSs satisfy commutative law, associative law, and distributive law.

Proposition 2.7([28]). Let $\widetilde{A}$ and $\widetilde{B}$ be two SVNSs in U. The following results hold:

(1) $\widetilde{A} \Subset \widetilde{A} \Cup \widetilde{B}$ and $\widetilde{B} \Subset \widetilde{A} \Cup \widetilde{B}$,

(2) $\widetilde{A} \Cap \widetilde{B} \Subset \widetilde{A}$ and $\widetilde{A} \Cap \widetilde{B} \Subset \widetilde{B}$,

(3) $(\widetilde{A}^c)^c = \widetilde{A}$,

(4) $(\widetilde{A} \Cup \widetilde{B})^c = \widetilde{A}^c \Cap \widetilde{B}^c$,

(5) $(\widetilde{A} \Cap \widetilde{B})^c = \widetilde{A}^c \Cup \widetilde{B}^c$.

2.3. *Single valued neutrosophic rough sets*

Definition 2.8([28]). A SVNS $\widetilde{R}$ in $U \times U$ is called a single valued neutrosophic relation (SVNR) in U, denoted by $\widetilde{R}= \{\langle (x,y), T_{\widetilde{R}}(x,y), I_{\widetilde{R}}(x,y), F_{\widetilde{R}}(x,y)\rangle \mid (x,y) \in U \times U\}$, where $T_{\widetilde{R}} : U \times U \longrightarrow [0,1]$, $I_{\widetilde{R}} : U \times U \longrightarrow [0,1]$, and $F_{\widetilde{R}} : U \times U \longrightarrow [0,1]$ denote the truth-membership function, indeterminacy membership function, and falsity-membership function of $\widetilde{R}$, respectively.

Let $\widetilde{R}$ be a SVNR in U, the complement $\widetilde{R}^c$ of $\widetilde{R}$ is defined as follows

$\widetilde{R}^c = \{\langle (x,y), T_{\widetilde{R}^c}(x,y), I_{\widetilde{R}^c}(x,y), F_{\widetilde{R}^c}(x,y)\rangle \mid (x,y) \in U \times U\}$,

where $\forall (x,y) \in U \times U$, $T_{\widetilde{R}^c}(x,y) = F_{\widetilde{R}}(x,y)$, $I_{\widetilde{R}^c}(x,y) = 1 - I_{\widetilde{R}}(x,y)$ and $F_{\widetilde{R}^c}(x,y) = T_{\widetilde{R}}(x,y)$.

Definition 2.9([28]). Let $\widetilde{R}$ be a SVNR in U. If $\forall x \in U$, $T_{\widetilde{R}}(x,x) = 1$ and $I_{\widetilde{R}}(x,x) = F_{\widetilde{R}}(x,x) = 0$, then $\widetilde{R}$ is called a reflexive SVNR. If $\forall x,y \in U$, $T_{\widetilde{R}}(x,y) = T_{\widetilde{R}}(y,x)$, $I_{\widetilde{R}}(x,y) = I_{\widetilde{R}}(y,x)$ and $F_{\widetilde{R}}(x,y) = F_{\widetilde{R}}(y,x)$, then $\widetilde{R}$ is called a symmetric SVNR. If $\forall x \in U$, $\bigvee_{y\in U} T_{\widetilde{R}}(x,y) = 1$ and $\bigwedge_{y\in U} I_{\widetilde{R}}(x,y) = \bigwedge_{y\in U} F_{\widetilde{R}}(x,y) = 0$, then $\widetilde{R}$ is called a serial SVNR. If $\forall x,y,z \in U$, $\bigvee_{y\in U}(T_{\widetilde{R}}(x,y) \wedge T_{\widetilde{R}}(y,z)) \le T_{\widetilde{R}}(x,z)$, $\bigwedge_{y\in U}(I_{\widetilde{R}}(x,y) \vee I_{\widetilde{R}}(y,z)) \ge I_{\widetilde{R}}(x,z)$ and $\bigwedge_{y\in U}(F_{\widetilde{R}}(x,y) \vee F_{\widetilde{R}}(y,z)) \ge F_{\widetilde{R}}(x,z)$, then $\widetilde{R}$ is called a transitive SVNR, where " $\vee$ " and " $\wedge$ " denote maximum and minimum, respectively.

Definition 2.10([28]). Let $\widetilde{R}$ be a SVNR in U, the tuple $(U, \widetilde{R})$ is called a single valued neutrosophic approximation space. $\forall \widetilde{A} \in \mathrm{SVNS}(U)$, the lower and upper approximations of $\widetilde{A}$ w.r.t. $(U, \widetilde{R})$, denoted by $\underline{\widetilde{R}}(\widetilde{A})$ and $\overline{\widetilde{R}}(\widetilde{A})$, are two SVNSs whose membership functions are defined as: $\forall x \in U$,

$T_{\underline{\widetilde{R}}(\widetilde{A})}(x) = \bigwedge_{y\in U}(F_{\widetilde{R}}(x,y) \vee T_{\widetilde{A}}(y))$,

$I_{\underline{\widetilde{R}}(\widetilde{A})}(x) = \bigvee_{y\in U}((1 - I_{\widetilde{R}}(x,y)) \wedge I_{\widetilde{A}}(y))$,

$F_{\underline{\widetilde{R}}(\widetilde{A})}(x) = \bigvee_{y\in U}(T_{\widetilde{R}}(x,y) \wedge F_{\widetilde{A}}(y))$;

$T_{\overline{\widetilde{R}}(\widetilde{A})}(x) = \bigvee_{y\in U}(T_{\widetilde{R}}(x,y) \wedge T_{\widetilde{A}}(y))$,

$I_{\overline{\widetilde{R}}(\widetilde{A})}(x) = \bigwedge_{y\in U}(I_{\widetilde{R}}(x,y) \vee I_{\widetilde{A}}(y))$,

$F_{\overline{\widetilde{R}}(\widetilde{A})}(x) = \bigwedge_{y\in U}(F_{\widetilde{R}}(x,y) \vee F_{\widetilde{A}}(y))$.

The pair $(\underline{\widetilde{R}}(\widetilde{A}), \overline{\widetilde{R}}(\widetilde{A}))$ is called the single valued neutrosophic rough set of $\widetilde{A}$ w.r.t. $(U, \widetilde{R})$. $\underline{\widetilde{R}}$ and $\overline{\widetilde{R}}$ are referred to as the single valued neutrosophic lower and upper approximation operators, respectively.

Yang et al. [28] studied the properties of single valued neutrosophic lower and upper approximation operators as follows.

Proposition 2.11([28]). Let $(U, \widetilde{R})$ be a single valued neutrosophic approximation space. The single valued neutrosophic lower and upper approximation operators have the following properties: $\forall \widetilde{A}, \widetilde{B} \in \mathrm{SVNS}(U)$, $\forall \alpha_1, \alpha_2, \alpha_3 \in [0,1]$,

(1) $\underline{\widetilde{R}}(\widetilde{U}) = \widetilde{U}$, $\overline{\widetilde{R}}(\widetilde{\emptyset}) = \widetilde{\emptyset}$;

(2) If $\widetilde{A} \Subset \widetilde{B}$, then $\underline{\widetilde{R}}(\widetilde{A}) \Subset \underline{\widetilde{R}}(\widetilde{B})$ and $\overline{\widetilde{R}}(\widetilde{A}) \Subset \overline{\widetilde{R}}(\widetilde{B})$;

(3) $\underline{\widetilde{R}}(\widetilde{A} \Cap \widetilde{B}) = \underline{\widetilde{R}}(\widetilde{A}) \Cap \underline{\widetilde{R}}(\widetilde{B})$, $\overline{\widetilde{R}}(\widetilde{A} \Cup \widetilde{B}) = \overline{\widetilde{R}}(\widetilde{A}) \Cup \overline{\widetilde{R}}(\widetilde{B})$;

(4) $\underline{\widetilde{R}}(\widetilde{A} \Cup \widetilde{B}) \Supset \underline{\widetilde{R}}(\widetilde{A}) \Cup \underline{\widetilde{R}}(\widetilde{B})$, $\overline{\widetilde{R}}(\widetilde{A} \Cap \widetilde{B}) \Subset \overline{\widetilde{R}}(\widetilde{A}) \Cap \overline{\widetilde{R}}(\widetilde{B})$;

(5) $\underline{\widetilde{R}}(\widetilde{A}^c) = (\overline{\widetilde{R}}(\widetilde{A}))^c$, $\overline{\widetilde{R}}(\widetilde{A}^c) = (\underline{\widetilde{R}}(\widetilde{A}))^c$;

(6) $\underline{\widetilde{R}}(A \Cup \widehat{\alpha_1, \alpha_2, \alpha_3}) = \underline{\widetilde{R}}(\widetilde{A}) \Cup \widehat{\alpha_1, \alpha_2, \alpha_3}$, $\overline{\widetilde{R}}(A \Cap \widehat{\alpha_1, \alpha_2, \alpha_3}) = \overline{\widetilde{R}}(\widetilde{A}) \Cap \widehat{\alpha_1, \alpha_2, \alpha_3}$;

(7) $\underline{\widetilde{R}}(\widehat{\alpha_1, \alpha_2, \alpha_3}) = \widehat{\alpha_1, \alpha_2, \alpha_3} \Longleftrightarrow \underline{\widetilde{R}}(\widetilde{\emptyset}) = \widetilde{\emptyset}$,

$\overline{\widetilde{R}}(\widehat{\alpha_1, \alpha_2, \alpha_3}) = \widehat{\alpha_1, \alpha_2, \alpha_3} \Longleftrightarrow \overline{\widetilde{R}}(\widetilde{U}) = \widetilde{U}$.

Proposition 2.12([28]). Let $(U, \widetilde{R})$ be a single valued neutrosophic approximation space. $\underline{\widetilde{R}}$ and $\overline{\widetilde{R}}$ are the lower and upper approximation operators, then we have

(1) $\widetilde{R}$ is serial $\Longleftrightarrow \underline{\widetilde{R}}(\widehat{\alpha_1, \alpha_2, \alpha_3}) = \widehat{\alpha_1, \alpha_2, \alpha_3}$, $\forall \alpha_1, \alpha_2, \alpha_3 \in [0,1]$,

$\Longleftrightarrow \underline{\widetilde{R}}(\widetilde{\emptyset}) = \widetilde{\emptyset}$,

$\Longleftrightarrow \overline{\widetilde{R}}(\widehat{\alpha_1, \alpha_2, \alpha_3}) = \widehat{\alpha_1, \alpha_2, \alpha_3}$, $\forall \alpha_1, \alpha_2, \alpha_3 \in [0,1]$,

$\Longleftrightarrow \overline{\overline{\widetilde{R}}}(\widetilde{U}) = \widetilde{U}$;

(2) $\widetilde{R}$ is reflexive $\Longleftrightarrow \underline{\widetilde{R}}(\widetilde{A}) \Subset \widetilde{A}$, $\forall \widetilde{A} \in \mathrm{SVNS}(U)$,

$\Longleftrightarrow \widetilde{A} \Subset \overline{\overline{\widetilde{R}}}(\widetilde{A})$, $\forall \widetilde{A} \in \mathrm{SVNS}(U)$;

(3) $\widetilde{R}$ is symmetric $\Longleftrightarrow \underline{\widetilde{R}}(1_{U-\{x\}})(y) = \underline{\widetilde{R}}(1_{U-\{y\}})(x)$, $\forall x, y \in U$,

$\Longleftrightarrow \overline{\overline{\widetilde{R}}}(1_x)(y) = \overline{\overline{\widetilde{R}}}(1_y)(x)$, $\forall x, y \in U$;

(4) $\widetilde{R}$ is transitive $\Longleftrightarrow \underline{\widetilde{R}}(\widetilde{A}) \Subset \underline{\widetilde{R}}(\underline{\widetilde{R}}(\widetilde{A}))$, $\forall \widetilde{A} \in \mathrm{SVNS}(U)$,

$\Longleftrightarrow \overline{\overline{\widetilde{R}}}(\overline{\overline{\widetilde{R}}}(\widetilde{A})) \Subset \overline{\overline{\widetilde{R}}}(\widetilde{A})$, $\forall \widetilde{A} \in \mathrm{SVNS}(U)$.

3 Single valued neutrosophic topological spaces

In this section, we will introduce the concept of single valued neutrosophic topological spaces and basis concepts related to single valued neutrosophic topological spaces.

We first introduce the concept of single valued neutrosophic topology as follows.

Definition 3.1. A single valued neutrosophic topology on a nonempty set U is a family τ of SVNSs in U that satisfies the following conditions:

(T_1) $\widetilde{\emptyset}, \widetilde{U} \in \tau$,

(T_2) $\widetilde{A} \Cap \widetilde{B} \in \tau$ for any $\widetilde{A}, \widetilde{B} \in \tau$,

(T_3) $\Cup_{i \in I} \widetilde{A}_i \in \tau$ for any $\widetilde{A}_i \in \tau$, $i \in I, I$ is an index set.

The pair (U, τ) is called a single valued neutrosophic topological space and each SVNS $\widetilde{A}$ in τ is referred to as a single valued neutrosophic open set in (U, τ). The complement of a single valued neutrosophic open set in (U, τ) is called a single valued neutrosophic closed set in (U, τ).

Example 3.2. Let $U = \{x_1, x_2\}$. $\widetilde{A}$, $\widetilde{B}$, $\widetilde{C}$, $\widetilde{D}$ are four SVNSs in U defined as follows:

$$\widetilde{A} = \{\langle x_1, 0.2, 0.8, 0.1\rangle, \langle x_2, 1, 0.3, 0.1\rangle\},$$

$$\widetilde{B} = \{\langle x_1, 0.2, 0.8, 0.6\rangle, \langle x_2, 0.5, 0.4, 1\rangle\},$$

$$\widetilde{C} = \{\langle x_1, 0.3, 0.7, 0.1\rangle, \langle x_2, 1, 0.2, 0.1\rangle\},$$

$$\widetilde{D} = \{\langle x_1, 0.1, 0.9, 0.8\rangle, \langle x_2, 0.4, 0.5, 1\rangle\}.$$

By Definitions 2.6 and 3.1, $\tau = \{\widetilde{\emptyset}, \widetilde{U}, \widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}\}$ is a single valued neutrosophic topology on U and (U, τ) is a single valued neutrosophic topological space.

Now, we define the single valued neutrosophic interior and closure operators in a single valued neutrosophic topological space.

Definition 3.3. Let (U, τ) be a single valued neutrosophic topological space. For any $\widetilde{A} \in \mathrm{SVNS}(U)$, the single valued neutrosophic interior and closure of $\widetilde{A}$ are defined as follows:

$$int(\widetilde{A}) = \Cup\{\widetilde{M} | \widetilde{M} \in \tau \text{ and } \widetilde{M} \Subset \widetilde{A}\},$$
$$cl(\widetilde{A}) = \Cap\{\widetilde{N} | \widetilde{N}^c \in \tau \text{ and } \widetilde{A} \Subset \widetilde{N}\},$$

where int and cl: $\mathrm{SVNS}(U) \longrightarrow \mathrm{SVNS}(U)$ are called the single valued neutrosophic interior and closure operators of τ, respectively.

Next, we discuss the properties of the single valued neutrosophic interior and closure operators.

Theorem 3.4. Let (U, τ) be a single valued neutrosophic topological space. For any $\widetilde{A} \in \mathrm{SVNS}(U)$, we have

(1) $\widetilde{A}$ is a single valued neutrosophic open set in (U, τ) iff $int(\widetilde{A}) = \widetilde{A}$;

(2) $\widetilde{A}$ is a single valued neutrosophic closed set in (U, τ) iff $cl(\widetilde{A}) = \widetilde{A}$.

Proof. It is straightforward from Definition 3.3.

Theorem 3.5. Let (U, τ) be a single valued neutrosophic topological space. For any $\widetilde{A}, \widetilde{B} \in \mathrm{SVNS}(U)$, the following results hold:

(Int0) $(int(\widetilde{A}))^c = cl(\widetilde{A}^c)$;

(Cl0) $(cl(\widetilde{A}))^c = int(\widetilde{A}^c)$;

(Int1) $int(\widetilde{U}) = \widetilde{U}$, $int(\widetilde{\emptyset}) = \widetilde{\emptyset}$;

(Cl1) $cl(\widetilde{U}) = \widetilde{U}$, $cl(\widetilde{\emptyset}) = \widetilde{\emptyset}$;

(Int2) $int(\widetilde{A}) \Subset \widetilde{A}$;

(Cl2) $\widetilde{A} \Subset cl(\widetilde{A})$;

(Int3) $int(int(\widetilde{A})) = int(\widetilde{A})$;

(Cl3) $cl(cl(\widetilde{A})) = cl(\widetilde{A})$;

(Int4) $int(\widetilde{A} \Cap \widetilde{B}) = int(\widetilde{A}) \Cap int(\widetilde{B})$;

(Cl4) $cl(\widetilde{A} \Cup \widetilde{B}) = cl(\widetilde{A}) \Cup cl(\widetilde{B})$;

(Int5) If $\widetilde{A} \Subset \widetilde{B}$, then $int(\widetilde{A}) \Subset int(\widetilde{B})$;

(Cl5) If $\widetilde{A} \Subset \widetilde{B}$, then $cl(\widetilde{A}) \Subset cl(\widetilde{B})$.

Proof. We only prove (Int0) and (Int4). For any $\widetilde{A}, \widetilde{B} \in \mathrm{SVNS}(U)$, by Definition 3.3 and Proposition 2.7, we have

$$\begin{aligned}
\text{(Int0)}\quad (int(\widetilde{A}))^c &= (\Cup\{\widetilde{M} \mid \widetilde{M} \in \tau \text{ and } \widetilde{M} \Subset \widetilde{A}\})^c \\
&= \Cap\{\widetilde{M}^c \mid \widetilde{M} \in \tau \text{ and } \widetilde{M} \Subset \widetilde{A}\} \\
&= \Cap\{\widetilde{N} \mid \widetilde{N}^c \in \tau \text{ and } \widetilde{N}^c \Subset \widetilde{A}\} \\
&= \Cap\{\widetilde{N} \mid \widetilde{N}^c \in \tau \text{ and } \widetilde{A}^c \Subset \widetilde{N}\} \\
&= cl(\widetilde{A}^c);
\end{aligned}$$

$$\begin{aligned}
\text{(Int4)}\quad int(\widetilde{A} \Cap \widetilde{B}) &= \Cup\{\widetilde{M} \mid \widetilde{M} \in \tau \text{ and } \widetilde{M} \Subset (\widetilde{A} \Cap \widetilde{B})\} \\
&= \Cup\{\widetilde{M} \mid \widetilde{M} \in \tau \text{ and } \widetilde{M} \Subset \widetilde{A} \text{ and } \widetilde{M} \Subset \widetilde{B}\} \\
&= (\Cup\{\widetilde{M} \mid \widetilde{M} \in \tau \text{ and } \widetilde{M} \Subset \widetilde{A}\}) \Cap (\Cup\{\widetilde{M} \mid \widetilde{M} \in \tau \text{ and } \widetilde{M} \Subset \widetilde{B}\}) \\
&= int(\widetilde{A}) \Cap int(\widetilde{B}).
\end{aligned}$$

The following Theorem 3.6 shows that under some conditions, a single valued neutrosophic operator is the single valued neutrosophic interior operator (the single valued neutrosophic closure operator) of a certain topology.

Theorem 3.6. (1) If a single valued neutrosophic operator $int : \mathrm{SVNS}(U) \longrightarrow \mathrm{SVNS}(U)$ satisfies the properties (Int1)–(Int4), then there exists a single valued neutrosophic topology τ_{int} on U such that $int_{\tau_{int}} = int$.

(2) If a single valued neutrosophic operator cl: $\mathrm{SVNS}(U) \longrightarrow \mathrm{SVNS}(U)$ satisfies the properties (Cl1)–(Cl4), then there exists a single valued neutrosophic topology τ_{cl} on U such that $cl_{\tau_{cl}} = cl$.

Proof. (1) Define $\tau_{int}= \{\widetilde{A} \in \mathrm{SVNS}(U) \mid int(\widetilde{A}) = \widetilde{A}\}$. Next, we show τ_{int} satisfies (T_1)–(T_3).

(T_1) By (Int1), $\widetilde{\emptyset}, \widetilde{U} \in \tau_{int}$.

(T_2) For any $\widetilde{A}, \widetilde{B} \in \tau_{int}$, $int(\widetilde{A}) = \widetilde{A}$ and $int(\widetilde{B}) = \widetilde{B}$. By (Int4), we have $int(\widetilde{A} \Cap \widetilde{B}) = int(\widetilde{A}) \Cap int(\widetilde{B}) = \widetilde{A} \Cap \widetilde{B}$. So $\widetilde{A} \Cap \widetilde{B} \in \tau_{int}$.

(T_3) Suppose that $\widetilde{A}_i \in \tau_{int}$, then $int(\widetilde{A}_i) = \widetilde{A}_i$ for any $i \in I$. By (Int2), we have $int(\Cup_{i\in I}\widetilde{A}_i) \Subset \Cup_{i\in I}\widetilde{A}_i$.

Conversely, $int(\widetilde{A}_i) \Subset \Cup_{i\in I} int(\widetilde{A}_i)$. By (Int3) and (Int5), we have $int(\Cup_{i\in I} int(\widetilde{A}_i)) \Supset int(int(\widetilde{A}_i)) = int(\widetilde{A}_i)$ for any $i \in I$. Thus $int(\Cup_{i\in I} int(\widetilde{A}_i)) \Supset \Cup_{i\in I} int(\widetilde{A}_i)$. Since $int(\widetilde{A}_i) = \widetilde{A_i}$, we have $int(\Cup_{i\in I}\widetilde{A}_i) \Supset \Cup_{i\in I}\widetilde{A}_i$.

Thus $int(\Cup_{i\in I}(\widetilde{A}_i)) = \Cup_{i\in I}\widetilde{A}_i$. So $\Cup_{i\in I}(\widetilde{A}_i) \in \tau_{int}$.

Hence τ_{int} is a single valued neutrosophic topology on U. It is obvious that $int_{\tau_{int}} = int$.

(2) Define $\tau_{cl} = \{\widetilde{A} \in \text{SVNS}(U) \mid cl(\widetilde{A}^c) = \widetilde{A}^c\}$. Next, we show τ_{cl} satisfies (T$_1$)–(T$_3$).

(T$_1$) By Definition 2.5, we have $\widetilde{\emptyset}^c = \widetilde{U}$ and $\widetilde{U}^c = \widetilde{\emptyset}$. Then by (Cl1), $cl(\widetilde{\emptyset}^c) = cl(\widetilde{U}) = \widetilde{U} = \widetilde{\emptyset}^c$ and $cl(\widetilde{U}^c) = cl(\widetilde{\emptyset}) = \widetilde{\emptyset} = \widetilde{U}^c$, which means that $\widetilde{\emptyset}, \widetilde{U} \in \tau_{cl}$.

(T$_2$) For any $\widetilde{A}, \widetilde{B} \in \tau_{cl}$, $cl(\widetilde{A}^c) = \widetilde{A}^c$ and $cl(\widetilde{B}^c) = \widetilde{B}^c$. By Proposition 2.7 and (Cl4), we have $cl((\widetilde{A} \Cap \widetilde{B})^c) = cl(\widetilde{A}^c \Cup \widetilde{B}^c) = cl(\widetilde{A}^c) \Cup cl(\widetilde{B}^c) = \widetilde{A}^c \Cup \widetilde{B}^c = (\widetilde{A} \Cap \widetilde{B})^c$. So $\widetilde{A} \Cap \widetilde{B} \in \tau_{cl}$.

(T$_3$) Suppose that $\widetilde{A}_i \in \tau_{cl}$, then $cl(\widetilde{A_i}^c) = \widetilde{A_i}^c$ for any $i \in I$. By (Cl2), we have $(\Cup_{i\in I} \widetilde{A}_i)^c \Subset cl((\Cup_{i\in I} \widetilde{A}_i)^c)$.

Conversely, by (Cl5), we have $cl((\Cup_{i\in I} \widetilde{A}_i)^c) = cl(\Cap_{i\in I} \widetilde{A_i}^c) \Subset cl(\widetilde{A_i}^c)$ for any $i \in I$. Thus $cl((\Cup_{i\in I} \widetilde{A}_i)^c) \Subset \Cap_{i\in I} cl(\widetilde{A_i}^c) = \Cap_{i\in I} \widetilde{A_i}^c = (\Cup_{i\in I} \widetilde{A}_i)^c$.

Thus $cl((\Cup_{i\in I} \widetilde{A}_i)^c) = (\Cup_{i\in I} \widetilde{A}_i)^c$. So $\Cup_{i\in I} \widetilde{A}_i \in \tau_{cl}$.

Hence τ_{cl} is a single valued neutrosophic topology on U. It is obvious that $cl_{\tau_{cl}} = cl$.

Theorem 3.7. (1) Let $int : \text{SVNS}(U) \longrightarrow \text{SVNS}(U)$ be a single valued neutrosophic operator satisfying the properties (Int1)–(Int4). Define

$$\tau'_{int} = \{int(\widetilde{A}) \mid \widetilde{A} \in \text{SVNS}(U)\},$$

then $\tau'_{int} = \tau_{int}$.

(2) Let $cl : \text{SVNS}(U) \longrightarrow \text{SVNS}(U)$ be a single valued neutrosophic operator satisfying the properties (Cl1)–(Cl4). Define

$$\tau'_{cl} = \{(cl(\widetilde{A}))^c \mid \widetilde{A} \in \text{SVNS}(U)\},$$

then $\tau'_{cl} = \tau_{cl}$.

Proof. (1) Obviously, $\tau_{int} = \{\widetilde{A} \in \text{SVNS}(U) \mid int(\widetilde{A}) = \widetilde{A}\} \Subset \tau'_{int}$. Conversely, for any $\widetilde{A} \in \text{SVNS}(U)$, by (Int3), $int(int(\widetilde{A})) = int(\widetilde{A})$, from which we have $int(\widetilde{A}) \in \tau_{int}$. Hence $\tau'_{int} \Subset \tau_{int}$. So $\tau'_{int} = \tau_{int}$.

(2) For any $\widetilde{A} \in \text{SVNS}(U)$, $(cl(\widetilde{A}))^c \in \tau'_{cl}$, we have $cl((((cl(\widetilde{A}))^c)^c) = cl(cl(\widetilde{A})) = cl(\widetilde{A}) = ((cl(\widetilde{A}))^c)^c$, which means that $(cl(\widetilde{A}))^c \in \tau_{cl}$. Then $\tau'_{cl} \Subset \tau_{cl}$. Conversely, for any $\widetilde{A} \in \tau_{cl}$, $\widetilde{A} = (cl(\widetilde{A}^c))^c$. Since $\widetilde{A}^c \in \text{SVNS}(U)$, we have $\widetilde{A} = (cl(\widetilde{A}^c))^c \in \tau'_{cl}$, which means that $\tau_{cl} \Subset \tau'_{cl}$. So $\tau_{cl} = \tau'_{cl}$.

Theorem 3.8. Let $int : \text{SVNS}(U) \longrightarrow \text{SVNS}(U)$ be a single valued neutrosophic operator satisfying the properties (Int0)–(Int4) and $cl : \text{SVNS}(U) \longrightarrow \text{SVNS}(U)$ be a single valued neutrosophic operator satisfying the properties (Cl0)–(Cl4). Then the following result holds:

$$\tau'_{int} = \tau_{int} = \tau'_{cl} = \tau_{cl}.$$

Proof. By Theorem 3.7, we have $\tau'_{int} = \tau_{int}$ and $\tau'_{cl} = \tau_{cl}$. Thus, we only need to prove that $\tau'_{int} = \tau'_{cl}$.

By (Int0) and (Cl0), we have

$$\begin{aligned}\tau'_{int} &= \{int(\widetilde{A}) \mid \widetilde{A} \in \mathrm{SVNS}(U)\} \\ &= \{(cl(\widetilde{A}^c))^c \mid \widetilde{A} \in \mathrm{SVNS}(U)\} \\ &= \{(cl(\widetilde{A}))^c \mid \widetilde{A}^c \in \mathrm{SVNS}(U)\} \\ &= \{(cl(\widetilde{A}))^c \mid \widetilde{A} \in \mathrm{SVNS}(U)\} \\ &= \tau'_{cl}.\end{aligned}$$

4 Relationships between single valued neutrosophic approximation spaces and single valued neutrosophic topological spaces

In this section, we will discuss the relationships between single valued neutrosophic approximation spaces and single valued neutrosophic topological spaces.

4.1. *From single valued neutrosophic approximation spaces to single valued neutrosophic topological spaces*

Let $(U, \widetilde{R})$ be a single valued neutrosophic approximation space. Define

$$\tau_{\widetilde{R}} = \{\widetilde{A} \in \mathrm{SVNS}(U) \mid \underline{\widetilde{R}}(\widetilde{A}) = \widetilde{A}\} \tag{1}$$

The following Theorem 4.1 shows that, by Equation (1), a reflexive SVNR $\widetilde{R}$ in U can induce a single valued neutrosophic topology $\tau_{\widetilde{R}}$ on U.

Theorem 4.1. If $\widetilde{R}$ is a reflexive SVNR in U, then $\tau_{\widetilde{R}}$ is a single valued neutrosophic topology on U.

Proof. We verify $\tau_{\widetilde{R}}$ satisfies (T_1)–(T_3).

(T_1) Since $\widetilde{R}$ is a reflexive relation, $\widetilde{R}$ is serial. By Propositions 2.11 and 2.12, we have $\underline{\widetilde{R}}(\widetilde{\emptyset}) = (\widetilde{\emptyset})$ and $\underline{\widetilde{R}}(\widetilde{U}) = (\widetilde{U})$. Hence $\widetilde{\emptyset}, \widetilde{U} \in \tau_{\widetilde{R}}$.

(T_2) According to the definition of $\tau_{\widetilde{R}}$, for any $\widetilde{A}, \widetilde{B} \in \tau_{\widetilde{R}}$, we have $\underline{\widetilde{R}}(\widetilde{A}) = \widetilde{A}$ and $\underline{\widetilde{R}}(\widetilde{B}) = \widetilde{B}$. By Proposition 2.11, we have $\underline{\widetilde{R}}(\widetilde{A} \Cap \widetilde{B}) = \underline{\widetilde{R}}(\widetilde{A}) \Cap \underline{\widetilde{R}}(\widetilde{B}) = \widetilde{A} \Cap \widetilde{B}$, which implies that $\widetilde{A} \Cap \widetilde{B} \in \tau_{\widetilde{R}}$.

Table 1: SVNR $\widetilde{R}$

$\widetilde{R}$	x_1	x_2	x_3
x_1	(0,0,1)	(0.3,0.1,0.6)	(1,0,0.4)
x_2	(0,0.2,0.4)	(0.6,0.5,1)	(0.6,0,1)
x_3	(1,0,1)	(1,0.5,1)	(1,0,0)

(T_3) For any $\widetilde{A}_i \in \tau_{\widetilde{R}}$, we have $\underline{\widetilde{R}}(\widetilde{A}_i) = \widetilde{A}_i$ for any $i \in I$. By Proposition 2.12 and the reflexivity of $\widetilde{R}$, we have $\underline{\widetilde{R}}(\Cup_{i\in I}\widetilde{A}_i) \Subset \Cup_{i\in I}\widetilde{A}_i$. Conversely, by Proposition 2.11, $\widetilde{A}_i = \underline{\widetilde{R}}(\widetilde{A_i}) \Subset \underline{\widetilde{R}}(\Cup_{i\in I}\widetilde{A}_i)$, which implies that $\Cup_{i\in I}\widetilde{A}_i \Subset \underline{\widetilde{R}}(\Cup_{i\in I}\widetilde{A}_i)$. Thus $\Cup_{i\in I}\widetilde{A}_i = \underline{\widetilde{R}}(\Cup_{i\in I}\widetilde{A}_i)$ and $\Cup_{i\in I}\widetilde{A}_i \in \tau_{\widetilde{R}}$.

So $\tau_{\widetilde{R}}$ is a single valued neutrosophic topology on U.

Remark 4.2. If the SVNR $\widetilde{R}$ is not reflexive, then $\tau_{\widetilde{R}}$ defined by Equations (1) may not be a single valued neutrosophic topology, as shown in the following example.

Example 4.3. Let $(U, \widetilde{R})$ be a single valued neutrosophic approximation space, where $U = \{x_1, x_2, x_3\}$ and $\widetilde{R} \in \mathrm{SVNR}(U \times U)$ is given in Table 1.

Obviously, $\widetilde{R}$ is not reflexive. According to Definition 2.10, we have

$T_{\underline{\widetilde{R}}(\widetilde{\emptyset})}(x_1) = \bigwedge_{y\in U}(F_{\widetilde{R}}(x_1, y) \vee T_{\widetilde{\emptyset}}(y)) = 0.4,$

$I_{\underline{\widetilde{R}}(\widetilde{\emptyset})}(x_1) = \bigvee_{y\in U}((1 - I_{\widetilde{R}}(x_1, y)) \wedge I_{\widetilde{\emptyset}}(y)) = 1,$

$F_{\underline{\widetilde{R}}(\widetilde{\emptyset})}(x_1) = \bigvee_{y\in U}(T_{\widetilde{R}}(x_1, y) \wedge F_{\widetilde{\emptyset}}(y)) = 1,$

$T_{\overline{\widetilde{R}}(\widetilde{U})}(x_1) = \bigvee_{y\in U}(T_{\widetilde{R}}(x_1, y) \wedge T_{\widetilde{U}}(y)) = 1,$

$I_{\overline{\widetilde{R}}(\widetilde{U})}(x_1) = \bigwedge_{y\in U}(I_{\widetilde{R}}(x_1, y) \vee I_{\widetilde{U}}(y)) = 0,$

$F_{\overline{\widetilde{R}}(\widetilde{U})}(x_1) = \bigwedge_{y\in U}(F_{\widetilde{R}}(x_1, y) \vee F_{\widetilde{U}}(y)) = 0.4.$

Hence $\underline{\widetilde{R}}(\widetilde{\emptyset})(x_1) = (0.4, 1, 1)$. Similarly, we can obtain $\underline{\widetilde{R}}(\widetilde{\emptyset})(x_2) = (0.4, 1, 0.6)$ and $\underline{\widetilde{R}}(\widetilde{\emptyset})(x_3) = (0, 1, 1)$. Thus $\underline{\widetilde{R}}(\widetilde{\emptyset}) = \{\langle x_1, 0.4, 1, 1\rangle, \langle x_2, 0.4, 1, 0.6\rangle, \langle x_3, 0, 1, 1\rangle\} \neq \widetilde{\emptyset}$, which means that $\widetilde{\emptyset} \notin \tau_{\widetilde{R}}$. So $\tau_{\widetilde{R}}$ do not form a single valued neutrosophic topology.

Lemma 4.4. If $\widetilde{R}$ is a reflexive and transitive SVNR in U, for any $\widetilde{A} \in \mathrm{SVNS}(U)$, we have

$$\underline{\widetilde{R}}(\underline{\widetilde{R}}(\widetilde{A})) = \underline{\widetilde{R}}(\widetilde{A}),$$
$$\overline{\widetilde{R}}(\overline{\widetilde{R}}(\widetilde{A})) = \overline{\widetilde{R}}(\widetilde{A}).$$

Proof. It can be easily verified by Proposition 2.12.

Theorem 4.5. Then the following conclusions hold:

(1) If $\widetilde{R}$ is a reflexive and transitive SVNR in U, then $\tau_{\widetilde{R}} = \{\underline{\widetilde{R}}(\widetilde{A}) \mid \widetilde{A} \in \mathrm{SVNS}(U)\}$;

(2) $\tau_{\widetilde{R}} = \{\widetilde{A} \in \mathrm{SVNS}(U) \mid \overline{\widetilde{R}}(\widetilde{A}^c) = \widetilde{A}^c\}$.

Proof. (1) It is obvious that $\tau_{\widetilde{R}} \Subset \{\underline{\widetilde{R}}(\widetilde{A}) \mid \widetilde{A} \in \mathrm{SVNS}(U)\}$. By Lemma 4.4, for any $\widetilde{A} \in \mathrm{SVNS}(U)$, we have $\underline{\widetilde{R}}(\underline{\widetilde{R}}(\widetilde{A})) = \underline{\widetilde{R}}(\widetilde{A})$, which implies that $\underline{\widetilde{R}}(\widetilde{A}) \in \tau_{\widetilde{R}}$ and $\{\underline{\widetilde{R}}(\widetilde{A}) \mid \widetilde{A} \in \mathrm{SVNS}(U)\} \Subset \tau_{\widetilde{R}}$. So $\tau_{\widetilde{R}} = \{\underline{\widetilde{R}}(\widetilde{A}) \mid \widetilde{A} \in \mathrm{SVNS}(U)\}$.

(2) It follows immediately from the duality of $\underline{\widetilde{R}}$ and $\overline{\widetilde{R}}$.

The above Theorem 4.5 (1) states a reflexive and transitive single valued neutrosophic relation can generate a single valued neutrosophic topology and this topology is just the family of all single valued neutrosophic lower approximations induced by the given single valued neutrosophic relation.

Proposition 4.6. Let I be an index set, and $\widetilde{A}_i \in \mathrm{SVNS}(U)$ for any $i \in I$. If $\widetilde{R}$ is a reflexive and transitive SVNR in U, then $\underline{\widetilde{R}}(\Cup_{i\in I}\underline{\widetilde{R}}(\widetilde{A}_i)) = \Cup_{i\in I}\underline{\widetilde{R}}(\widetilde{A}_i)$.

Proof. By the reflexivity of $\widetilde{R}$, we have $\underline{\widetilde{R}}(\Cup_{i\in I}\underline{\widetilde{R}}(\widetilde{A}_i)) \Subset \Cup_{i\in I}\underline{\widetilde{R}}(\widetilde{A}_i)$.

On the other hand, since $\Cup_{i\in I}\underline{\widetilde{R}}(\widetilde{A}_i) \Supset \underline{\widetilde{R}}(\widetilde{A}_i)$, $\underline{\widetilde{R}}(\Cup_{i\in I}\underline{\widetilde{R}}(\widetilde{A}_i)) \Supset \underline{\widetilde{R}}(\underline{\widetilde{R}}(\widetilde{A}_i))$. By Lemma 4.4, we have $\underline{\widetilde{R}}(\Cup_{i\in I}\underline{\widetilde{R}}(\widetilde{A}_i)) \Supset \underline{\widetilde{R}}(\widetilde{A}_i)$, which means that $\underline{\widetilde{R}}(\Cup_{i\in I}\underline{\widetilde{R}}(\widetilde{A}_i)) \Supset \Cup_{i\in I}\underline{\widetilde{R}}(\widetilde{A}_i)$.

Thus $\underline{\widetilde{R}}(\Cup_{i\in I}\underline{\widetilde{R}}(\widetilde{A}_i)) = \Cup_{i\in I}\underline{\widetilde{R}}(\widetilde{A}_i)$.

The following Theorem 4.7 shows that the single valued neutrosophic lower and upper approximation operators are the interior and closure operators of a single valued topological space induced by a reflexive and transitive SVNR, respectively.

Theorem 4.7. Let $(U, \tau_{\widetilde{R}})$ be a single valued neutrosophic topological space induced by a reflexive and transitive SVNR w.r.t. $(U, \widetilde{R})$, i.e., $\tau_{\widetilde{R}} = \{\underline{\widetilde{R}}(\widetilde{A}) \mid \widetilde{A} \in \mathrm{SVNS}(U)\}$, then $\forall \widetilde{A} \in \mathrm{SVNS}(U)$,

(1) $$\begin{aligned}\underline{\widetilde{R}}(\widetilde{A}) &= int_{\tau_{\widetilde{R}}}(A)\\ &= \Cup\{\underline{\widetilde{R}}(\widetilde{B}) \mid \underline{\widetilde{R}}(\widetilde{B}) \Subset \widetilde{A}, \widetilde{B} \in \mathrm{SVNS}(U)\},\end{aligned}$$

(2) $$\begin{aligned}\overline{\widetilde{R}}(\widetilde{A}) &= cl_{\tau_{\widetilde{R}}}(\widetilde{A})\\ &= \Cap\{(\underline{\widetilde{R}}(\widetilde{B}))^c \mid (\underline{\widetilde{R}}(\widetilde{B}))^c \Supset \widetilde{A}, \widetilde{B} \in \mathrm{SVNS}(U)\}\\ &= \Cap\{\overline{\widetilde{R}}(\widetilde{B}^c) \mid \overline{\widetilde{R}}(\widetilde{B}^c) \Supset \widetilde{A}, \widetilde{B} \in \mathrm{SVNS}(U)\}.\end{aligned}$$

Proof. (1) By reflexivity of $\widetilde{R}$ and Proposition 2.12, we have $\underline{\widetilde{R}}(\widetilde{A}) \Subset \widetilde{A}$. So $\underline{\widetilde{R}}(\widetilde{A}) \Subset$

$\Cup\{\underline{\widetilde{R}}(\widetilde{B}) \mid \underline{\widetilde{R}}(\widetilde{B}) \Subset \widetilde{A}, \widetilde{B} \in \text{SVNS}(U)\}$. On the other hand, we have $\Cup\{\underline{\widetilde{R}}(\widetilde{B}) \mid \underline{\widetilde{R}}(\widetilde{B}) \Subset \widetilde{A}, \widetilde{B} \in \text{SVNS}(U)\} \Subset \widetilde{A}$, from which it follows that $\underline{\widetilde{R}}(\Cup\{\underline{\widetilde{R}}(\widetilde{B}) \mid \underline{\widetilde{R}}(\widetilde{B}) \Subset \widetilde{A}, \widetilde{B} \in \text{SVNS}(U)\}) \Subset \underline{\widetilde{R}}(\widetilde{A})$. By Proposition 4.6, $\Cup\{\underline{\widetilde{R}}(\widetilde{B}) \mid \underline{\widetilde{R}}(\widetilde{B}) \Subset \widetilde{A}, \widetilde{B} \in \text{SVNS}(U)\} = \underline{\widetilde{R}}(\Cup\{\underline{\widetilde{R}}(\widetilde{B}) \mid \underline{\widetilde{R}}(\widetilde{B})) \Subset \underline{\widetilde{R}}(\widetilde{A})$. Hence we can conclude that $\underline{\widetilde{R}}(\widetilde{A}) = \Cup\{\underline{\widetilde{R}}(\widetilde{B}) \mid \underline{\widetilde{R}}(\widetilde{B}) \Subset \widetilde{A}, \widetilde{B} \in \text{SVNS}(U)\}$.

(2) It follows immediately from the duality of $\underline{\widetilde{R}}$ and $\overline{\overline{\widetilde{R}}}$.

Proposition 4.8. Let $\widetilde{R} \in \text{SVNR}(U \times U)$, then $\forall x, y \in U$, we have

$$\underline{\widetilde{R}}_{(1_y)}(x) = (1, 0, 0) \text{ and } \overline{\overline{\widetilde{R}}}_{(1_y)}(x) = \widetilde{R}(x, y).$$

Proof. It can easily be verified by Definition 2.10.

Theorem 4.9. Let $(U, \widetilde{R})$ be a single valued neutrosophic approximation space and $(U, \tau_{\widetilde{R}})$ be a single valued neutrosophic topological space induced by $(U, \widetilde{R})$, where $\widetilde{R}$ is a reflexive and transitive SVNR in U, then

$$\widetilde{R} = \{\langle (x, y), T_{\widetilde{R}}(x, y), I_{\widetilde{R}}(x, y), F_{\widetilde{R}}(x, y)\rangle \mid (x, y) \in U \times U\},$$

where $\forall x, y \in U$, $T_{\widetilde{R}}(x, y) = \bigwedge_{\widetilde{B} \in (y)_{\tau_{\widetilde{R}}}} T_{\widetilde{B}}(x)$, $I_{\widetilde{R}}(x, y) = \bigvee_{\widetilde{B} \in (y)_{\tau_{\widetilde{R}}}} I_{\widetilde{B}}(x)$, $F_{\widetilde{R}}(x, y) = \bigvee_{\widetilde{B} \in (y)_{\tau_{\widetilde{R}}}} F_{\widetilde{B}}(x)$ and $(y)_{\tau_{\widetilde{R}}} = \{\widetilde{B} \in \text{SVNS}(U) \mid \widetilde{B}^c \in \tau_{\widetilde{R}}, T_{\widetilde{B}}(y) = 1, I_{\widetilde{B}}(y) = 0, F_{\widetilde{B}}(y) = 0\}$.

Proof. For any $x, y \in U$, by Theorem 4.7, we have $\overline{\overline{\widetilde{R}}}(1_y) = cl_{\tau_{\widetilde{R}}}(1_y)$. Moreover, by Proposition 4.8, we have $T_{\widetilde{R}}(x, y) = T_{\overline{\overline{\widetilde{R}}}(1_y)}(x) = T_{cl_{\tau_{\widetilde{R}}}(1_y)}(x)$, $I_{\widetilde{R}}(x, y) = I_{\overline{\overline{\widetilde{R}}}(1_y)}(x) = I_{cl_{\tau_{\widetilde{R}}}(1_y)}(x)$ and $F_{\widetilde{R}}(x, y) = F_{\overline{\overline{\widetilde{R}}}(1_y)}(x) = F_{cl_{\tau_{\widetilde{R}}}(1_y)}(x)$.

Notice that $cl_{\tau_{\widetilde{R}}}(1_y) = \Cap\{\widetilde{B} \in \text{SVNS}(U) \mid \widetilde{B}^c \in \tau_{\widetilde{R}}, 1_y \Subset \widetilde{B}\}$. Then for any $x, y \in U$, we have

$$\begin{aligned}
T_{\widetilde{R}}(x, y) &= T_{cl_{\tau_{\widetilde{R}}}(1_y)}(x) \\
&= T_{\Cap\{\widetilde{B} \in SVNS(U) \mid \widetilde{B}^c \in \tau_{\widetilde{R}}, 1_y \Subset \widetilde{B}\}}(x) \\
&= \bigwedge\{T_{\widetilde{B}}(x) \mid \widetilde{B}^c \in \tau_{\widetilde{R}}, T_{1_y}(t) \leq T_{\widetilde{B}}(t) \text{ for any } t \in U\} \\
&= \bigwedge\{T_{\widetilde{B}}(x) \mid \widetilde{B}^c \in \tau_{\widetilde{R}}, T_{\widetilde{B}}(y) = 1\} \\
&= \bigwedge\nolimits_{\widetilde{B} \in (y)_{\tau_{\widetilde{R}}}} T_{\widetilde{B}}(x), \\
I_{\widetilde{R}}(x, y) &= I_{cl_{\tau_{\widetilde{R}}}(1_y)}(x) \\
&= I_{\Cap\{\widetilde{B} \in SVNS(U) \mid \widetilde{B}^c \in \tau_{\widetilde{R}}, 1_y \Subset \widetilde{B}\}}(x) \\
&= \bigvee\{I_{\widetilde{B}}(x) \mid \widetilde{B}^c \in \tau_{\widetilde{R}}, I_{1_y}(t) \geq I_{\widetilde{B}}(t) \text{ for any } t \in U\} \\
&= \bigvee\{I_{\widetilde{B}}(x) \mid \widetilde{B}^c \in \tau_{\widetilde{R}}, I_{\widetilde{B}}(y) = 0\} \\
&= \bigvee\nolimits_{\widetilde{B} \in (y)_{\tau_{\widetilde{R}}}} I_{\widetilde{B}}(x),
\end{aligned}$$

$$
\begin{aligned}
F_{\widetilde{R}}(x,y) &= F_{cl_{\tau_{\widetilde{R}}}(1_y)}(x) \\
&= F_{\Cap\{\widetilde{B}\in SVNS(U)\mid \widetilde{B}^c\in\tau_{\widetilde{R}},1_y\Subset\widetilde{B}\}}(x) \\
&= \bigvee\{F_{\widetilde{B}}(x)\mid \widetilde{B}^c\in\tau_{\widetilde{R}}, F_{1_y}(t)\geq F_{\widetilde{B}}(t) \text{ for any } t\in U\} \\
&= \bigvee\{F_{\widetilde{B}}(x)\mid \widetilde{B}^c\in\tau_{\widetilde{R}}, F_{\widetilde{B}}(y)=0\} \\
&= \bigvee\nolimits_{\widetilde{B}\in(y)_{\tau_{\widetilde{R}}}} F_{\widetilde{B}}(x).
\end{aligned}
$$

This completes the proof.

Theorem 4.9 above shows that a reflexive and transitive SVNR can be represented by its induced single valued neutrosophic topology.

4.2. *From single valued neutrosophic topological spaces to single valued neutrosophic approximation spaces*

In subsection 4.1, we obtain a reflexive single valued neutrosophic relation derived from a single valued neutrosophic approximation space can generate a single valued neutrosophic topology. Furthermore, a reflexive and transitive single valued neutrosophic relation can induce a single valued neutrosophic topological space such that its single valued neutrosophic interior and closure operators are the lower and upper approximation operators induced by this single valued neutrosophic relation, respectively. In this subsection, we discuss the reverse problem—if a single valued neutrosophic topological space can induce a single valued neutrosophic approximation space under some specifical conditions?

Theorem 4.10. Let (U,τ) be a single valued neutrosophic topological space and int, cl : SVNS(U) $\longrightarrow$ SVNS(U) be its single valued neutrosophic interior and closure operators, respectively. Then there exists a reflexive and transitive SVNR $\widetilde{R_\tau}$ in U such that $\underline{\widetilde{R_\tau}}(\widetilde{A}) = int(\widetilde{A})$ and $\overline{\widetilde{R_\tau}}(\widetilde{A}) = cl(\widetilde{A})$ for all $\widetilde{A}\in$ SVNS(U) iff int satisfies the axioms (I1) and (I2), or equivalently, cl satisfies the axioms (C1) and (C2): $\forall \widetilde{A},\widetilde{B}\in$ SVNS$(U), \forall \alpha_1,\alpha_2,\alpha_3\in[0,1]$,

(I1) $int(\widetilde{A}\Cup \widehat{\alpha_1,\alpha_2,\alpha_3}) = int(\widetilde{A})\Cup \widehat{\alpha_1,\alpha_2,\alpha_3}$,

(I2) $int(\widetilde{A}\Cap\widetilde{B}) = int(\widetilde{A})\Cap int(\widetilde{B})$,

(C1) $cl(\widetilde{A}\Cap \widehat{\alpha_1,\alpha_2,\alpha_3}) = cl(\widetilde{A})\Cap \widehat{\alpha_1,\alpha_2,\alpha_3}$,

(C2) $cl(\widetilde{A}\Cup\widetilde{B}) = cl(\widetilde{A})\Cup cl(\widetilde{B})$.

Proof. " $\Longrightarrow$ " Suppose that there exists a reflexive and transitive SVNR $\widetilde{R_\tau}$ in U such that $\underline{\widetilde{R_\tau}}(\widetilde{A}) = int(\widetilde{A})$ and $\overline{\widetilde{R_\tau}}(\widetilde{A}) = cl(\widetilde{A})$ for all $\widetilde{A}\in$ SVNS(U), then it can be easily observed that the axioms (I1), (I2), (C1) and (C2) hold.

" $\Longleftarrow$ " Assume that the operator cl satisfies the axioms (C1) and (C2). By cl, we define a single valued neutrosophic relation $\widetilde{R_\tau}=\{\langle(x,y),T_{\widetilde{R_\tau}}(x,y),I_{\widetilde{R_\tau}}(x,y),F_{\widetilde{R_\tau}}(x,y)\rangle\mid(x,y)\in U\times U\}$ in $U\times U$ as follows:

$$T_{\widetilde{R_\tau}}(x,y)=T_{cl(1_y)}(x),\ I_{\widetilde{R_\tau}}(x,y)=I_{cl(1_y)}(x),\ F_{\widetilde{R_\tau}}(x,y)=F_{cl(1_y)}(x) \tag{2}$$

Moreover, we can prove that for any $\widetilde{A}\in\text{SVNS}(U)$,

$$\widetilde{A}=\Cup_{y\in U}(1_y\Cap T_{\widetilde{A}}(y),\widehat{I_{\widetilde{A}}(y)},F_{\widetilde{A}}(y)).$$

In fact, for any $x\in U$, by Definition 2.6, we have

$$\begin{aligned}
T_{\Cup_{y\in U}(1_y\Cap T_{\widetilde{A}}(y),\widehat{I_{\widetilde{A}}(y)},F_{\widetilde{A}}(y)}(x)&=\textstyle\bigvee_{y\in U}T_{1_y\Cap T_{\widetilde{A}}(y),\widehat{I_{\widetilde{A}}(y)},F_{\widetilde{A}}(y))}(x)\\
&=\textstyle\bigvee_{y\in U}(T_{1_y}(x)\wedge T_{T_{\widetilde{A}}(y),\widehat{I_{\widetilde{A}}(y)},F_{\widetilde{A}}(y)}(x))\\
&=(1\wedge T_{\widetilde{A}}(x))\textstyle\bigvee_{y\in U,y\neq x}(0\wedge T_{\widetilde{A}}(y))\\
&=T_{\widetilde{A}}(x).
\end{aligned}$$

$$\begin{aligned}
I_{\Cup_{y\in U}(1_y\Cap T_{\widetilde{A}}(y),\widehat{I_{\widetilde{A}}(y)},F_{\widetilde{A}}(y))}(x)&=\textstyle\bigwedge_{y\in U}I_{1_y\Cap T_{\widetilde{A}}(y),\widehat{I_{\widetilde{A}}(y)},F_{\widetilde{A}}(y))}(x)\\
&=\textstyle\bigwedge_{y\in U}(I_{1_y}(x)\vee I_{T_{\widetilde{A}}(y),\widehat{I_{\widetilde{A}}(y)},F_{\widetilde{A}}(y)}(x))\\
&=(0\vee I_{\widetilde{A}}(x))\textstyle\bigwedge_{y\in U,y\neq x}(1\vee I_{\widetilde{A}}(y))\\
&=I_{\widetilde{A}}(x).
\end{aligned}$$

$$\begin{aligned}
F_{\Cup_{y\in U}(1_y\Cap T_{\widetilde{A}}(y),\widehat{I_{\widetilde{A}}(y)},F_{\widetilde{A}}(y))}(x)&=\textstyle\bigwedge_{y\in U}F_{1_y\Cap T_{\widetilde{A}}(y),\widehat{I_{\widetilde{A}}(y)},F_{\widetilde{A}}(y))}(x)\\
&=\textstyle\bigwedge_{y\in U}(F_{1_y}(x)\vee F_{T_{\widetilde{A}}(y),\widehat{I_{\widetilde{A}}(y)},F_{\widetilde{A}}(y)}(x))\\
&=(0\vee F_{\widetilde{A}}(x))\textstyle\bigwedge_{y\in U,y\neq x}(1\vee F_{\widetilde{A}}(y))\\
&=F_{\widetilde{A}}(x).
\end{aligned}$$

So $\widetilde{A}=\Cup_{y\in U}(1_y\Cap T_{\widetilde{A}}(y),\widehat{I_{\widetilde{A}}(y)},F_{\widetilde{A}}(y))$ for any $\widetilde{A}\in\text{SVNS}(U)$.

Next, we prove $cl(\widetilde{A})=\overline{\widetilde{R_\tau}}(\widetilde{A})$ and $int(\widetilde{A})=\underline{\widetilde{R_\tau}}(\widetilde{A})$.

For any $x\in U$, by definition 2.10 and the axioms (I1), (I2), (C1) and (C2), we have

$$\begin{aligned}
T_{\overline{\widetilde{R_\tau}}(\widetilde{A})}(x)&=\textstyle\bigvee_{y\in U}(T_{\widetilde{R_\tau}}(x,y)\wedge T_{\widetilde{A}}(y))\\
&=\textstyle\bigvee_{y\in U}(T_{cl(1_y)}(x)\wedge T_{T_{\widetilde{A}}(y),\widehat{I_{\widetilde{A}}(y)},F_{\widetilde{A}}(y)}(x))\\
&=\textstyle\bigvee_{y\in U}(T_{cl(1_y)\Cap T_{\widetilde{A}}(y),\widehat{I_{\widetilde{A}}(y)},F_{\widetilde{A}}(y)}(x))\\
&=\textstyle\bigvee_{y\in U}(T_{cl(1_y\Cap T_{\widetilde{A}}(y),\widehat{I_{\widetilde{A}}(y)},F_{\widetilde{A}}(y))}(x))\\
&=T_{\Cup_{y\in U}cl(1_y\Cap T_{\widetilde{A}}(y),\widehat{I_{\widetilde{A}}(y)},F_{\widetilde{A}}(y))}(x)\\
&=T_{cl(\Cup_{y\in U}(1_y\Cap T_{\widetilde{A}}(y),\widehat{I_{\widetilde{A}}(y)},F_{\widetilde{A}}(y)))}(x)\\
&=T_{cl(\widetilde{A})}(x).
\end{aligned}$$

$$\begin{aligned}
I_{\overline{\widetilde{R_\tau}}(\widetilde{A})}(x)&=\textstyle\bigwedge_{y\in U}(I_{\widetilde{R_\tau}}(x,y)\vee I_{\widetilde{A}}(y))\\
&=\textstyle\bigwedge_{y\in U}(I_{cl(1_y)}(x)\vee I_{T_{\widetilde{A}}(y),\widehat{I_{\widetilde{A}}(y)},F_{\widetilde{A}}(y)}(x))
\end{aligned}$$

$$
\begin{aligned}
&= \bigwedge_{y\in U}(I_{cl(1_y)\Cap T_{\widetilde{A}}(y),\widehat{I_{\widetilde{A}}(y)},F_{\widetilde{A}}(y)}(x)) \\
&= \bigwedge_{y\in U}(I_{cl(1_y\Cap T_{\widetilde{A}}(y),\widehat{I_{\widetilde{A}}(y)},F_{\widetilde{A}}(y))}(x)) \\
&= I_{\Cup_{y\in U} cl(1_y\Cap T_{\widetilde{A}}(y),\widehat{I_{\widetilde{A}}(y)},F_{\widetilde{A}}(y))}(x) \\
&= I_{cl(\Cup_{y\in U}(1_y\Cap T_{\widetilde{A}}(y),\widehat{I_{\widetilde{A}}(y)},F_{\widetilde{A}}(y)))}(x) \\
&= I_{cl(\widetilde{A})}(x). \\
F_{\overline{\widetilde{R_\tau}}(\widetilde{A})}(x) &= \bigwedge_{y\in U}(F_{\widetilde{R_\tau}}(x,y)\vee F_{\widetilde{A}}(y)) \\
&= \bigwedge_{y\in U}(F_{cl(1_y)}(x)\vee F_{T_{\widetilde{A}}(y),\widehat{I_{\widetilde{A}}(y)},F_{\widetilde{A}}(y)}(x)) \\
&= \bigwedge_{y\in U}(F_{cl(1_y)\Cap T_{\widetilde{A}}(y),\widehat{I_{\widetilde{A}}(y)},F_{\widetilde{A}}(y)}(x)) \\
&= \bigwedge_{y\in U}(F_{cl(1_y\Cap T_{\widetilde{A}}(y),\widehat{I_{\widetilde{A}}(y)},F_{\widetilde{A}}(y))}(x)) \\
&= F_{\Cup_{y\in U} cl(1_y\Cap T_{\widetilde{A}}(y),\widehat{I_{\widetilde{A}}(y)},F_{\widetilde{A}}(y))}(x) \\
&= F_{cl(\Cup_{y\in U}(1_y\Cap T_{\widetilde{A}}(y),\widehat{I_{\widetilde{A}}(y)},F_{\widetilde{A}}(y)))}(x) \\
&= F_{cl(\widetilde{A})}(x).
\end{aligned}
$$

Thus $cl(\widetilde{A}) = \overline{\widetilde{R_\tau}}(\widetilde{A})$. Note that cl and int are dual to each other, we have $int(\widetilde{A}) = \underline{\widetilde{R_\tau}}(\widetilde{A})$. By Theorem 3.5, we have $\underline{\widetilde{R_\tau}}(\widetilde{A}) \Subset \widetilde{A}$. Then, by Proposition 2.12, $\widetilde{R_\tau}$ is reflexive. Moreover, by Theorem 3.5 again, we have $\underline{\widetilde{R_\tau}}(\underline{\widetilde{R_\tau}}(\widetilde{A})) = \underline{\widetilde{R_\tau}}(\widetilde{A})$, which means that $\underline{\widetilde{R_\tau}}(\widetilde{A}) \Subset \underline{\widetilde{R_\tau}}(\underline{\widetilde{R_\tau}}(\widetilde{A}))$. By Proposition 2.12, $\widetilde{R_\tau}$ is transitive. Hence $\widetilde{R_\tau}$ is a reflexive and transitive SVNR.

The above Theorem 4.10 gives the sufficient and necessary conditions that a single valued neutrosophic interior (closure, respectively) operator derived from a single valued neutrosophic topological space is just the single valued neutrosophic lower (upper, respectively) approximation operator induced by a reflexive and transitive single valued neutrosophic relation. Based on Theorem 4.10, we give the following definition.

Definition 4.11. Let (U,τ) be a single valued neutrosophic topological space and int, cl: SVNS(U) $\longrightarrow$ SVNS(U) be the single valued neutrosophic interior and closure operators of τ, respectively. If int satisfies the axioms (I1) and (I2) (or equivalently, cl satisfies the axioms (C1) and (C2)), then we call (U,τ) a single valued neutrosophic rough topological space and τ a single valued neutrosophic rough topology.

Let $\widetilde{\mathcal{R}}$ be the set of all reflexive and transitive SVNRs in U and $\mathcal{T}$ be the set of all single valued neutrosophic rough topologies. We can obtain the following results.

Theorem 4.12. (1) If $\widetilde{R} \in \widetilde{\mathcal{R}}$, $\tau_{\widetilde{R}}$ is defined by Equation (1) and $\widetilde{R}_{\tau_{\widetilde{R}}}$ is defined by Equation (2), then $\widetilde{R}_{\tau_{\widetilde{R}}} = \widetilde{R}$.

(2) If $\tau \in \mathcal{T}$, $\widetilde{R}_\tau$ is defined by Equation (2) and $\tau_{\widetilde{R}_\tau}$ is defined by Equation (1), then $\tau_{\widetilde{R}_\tau} = \tau$.

Proof. (1) By Theorem 4.7, and the reflexivity and transitivity of $\widetilde{R}$, we have $\underline{\widetilde{R}} = int_{\tau_{\widetilde{R}}}$ and $\overline{\widetilde{R}} = cl_{\tau_{\widetilde{R}}}$. According to Proposition 4.8, for any $x, y \in U$, we have

$T_{\widetilde{R}_{\tau_{\widetilde{R}}}}(x,y) = T_{cl_{\tau_{\widetilde{R}}}(1_y)}(x) = T_{\overline{\widetilde{R}}(1_y)}(x) = T_{\widetilde{R}}(x,y)$,

$I_{\widetilde{R}_{\tau_{\widetilde{R}}}}(x,y) = I_{cl_{\tau_{\widetilde{R}}}(1_y)}(x) = I_{\overline{\widetilde{R}}(1_y)}(x) = I_{\widetilde{R}}(x,y)$,

$F_{\widetilde{R}_{\tau_{\widetilde{R}}}}(x,y) = F_{cl_{\tau_{\widetilde{R}}}(1_y)}(x) = F_{\overline{\widetilde{R}}(1_y)}(x) = F_{\widetilde{R}}(x,y)$.

Thus $\widetilde{R}_{\tau_{\widetilde{R}}} = \widetilde{R}$.

(2) By Equation (1) and Theorem 4.10, we have

$\tau_{\widetilde{R}_\tau} = \{\widetilde{A} \in \text{SVNS}(U) \mid \underline{\widetilde{R_\tau}}(\widetilde{A}) = \widetilde{A}\} = \{\widetilde{A} \in \text{SVNS}(U) \mid int(\widetilde{A}) = \widetilde{A}\} = \tau$.

Theorem 4.13. There exists a one-to-one correspondence between $\widetilde{\mathcal{R}}$ and $\mathcal{T}$.

Proof. Define a mapping $f : \widetilde{\mathcal{R}} \longrightarrow \mathcal{T}$ as $\forall \widetilde{R} \in \mathcal{R}$, $f(\widetilde{R}) = \tau_{\widetilde{R}}$.

On the other hand, define a mapping $g : \mathcal{T} \longrightarrow \widetilde{\mathcal{R}}$ as $\forall \tau \in \mathcal{T}$, $g(\tau) = \widetilde{R_\tau}$. Then, by Theorem 4.12, it is easy to verify that both f and g are one-to-one correspondences between $\mathcal{R}$ and $\mathcal{T}$.

Theorem 4.13 shows that there exists a one-to-one correspondence between the set of all reflexive and transitive single valued neutrosophic relations and the set of all single valued neutrosophic rough topologies such that the single valued neutrosophic lower and upper approximation operators induced by the reflexive and transitive single valued neutrosophic relations are the single valued neutrosophic interior and closure operators of single valued neutrosophic rough topologies, respectively.

5 Conclusion

In this paper, we study the topological structures of single valued neutrosophic rough sets. Firstly, we prove that a reflexive and transition single valued neutrosophic relation can induce a single valued neutrosophic topological space such that its single valued neutrosophic interior and closure operators are the lower and upper approximation operators induced by this single valued neutrosophic relation, respectively. Then, we investigate the sufficient and necessary conditions that a single valued neutrosophic interior (closure, respectively) operator derived from a single valued neutrosophic topological space is just the single valued neutrosophic lower (upper, respectively) approximation operator derived from a single valued neutrosophic approximation space. Finally, we show there exists a one-to-one correspondence between the set of all reflexive and transitive single valued neutrosophic relations and the set of all single valued neutrosophic rough topologies.

Acknowledgements

This work is partially supported by the National Natural Science Foundation of China (No. 61473181) and the Fundamental Research Funds For the Central Universities (GK201702008).

Compliance with ethical standards

Conflict of interest Authors declares that they have no conflict of interest.

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

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