Interval Neutrosophic Muirhead mean Operators and Their Application in Multiple Attribute Group Decision Making

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Abstract: In recent years, neutrosophic sets (NSs) have attracted widespread attentions and been widely applied to multiple attribute decision-making (MADM). The interval neutrosophic set (INS) is an extension of NS, in which the truth-membership, indeterminacy-membership and falsity-membership degree are expressed by interval values, respectively. Obviously, INS can conveniently describe complex information. At the same time, Muirhead mean (MM) can capture the interrelationships among the multi-input arguments, which is a generalization of some existing aggregation operators. In this paper, we extend MM to INS, and develop some interval neutrosophic Muirhead mean (INMM) operators, and then we prove their some properties and discuss some special cases with respect to the parameter vector P. Moreover, we propose two new methods to deal with MADM problems based on the proposed operators. Finally, we verify the validity of our methods by an illustrative example, and analyze the advantages of our methods by comparing with other existing methods.

Keywords: interval neutrosophic sets; Muirhead mean operators; multiple attribute decision making.

1. Introduction

In real decision makings, because of complexity and fuzziness of decision making problems, it is difficult for decision-makers to express a preference precisely by crisp numbers for multiple attribute decision-making (MADM) and multiple attribute group decision-making (MAGDM) problems with incomplete, indeterminate and inconsistent information. Under these circumstances, Zadeh [35] proposed the fuzzy set (FS) theory, which is an effective tool to describe fuzzy information and then is used to solve MADM and MAGDM problems [5, 30]. Since FS only has one membership, and it cannot handle some complicated fuzzy information. Then, Atanassov [1,2] proposed the intuitionistic fuzzy set (IFS) by adding the non-membership on the basis of Zadeh's FS, so it is composed of truth-membership $T_A(x)$ and falsity-membership $F_A(x)$. In the IFS, membership degree and non-membership degree are expressed by real numbers, sometimes, it is insufficient or inadequate to express more complex qualitative information, then Chen and Liu [7] proposed a concept called linguistic intuitionistic fuzzy numbers (LIFNs) in which the membership degree and non-membership degree are expressed by linguistic terms. However, these concepts aforementioned can only process incomplete information but not the indeterminate information and inconsistent information.

Therefore, based on the neutrosophy, Smarandache proposed the concept of the neutrosophic set (NS) [25, 26] in 1999, which added an independent indeterminacy-membership on the basis of intuitionistic fuzzy set. Obviously, NS is composed of truth-membership, indeterminacy-membership and falsity-membership, respectively, and it is a generalization of fuzzy set, paraconsistent set, intuitionistic fuzzy set, paradoxist set etc. Nowadays, many great achievements about NS have been made. For example, Wang et al. [28, 29] proposed a single valued neutrosophic set (SVNS) with extension of NS. Majumdar and Samant [19] proposed a measure of entropy of SVNSs. And that, Ye [32] proposed simplified neutrosophic sets (SNSs). Wang and Li [27] proposed multi-valued neutrosophic sets (MVNSs). In fact, sometimes it may be difficult to express the degrees of truth-membership, falsity-membership, and indeterminacy-membership by real values, similar to interval valued intuitionistic fuzzy set (IVIFS) introduced by Atanassov [3], Wang et al. [28] proposed the concept of interval neutrosophic set (INS) which is more convenient to deal with complex information.

In recent years, information aggregation operators [12, 14, 15, 17] have attracted wide attentions of researchers and have become an important research topic in MADM or MAGDM fields. Because they have more advantages than some traditional approaches such as TOPSIS [10], VIKOR [13], ELECTRE [16] and so on. For instance, aggregation operators can provide the comprehensive values of the alternatives and then do ranking on the basis of them while traditional approaches can only give the ranking results. Now many different operators were developed for some special functions, such as power average (PA) operator [31] which can aggregate the input data by assigning the weighted vector based on the support degree between the input arguments; Heronian mean [4] and Bonferroni mean [6] can consider the interrelationship of the input arguments. Yu and Wu [34] explained the advantages of HM over BM are that HM can consider the correlation between an attribute and itself, and can relieve the calculation redundancy. However, in real decision making, because of the complexity of decision making problems, there exist the interrelationships among more than two attributes, and BM or HM can only consider the interrelationship between any two input arguments. Obviously, they are difficult to deal with this situation. Muirhead mean (MM) [21] is a well-known aggregation operator which can consider interrelationships among any number of arguments assigned by a variable vector P, and some existing operators, such as arithmetic and geometric operators (not considering the interrelationships), BM operator and Maclaurin symmetric mean [22], are its special cases. Therefore, the MM can offer a flexible and robust mechanism to process the information fusion problem and make it more adequate to solve MADM problems. However, the original MM can only deal with the numeric arguments, in order to make the MM operator to process the fuzzy information, Qin and Liu [23] extended the MM operator to process the 2-tuple linguistic information, and proposed some 2-tuple linguistic MM operators, then applied them to solve the MAGDM problems.

Due to the increasing complexity of the decision-making environment, it is usually difficult for decision makers to give the evaluation information by crisp numbers. Because INSs can better deal with fuzzy, incomplete, indeterminate and inconsistent information, and the MM operator can consider interrelationships among any number of arguments, it is meaningful to extend the MM operator to process interval neutrosophic information. So, the aims of this paper are (1) to propose some new interval neutrosophic MM operators by combining MM operator and INS; (2) to explore some desirable properties and special cases of the proposed operators; (3) to propose a multiple-attribute group decision making (MAGDM) methods based on the proposed operators; (4) to show the effectiveness and advantages of the proposed methods.

So the rest of this paper is organized as follows. In the next Section, we briefly review some basic concepts, and operational rules, comparison method and distance of INNs, Muirhead mean (MM) operator. In Section 3, we propose the some interval neutrosophic MM operators, and study some properties and some special cases of these operators. In Section 4, we develop two MADM methods for INSs based on the proposed interval neutrosophic MM operators. In Section 5, an illustrative example is given to verify the validity of the proposed methods and to show their advantages. In Section 6, we give some conclusions of this study.

2. Preliminaries

2.1 The interval neutrosophic set

Definition 1 [25]. Let X be a space of points (objects), with a generic element in X denoted by x. A neutrosophic set A in X is denoted by:

$$A=\{x(T_A(x),I_A(x),F_A(x))|x\in X\}$$

where $T_A(x)$, $I_A(x)$ and $F_A(x)$ denotes the truth-membership function, the indeterminacy-membership function and the falsity-membership function of the element $x\in X$ to the set A respectively. For each point x in X, we have $T_A(x),I_A(x),F_A(x)\in\ ]0^-,1^+[$, and $0^-\leq T_A(x)+I_A(x)+F_A(x)\leq 3^+$.

The neutrosophic set was mainly proposed from the philosophical point of view, it is difficult to apply to the real application. To solve this problem, Wang et al. [28] further proposed a single-valued neutrosophic set from a scientific or engineering perspective, which is an extension of fuzzy set, intuitionistic fuzzy set, paraconsistent set, paradoxist set etc. The definition of a single-valued set is given as follows.

Definition 2 [29]. Let X be a space of points (objects), with a generic element in X denoted by x. A single valued neutrosophic set (SVNS) A in X is denoted by: $A=\{x(T_A(x),I_A(x),F_A(x))|x\in X\}$, where $T_A(x)$, $I_A(x)$ and $F_A(x)$ denotes the truth-membership function, the indeterminacy-membership function and the falsity-membership function of the element $x\in X$ to the set A respectively. For each point x in X, we have $T_A(x),I_A(x),F_A(x)\in[0,1]$, and $0\leq T_A(x)+I_A(x)+F_A(x)\leq 3$.

For simplicity, we can use $x=(T,I,F)$ to represent an element x in SVNS, and the element x is called a single valued neutrosophic number (SVNN).

In order to describe more complex information, Wang et al. [27] further define the concept of interval valued neutrosophic set shown as follows.

Definition 3 [28]. Let X be a space of points (objects), with a generic element in X denoted by x. A interval neutrosophic set (INS) A in X is denoted by $A=\{x(T_A(x),I_A(x),F_A(x))|x\in X\}$, where $T_A(x)$, $I_A(x)$ and $F_A(x)$ denotes the truth-membership function, the indeterminacy-membership function and the falsity-membership function of the element $x\in X$ to the set A respectively. For each point x in X, meets $T_A(x),I_A(x),F_A(x)\subset[0,1]$, and $0\leq\sup(T_A(x))+\sup(I_A(x))+\sup(F_A(x))\leq 3$.

For simplicity, we can use $x=([T^L,T^U],[I^L,I^U],[F^L,F^U])$ to represent an element x in INS, and it is called interval neutrosophic number (INN). where $[T^L,T^U]\subseteq[0,1]$, $[I^L,I^U]\subseteq[0,1]$, $[F^L,F^U]\subseteq[0,1]$ and $T^U+I^U+F^U\leq 3$.

Definition 4 [33]. Let $x=([T_1^L,T_1^U],[I_1^L,I_1^U],[F_1^L,F_1^U])$ and $y=([T_2^L,T_2^U],[I_2^L,I_2^U],[F_2^L,F_2^U])$ be two INNs and $k>0$. The operations of INNs can be defined as follows:

(1) $x \oplus y = \left(\left[T_1^L + T_2^L - T_1^L T_2^L, T_1^U + T_2^U - T_1^U T_2^U\right], \left[I_1^L I_2^L, I_1^U I_2^U\right], \left[F_1^L F_2^L, F_1^U F_2^U\right]\right)$ (1)

(2) $x \otimes y = \left(\left[T_1^L T_2^L, T_1^U T_2^U\right], \left[I_1^L + I_2^L - I_1^L I_2^L, I_1^U + I_2^U - I_1^U I_2^U\right], \left[F_1^L + F_2^L - F_1^L F_2^L, F_1^U + F_2^U - F_1^U F_2^U\right]\right)$ (2)

(3) $kx = (\left[1-\left(1-T_1^L\right)^k, 1-\left(1-T_1^U\right)^k\right], \left[\left(I_1^L\right)^K, \left(I_1^U\right)^K\right], \left[\left(F_1^L\right)^K, \left(F_1^U\right)^K\right])$ (3)

(4) $x^k = (\left[\left(T_1^L\right)^k, \left(T_1^U\right)^k\right], \left[1-\left(1-I_1^L\right)^K, 1-\left(1-I_1^U\right)^K\right], \left[1-\left(1-F_1^L\right)^K, 1-\left(1-F_1^U\right)^K\right])$ (4)

Example 1. Let $A = \left([0.7,0.8],[0.0,0.1],[0.1,0.2]\right)$ and $B = \left([0.4,0.5],[0.2,0.3],[0.3,0.4]\right)$ be two INNs, and $k = 2$, then we can get the following operational results.

(1) $2 \cdot A = \left([0.91,0.96],[0,0.01],[0.01,0.04]\right)$,

(2) $A^2 = \left([0.49,0.64],[0,0.19],[0.19,0.36]\right)$,

(3) $A \oplus B = \left([0.82,0.90],[0,0.05],[0.03,0.08]\right)$,

(4) $A \cdot B = \left([0.28,0.40],[0.20,0.37],[0.37,0.52]\right)$.

Theorem 1. Let $A_1 = \left(\left[T_1^L, T_1^U\right], \left[I_1^L, I_1^U\right], \left[F_1^L, F_1^U\right]\right)$, $A_2 = \left(\left[T_2^L, T_2^U\right], \left[I_2^L, I_2^U\right], \left[F_2^L, F_2^U\right]\right)$, and $A = \left(\left[T^L, T^U\right], \left[I^L, I^U\right], \left[F^L, F^U\right]\right)$ be three INNs, then operational rules of INNs have the following properties.

(1) $A_1 \oplus A_2 = A_2 \oplus A_1$; (5)

(2) $A_1 \otimes A_2 = A_2 \otimes A_1$; (6)

(3) $\lambda(A_1 \oplus A_2) = \lambda A_1 \oplus \lambda A_2, \lambda > 0$; (7)

(4) $(A_1 \otimes A_2)^\lambda = A_1^\lambda \otimes A_2^\lambda, \lambda > 0$; (8)

(5) $\lambda_1 A \oplus \lambda_2 A = (\lambda_1 + \lambda_2)A, \lambda_1 > 0, \ \lambda_2 > 0$; (9)

(6) $A^{\lambda_1} \otimes A^{\lambda_2} = A^{\lambda_1 + \lambda_2}, \ \lambda_1 > 0, \lambda_2 > 0$; (10)

According to the score function and accuracy function of IFSs [8,9,24], the score function and accuracy function of an INN can be defined as follows.

Definition 5. Let $x = \left(\left[T^L, T^U\right], \left[I^L, I^U\right], \left[F^L, F^U\right]\right)$ be an INN, and then score function $s(x)$ and accuracy function $a(x)$ of an INN can be defined as follows:

(1) $s(x) = \frac{T^L + T^U}{2} + 1 - \frac{I^L + I^U}{2} + 1 - \frac{F^L + F^U}{2}$; (11)

(2) $a(x) = \frac{T^L + T^U}{2} + 1 - \frac{I^L + I^U}{2} + \frac{F^L + F^U}{2}$. (12)

Then according to score function $s(x)$ and accuracy function $a(x)$ of INNs, we can give the comparison method for INNs as follows.

Definition 6. Let $x=\left(\left[T_1^L,T_1^U\right],\left[I_1^L,I_1^U\right],\left[F_1^L,F_1^U\right]\right)$ and $y=\left(\left[T_2^L,T_2^U\right],\left[I_2^L,I_2^U\right],\left[F_2^L,F_2^U\right]\right)$ be two INNs, then we have

(1) If $s(x)>s(y)$, then x is superior to y, denoted by $x\succ y$;

(2) If $s(x)=s(y)$ and $a(x)\succ a(y)$, then $x\succ y$;

(3) If $s(x)=s(y)$ and $a(x)=a(y)$, then denoted by $x=y$.

2.2 The Muirhead mean (MM) operator

The MM was firstly introduced by Muirhead [21] in 1902, which was defined as follows:

Definition 7 [21]. Let $\alpha_i\ (i=1,2,...,n)$ be a collection of nonnegative real numbers, and $P=(p_1,p_2,\cdots,\ p_n)\in R^n$ be a vector of parameters. If

$$MM^P\left(\alpha_1,\alpha_2,...,\alpha_n\right)=\left(\frac{1}{n!}\sum_{\vartheta\in S_n}\prod_{j=1}^{n}\alpha_{\vartheta(j)}^{p_j}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}} \tag{13}$$

Then we call MM^P the Muirhead mean (MM), where $\vartheta(j)(j=1,2,\cdots,n)$ is any a permutation of $(1,2,\cdots,n)$, and S_n is the collection of all permutations of $(1,2,\cdots,n)$.

In addition, From Eq. (13), we can know that

(1) When $P=(1,0,\cdots,0)$, the MM reduces to $MM^{(1,0,\cdots,0)}\left(\alpha_1,\alpha_2,...,\alpha_n\right)=\frac{1}{n}\sum_{j=1}^{n}\alpha_j$ which is the arithmetic averaging operator.

(2) When $P=\left(1/n,1/n,\cdots,1/n\right)$, the MM reduces to $MM^{(1/n,1/n,\cdots,1/n)}\left(\alpha_1,\alpha_2,...,\alpha_n\right)=\prod_{j=1}^{n}\alpha_j^{1/n}$ which is the geometric averaging operator.

(3) When $P=(1,1,0,0,\cdots,0)$, the MM reduces to $MM^{(1,1,0,0,\cdots,0)}\left(\alpha_1,\alpha_2,...,\alpha_n\right)=\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\alpha_i\alpha_j\right)^{\frac{1}{2}}$ which is the BM operator [6].

(4) When $P=(\overbrace{1,1,\cdots,1}^{k},\overbrace{0,0,\cdots,0}^{n-k})$, the MM reduces to

$$MM^{(\overbrace{1,1,\cdots,1}^{k},\overbrace{0,0,\cdots,0}^{n-k})}\left(\alpha_1,\alpha_2,...,\alpha_n\right)=\left(\frac{\bigoplus_{1\le i_1\prec\ldots\prec i_k\le n}\bigotimes_{j=1}^{k}\alpha_{i_j}}{C_n^k}\right)^{\frac{1}{k}}$$

which is the Maclaurin symmetric mean (MSM) operator [18].

From the definition 7 and the special cases of MM operator mentioned-above, we can know that the advantage of the MM operator is that it can capture the overall interrelationships among the multiple input arguments and it is a generalization of some existing aggregation operators.

3.Interval neutrosophic Muirheadmean(INMM) operators

Because the traditional MM can only process the crisp number, and INNs can easily deal indeterminate and inconsistent information, it is necessary to extend MM to process INNs. In this section, we will propose some INMM operators for the interval neutrosophic information, and discuss some properties of the new operators.

3.1Interval neutrosophic Muirhead mean (INMM) operator

Definition 8. Let $\alpha_i = \left(\left[T_i^L, T_i^U\right], \left[I_i^L, I_i^U\right], \left[F_i^L, F_i^U\right]\right) (i = 1, 2, ..., n)$ be a collection of INNs, and $P = \left(p_1, p_2, \cdots, p_n\right) \in R^n$ be a vector of parameters. If

$$INMM^P\left(\alpha_1, \alpha_2, ..., \alpha_n\right) = \left(\frac{1}{n!}\sum_{\vartheta \in S_n}\prod_{j=1}^{n}\alpha_{\vartheta(j)}^{p_j}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}} \tag{14}$$

Then we call $INMM^P$ the interval neutrosophic MM (INMM) operator, where $\vartheta(j)(j = 1, 2, \cdots, n)$ is any a permutation of $\left(1, 2, \cdots, n\right)$, and S_n is the collection of all permutations of $\left(1, 2, \cdots, n\right)$.

Theorem 2. Let $\alpha_i = \left(\left[T_i^L, T_i^U\right], \left[I_i^L, I_i^U\right], \left[F_i^L, F_i^U\right]\right) (i = 1, 2, ..., n)$ be a collection of the INNs, then the aggregation result from Definition 8 is still an INN, and has

$$INMM^P\left(\alpha_1, \alpha_2, ..., \alpha_n\right) = \left(\left[\left(1-\left(\prod_{\vartheta \in S_n}\left(1-\prod_{j=1}^{n}\left(T_{\vartheta(j)}^L\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}, \left(1-\left(\prod_{\vartheta \in S_n}\left(1-\prod_{j=1}^{n}\left(T_{\vartheta(j)}^U\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}\right],\right.$$
$$\left[1-\left(1-\left(\prod_{\vartheta \in S_n}\left(1-\prod_{j=1}^{n}(1-I_{\vartheta(j)}^L)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}, 1-\left(1-\left(\prod_{\vartheta \in S_n}\left(1-\prod_{j=1}^{n}(1-I_{\vartheta(j)}^U)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}\right], \tag{15}$$
$$\left.\left[1-\left(1-\left(\prod_{\vartheta \in S_n}\left(1-\prod_{j=1}^{n}(1-F_{\vartheta(j)}^L)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}, 1-\left(1-\left(\prod_{\vartheta \in S_n}\left(1-\prod_{j=1}^{n}(1-F_{\vartheta(j)}^U)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}\right]\right)$$

Proof.

We need to prove (1) Eq. (15) is right; (2) Eq.(15) is an INN.

(1) Firstly, we prove the Eq. (15) is kept.

According to the operational laws of INNs, we get

$$\alpha_{\vartheta(j)}^{p_j} = \left(\left[\left(T_{\vartheta(j)}^L\right)^{p_j}, \left(T_{\vartheta(j)}^U\right)^{p_j}\right], \left[1-(1-I_{\vartheta(j)}^L)^{p_j}, 1-(1-I_{\vartheta(j)}^U)^{p_j}\right], \left[1-(1-F_{\vartheta(j)}^L)^{p_j}, 1-(1-F_{\vartheta(j)}^U)^{p_j}\right]\right),$$

and

$$\prod_{j=1}^{n}\alpha_{\vartheta(j)}^{p_j} = \left(\left[\prod_{j=1}^{n}\left(T_{\vartheta(j)}^L\right)^{p_j}, \prod_{j=1}^{n}\left(T_{\vartheta(j)}^U\right)^{p_j}\right], \left[1-\prod_{j=1}^{n}(1-I_{\vartheta(j)}^L)^{p_j}, 1-\prod_{j=1}^{n}(1-I_{\vartheta(j)}^U)^{p_j}\right],\right.$$
$$\left.\left[1-\prod_{j=1}^{n}(1-F_{\vartheta(j)}^L)^{p_j}, 1-\prod_{j=1}^{n}(1-F_{\vartheta(j)}^U)^{p_j}\right]\right),$$

then

$$\sum_{\vartheta\in S_n}\prod_{j=1}^{n}\alpha_{\vartheta(j)}^{p_j}=\left(\left[1-\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(T_{\vartheta(j)}^{L}\right)^{p_j}\right),1-\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(T_{\vartheta(j)}^{U}\right)^{p_j}\right)\right],\right.$$

$$\left.\left[\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-I_{\vartheta(j)}^{L})^{p_j}\right),\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-I_{\vartheta(j)}^{U})^{p_j}\right)\right],\left[\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-F_{\vartheta(j)}^{L})^{p_j}\right),\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-F_{\vartheta(j)}^{U})^{p_j}\right)\right]\right),$$

further,

$$\frac{1}{n!}\sum_{\vartheta\in S_n}\prod_{j=1}^{n}\alpha_{\vartheta(j)}^{p_j}=\left(\left[1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(T_{\vartheta(j)}^{L}\right)^{p_j}\right)\right)^{\frac{1}{n!}},1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(T_{\vartheta(j)}^{U}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right],\right.$$

$$\left[\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-I_{\vartheta(j)}^{L})^{p_j}\right)\right)^{\frac{1}{n!}},\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-I_{\vartheta(j)}^{U})^{p_j}\right)\right)^{\frac{1}{n!}}\right],\quad ,$$

$$\left.\left[\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-F_{\vartheta(j)}^{L})^{p_j}\right)\right)^{\frac{1}{n!}},\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-F_{\vartheta(j)}^{U})^{p_j}\right)\right)^{\frac{1}{n!}}\right]\right)$$

so,

$$\left(\frac{1}{n!}\sum_{\vartheta\in S_n}\prod_{j=1}^{n}\alpha_{\vartheta(j)}^{p_j}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}=\left(\left[\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(T_{\vartheta(j)}^{L}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}},\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(T_{\vartheta(j)}^{U}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}\right],\right.$$

$$\left[1-\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-I_{\vartheta(j)}^{L})^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}},1-\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-I_{\vartheta(j)}^{U})^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}\right],$$

$$\left.\left[1-\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-F_{\vartheta(j)}^{L})^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}},1-\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-F_{\vartheta(j)}^{U})^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}\right]\right)$$

i.e., (15) is kept.

(2) Then we will prove that (15) is an INN.

Let $T^{L}=\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(T_{\vartheta(j)}^{L}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}},T^{U}=\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(T_{\vartheta(j)}^{U}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}$,

$$I^{L}=1-\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-I_{\vartheta(j)}^{L})^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}},I^{U}=1-\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-I_{\vartheta(j)}^{U})^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}},$$

$$F^{L}=1-\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-F_{\vartheta(j)}^{L})^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}},F^{U}=1-\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-F_{\vartheta(j)}^{U})^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}$$

Then we need prove the following two conditions.

(i) $\left[T^L, T^U\right] \subseteq [0,1], \left[I^L, I^U\right] \subseteq [0,1], \left[F^L, F^U\right] \subseteq [0,1]$;

(ii) $0 \le T^U + I^U + F^U \le 3$.

(i) Since $T^L_{\vartheta(j)} \in [0,1]$, we can get

$$\left(T^L_{\vartheta(j)}\right)^{p_j} \in [0,1] \text{ and } \prod_{j=1}^{n}\left(T^L_{\vartheta(j)}\right)^{p_j} \in [0,1],$$

then $1-\prod_{j=1}^{n}\left(T^L_{\vartheta(j)}\right)^{p_j} \in [0,1]$, $\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(T^L_{\vartheta(j)}\right)^{p_j}\right) \in [0,1]$, and $\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(T^L_{\vartheta(j)}\right)^{p_j}\right)\right)^{\frac{1}{n!}} \in [0,1]$,

further,

$$1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(T^L_{\vartheta(j)}\right)^{p_j}\right)\right)^{\frac{1}{n!}} \in [0,1], \text{ and } \left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(T^L_{\vartheta(j)}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}} \in [0,1],$$

i.e., $0 \le T^L \le 1$.

Similarly, we can get $0 \le T^U \le 1$, $0 \le I^L \le 1$, $0 \le I^U \le 1$, $0 \le F^L \le 1$, $0 \le F^U \le 1$.

So, condition (i) is met.

(ii) Since $0 \le T^U \le 1$, $0 \le I^U \le 1$, $0 \le F^U \le 1$, then we can get $0 \le T^U + I^U + F^U \le 3$.

According to (i) and (ii), we can know the aggregation result from (15) is still an INN.

Then according to (1) and (2), Theorem 2 is kept.

Example 2. Let $x = ([0.3,0.4],[0.1,0.3],[0.4,0.5])$, $y = ([0.5,0.6],[0.1,0.4],[0.1,0.3])$ and $z = ([0.4,0.5],[0.1,0.3],[0.3,0.4])$ be three INNs, and $P = (1.0, 0.5, 0.4)$, then according to (15), we have

$$INMM^{(1.0,0.5,0.4)}(x,y,z) = \left(\left[\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(T^L_{\vartheta(j)}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}, \left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(T^U_{\vartheta(j)}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}\right],\right.$$

$$\left[1-\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-I^L_{\vartheta(j)})^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}, 1-\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-I^U_{\vartheta(j)})^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}\right],$$

$$\left.\left[1-\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-F^L_{\vartheta(j)})^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}, 1-\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-F^U_{\vartheta(j)})^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}\right]\right)$$

$$= \left(\left[\begin{pmatrix} 1-\left((1-0.3^{1.0}\times 0.5^{0.5}\times 0.4^{0.4})\times(1-0.3^{1.0}\times 0.4^{0.5}\times 0.5^{0.4})\times(1-0.4^{1.0}\times 0.5^{0.5}\times 0.3^{0.4})\times\right. \\ \left.(1-0.4^{1.0}\times 0.3^{0.5}\times 0.5^{0.4})\times(1-0.5^{1.0}\times 0.3^{0.5}\times 0.4^{0.4})\times(1-0.5^{1.0}\times 0.4^{0.5}\times 0.3^{0.4})\right)^{\frac{1}{3!}} \end{pmatrix}^{\frac{1}{1.0+0.5+0.4}}, \right.\right.$$

$$\left.\begin{pmatrix} 1-\left((1-0.4^{1.0}\times 0.6^{0.5}\times 0.5^{0.4})\times(1-0.4^{1.0}\times 0.5^{0.5}\times 0.6^{0.4})\times(1-0.5^{1.0}\times 0.4^{0.5}\times 0.6^{0.4})\times\right. \\ \left.(1-0.5^{1.0}\times 0.6^{0.5}\times 0.4^{0.4})\times(1-0.6^{1.0}\times 0.4^{0.5}\times 0.5^{0.4})\times(1-0.6^{1.0}\times 0.5^{0.5}\times 0.4^{0.4})\right)^{\frac{1}{3!}} \end{pmatrix}^{\frac{1}{1.0+0.5+0.4}}\right],$$

$$\left[1-\left(1-\left(\left(1-(1-0.1)^{1.0}\times(1-0.1)^{0.5}\times(1-0.1)^{0.4}\right)^{6}\right)^{\frac{1}{3!}}\right)^{\frac{1}{1.0+0.5+0.4}},\right.$$

$$\left.1-\left(1-\left(\begin{array}{l}\left(1-(1-0.3)^{1.0}\times(1-0.4)^{0.5}\times(1-0.3)^{0.4}\right)^{2}\times\left(1-(1-0.3)^{1.0}\times(1-0.3)^{0.5}\times(1-0.4)^{0.4}\right)^{2}\times\\ \left(1-(1-0.4)^{1.0}\times(1-0.3)^{0.5}\times(1-0.3)^{0.4}\right)^{2}\end{array}\right)^{\frac{1}{3!}}\right)^{\frac{1}{1.0+0.5+0.4}}\right],$$

$$\left[1-\left(1-\left(\begin{array}{l}\left(1-(1-0.4)^{1.0}\times(1-0.1)^{0.5}\times(1-0.3)^{0.4}\right)\times\left(1-(1-0.4)^{1.0}\times(1-0.3)^{0.5}\times(1-0.1)^{0.4}\right)\times\\ \left(1-(1-0.1)^{1.0}\times(1-0.4)^{0.5}\times(1-0.3)^{0.4}\right)\times\left(1-(1-0.1)^{1.0}\times(1-0.3)^{0.5}\times(1-0.4)^{0.4}\right)\times\\ \left(1-(1-0.3)^{1.0}\times(1-0.1)^{0.5}\times(1-0.4)^{0.4}\right)\times\left(1-(1-0.3)^{1.0}\times(1-0.4)^{0.5}\times(1-0.1)^{0.4}\right)\end{array}\right)^{\frac{1}{3!}}\right)^{\frac{1}{1.0+0.5+0.4}},\right.$$

$$\left.1-\left(1-\left(\begin{array}{l}\left(1-(1-0.5)^{1.0}\times(1-0.3)^{0.5}\times(1-0.4)^{0.4}\right)\times\left(1-(1-0.5)^{1.0}\times(1-0.4)^{0.5}\times(1-0.3)^{0.4}\right)\times\\ \left(1-(1-0.3)^{1.0}\times(1-0.4)^{0.5}\times(1-0.5)^{0.4}\right)\times\left(1-(1-0.3)^{1.0}\times(1-0.5)^{0.5}\times(1-0.4)^{0.4}\right)\times\\ \left(1-(1-0.4)^{1.0}\times(1-0.5)^{0.5}\times(1-0.3)^{0.4}\right)\times\left(1-(1-0.4)^{1.0}\times(1-0.3)^{0.5}\times(1-0.5)^{0.4}\right)\end{array}\right)^{\frac{1}{3!}}\right)^{\frac{1}{1.0+0.5+0.4}}\right]\Bigg)$$

$$=\left([0.393,0.495],[0.181,0.335],[0.273,0.404]\right)$$

Next, we will discuss some properties of INMM operator.

Property 1 (Idempotency). If all $\alpha_i\ (i=1,2,...,n)$ are equal, i.e., $\alpha_i=\alpha=\left(\left[T^L,T^U\right],\left[I^L,I^U\right],\left[F^L,F^U\right]\right)$, then

$$INMM^{P}\left(\alpha_1,\alpha_2,...,\alpha_n\right)=\alpha\ .$$

Proof.

Since $\alpha_i=\alpha=(\left[T^L,T^U\right],\left[I^L,I^U\right],\left[F^L,F^U\right])$, based on Theorem 2, we get

$$INMM^{P}\left(\alpha,\alpha,...,\alpha\right)=\left(\left[\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(T^L\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}},\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(T^U\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}\right],\right.$$

$$\left[1-\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-I^L)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}1-\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-I^U)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}\right],$$

$$\left.\left[1-\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-F^L)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}},1-\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-F^U)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}\right]\right)$$

$$=\left(\left[\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\left(T^L\right)^{\sum_{j=1}^{n}p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}},\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\left(T^U\right)^{\sum_{j=1}^{n}p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}\right],\right.$$

$$\left[1-\left(1-\left(\prod_{\vartheta\in S_n}\left(1-(1-I^L)^{\sum_{j=1}^{n}p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}},1-\left(1-\left(\prod_{\vartheta\in S_n}\left(1-(1-I^U)^{\sum_{j=1}^{n}p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}\right],$$

$$\left[1-\left(1-\left(\prod_{\vartheta\in S_n}\left(1-(1-F^L)^{\sum_{j=1}^n p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^n p_j}},1-\left(1-\left(\prod_{\vartheta\in S_n}\left(1-(1-F^U)^{\sum_{j=1}^n p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^n p_j}}\right]\Bigg)$$

$$=\left(\left[\left(1-\left(\left(1-\left(T^L\right)^{\sum_{j=1}^n p_j}\right)^{n!}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^n p_j}},\left(1-\left(\left(1-\left(T^U\right)^{\sum_{j=1}^n p_j}\right)^{n!}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^n p_j}}\right],\right.$$

$$\left[1-\left(1-\left(\left(1-(1-I^L)^{\sum_{j=1}^n p_j}\right)^{n!}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^n p_j}},1-\left(1-\left(\left(1-(1-I^U)^{\sum_{j=1}^n p_j}\right)^{n!}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^n p_j}}\right],$$

$$\left.\left[1-\left(1-\left(\left(1-(1-F^L)^{\sum_{j=1}^n p_j}\right)^{n!}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^n p_j}},1-\left(1-\left(\left(1-(1-F^U)^{\sum_{j=1}^n p_j}\right)^{n!}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^n p_j}}\right]\right)$$

$$=\left(\left[\left(\left(T^L\right)^{\sum_{j=1}^n p_j}\right)^{\frac{1}{\sum_{j=1}^n p_j}},\left(\left(T^U\right)^{\sum_{j=1}^n p_j}\right)^{\frac{1}{\sum_{j=1}^n p_j}}\right],\left[1-\left((1-I^L)^{\sum_{j=1}^n p_j}\right)^{\frac{1}{\sum_{j=1}^n p_j}},1-\left((1-I^U)^{\sum_{j=1}^n p_j}\right)^{\frac{1}{\sum_{j=1}^n p_j}}\right],\right.$$

$$\left.\left[1-\left((1-F^L)^{\sum_{j=1}^n p_j}\right)^{\frac{1}{\sum_{j=1}^n p_j}},1-\left((1-F^U)^{\sum_{j=1}^n p_j}\right)^{\frac{1}{\sum_{j=1}^n p_j}}\right]\right)$$

$$=\left(\left[T^L,T^U\right],\left[1-(1-I^L),1-(1-I^U)\right],\left[1-(1-F^L),1-(1-F^U)\right]\right)=\left(\left[T^L,T^U\right],\left[I^L,I^U\right],\left[F^L,F^U\right]\right).$$

Property 2 (Monotonicity). Let $\alpha_i=\left(\left[T_i^L,T_i^U\right],\left[I_i^L,I_i^U\right],\left[F_i^L,F_i^U\right]\right)$ and $\alpha_i'=\left(\left[T_i'^L,T_i'^U\right],\left[I_i'^L,I_i'^U\right],\left[F_i'^L,F_i'^U\right]\right)$ $(i=1,2,...,n)$ be two sets of INNs. If $T_i^L\ge T_i'^L, T_i^U\ge T_i'^U$, $I_i^L\le I_i'^L, I_i^U\le I_i'^U$, $F_i^L\le F_i'^L, F_i^U\le F_i'^U$ for all i, then

$$INMM^P\left(\alpha_1,\alpha_2,...,\alpha_n\right)\ge INMM^P\left(\alpha_1',\alpha_2',...,\alpha_n'\right).$$

Proof.

Let $\begin{array}{l} INMM^P\left(\alpha_1,\alpha_2,...,\alpha_n\right)=(\left[T^L,T^U\right],\left[I^L,I^U\right],\left[F^L,F^U\right]), \\ IFMM^P\left(\alpha_1',\alpha_2',...,\alpha_n'\right)=(\left[T'^L,T'^U\right],\left[I'^L,I'^U\right],\left[F'^L,F'^U\right]) \end{array}$,

where

$$T^L=\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^n\left(T_{\vartheta(j)}^L\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^n p_j}},T'^L=\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^n\left(T'^L_{\vartheta(j)}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^n p_j}},$$

$$T^U=\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^n\left(T_{\vartheta(j)}^U\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^n p_j}},T'^U=\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^n\left(T'^U_{\vartheta(j)}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^n p_j}},$$

and $I^{L}=1-\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-I^{L}_{\vartheta(j)})^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}$, $I'^{L}=1-\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-I'^{L}_{\vartheta(j)})^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}$

$$I^{U}=1-\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-I^{U}_{\vartheta(j)})^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}},\ I'^{U}=1-\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-I'^{U}_{\vartheta(j)})^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}},$$

$$F^{L}=1-\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-F^{L}_{\vartheta(j)})^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}},\ F'^{L}=1-\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-F'^{L}_{\vartheta(j)})^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}$$

$$F^{U}=1-\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-F^{U}_{\vartheta(j)})^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}},\ F'^{U}=1-\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-F'^{U}_{\vartheta(j)})^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}$$

Since $T_i^{L}\geq T_i'^{L}, T_i^{U}\geq T_i'^{U}$ we can get

$\left(T^{L}_{\vartheta(j)}\right)^{p_j}\geq\left(T'^{L}_{\vartheta(j)}\right)^{p_j},\left(T^{U}_{\vartheta(j)}\right)^{p_j}\geq\left(T'^{U}_{\vartheta(j)}\right)^{p_j}$ and $\prod_{j=1}^{n}\left(T^{L}_{\vartheta(j)}\right)^{p_j}\geq\prod_{j=1}^{n}\left(T'^{L}_{\vartheta(j)}\right)^{p_j},\prod_{j=1}^{n}\left(T^{U}_{\vartheta(j)}\right)^{p_j}\geq\prod_{j=1}^{n}\left(T'^{U}_{\vartheta(j)}\right)^{p_j}$,

then $1-\prod_{j=1}^{n}\left(T^{L}_{\vartheta(j)}\right)^{p_j}\leq 1-\prod_{j=1}^{n}\left(T'^{L}_{\vartheta(j)}\right)^{p_j}, 1-\prod_{j=1}^{n}\left(T^{U}_{\vartheta(j)}\right)^{p_j}\leq 1-\prod_{j=1}^{n}\left(T'^{U}_{\vartheta(j)}\right)^{p_j}$,

$$\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(T^{L}_{\vartheta(j)}\right)^{p_j}\right)\leq\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(T'^{L}_{\vartheta(j)}\right)^{p_j}\right),\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(T^{U}_{\vartheta(j)}\right)^{p_j}\right)\leq\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(T'^{U}_{\vartheta(j)}\right)^{p_j}\right),$$

and

$$\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(T^{L}_{\vartheta(j)}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\leq\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(T'^{L}_{\vartheta(j)}\right)^{p_j}\right)\right)^{\frac{1}{n!}},\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(T^{U}_{\vartheta(j)}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\leq\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(T'^{U}_{\vartheta(j)}\right)^{p_j}\right)\right)^{\frac{1}{n!}},$$

further,

$$1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(T^{L}_{\vartheta(j)}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\geq 1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(T'^{L}_{\vartheta(j)}\right)^{p_j}\right)\right)^{\frac{1}{n!}},1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(T^{U}_{\vartheta(j)}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\geq 1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(T'^{U}_{\vartheta(j)}\right)^{p_j}\right)\right)^{\frac{1}{n!}}$$

and

$$\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(T^{L}_{\vartheta(j)}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}\geq\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(T'^{L}_{\vartheta(j)}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}},$$

$$\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(T^{U}_{\vartheta(j)}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}\geq\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(T'^{U}_{\vartheta(j)}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}.$$

i.e., $T^{L}\geq T'^{L}, T^{U}\geq T'^{U}$.

Similarly, we also have $I^L \le I'^L, I^U \le I'^U$, $F^L \le F'^L, F^U \le F'^U$.

In the following, we will discuss three situations as follows.

(1) If $T^L \ge T'^L, T^U \ge T'^U$, and $I^L \le I'^L, I^U \le I'^U$, $F^L \le F'^L, F^U \le F'^U$, then

$INMM^P(\alpha_1, \alpha_2, ..., \alpha_n) > INMM^P(\alpha'_1, \alpha'_2, ..., \alpha'_n)$;

(2)If $T^L = T'^L, T^U = T'^U$, and $I^L \le I'^L, I^U \le I'^U$, $F^L \le F'^L, F^U \le F'^U$, then

$INMM^P(\alpha_1, \alpha_2, ..., \alpha_n) > INMM^P(\alpha'_1, \alpha'_2, ..., \alpha'_n)$;

(3)If $T^L = T'^L, T^U = T'^U$ and $I^L = I'^L, I^U = I'^U$, $F^L = F'^L, F^U = F'^U$, then

$INMM^P(\alpha_1, \alpha_2, ..., \alpha_n) = INMM^P(\alpha'_1, \alpha'_2, ..., \alpha'_n)$.

So, Property 2 is right.

Property 3 (Boundedness). Let $\alpha_i = \left(\left[T_i^L, T_i^U\right], \left[I_i^L, I_i^U\right], \left[F_i^L, F_i^U\right]\right)$ $(i = 1, 2, ..., n)$ be a collections of INNs, and $\alpha^- = (\min(T_i), \max(I_i), \max(F_i))$, $\alpha^+ = (\max(T_i), \min(I_i), \min(F_i))$, then

$\alpha^- \le INMM^P(\alpha_1, \alpha_2, ..., \alpha_n) \le \alpha^+$.

Proof.

Based on Properties 1 and 2, we have

$$INMM^P(\alpha_1, \alpha_2, ..., \alpha_n) \ge INMM^P(\alpha^-, \alpha^-, ..., \alpha^-) = \alpha^-$$

and $INMM^P(\alpha_1, \alpha_2, ..., \alpha_n) \le INMM^P(\alpha^+, \alpha^+, ..., \alpha^+) = \alpha^+$.

So, we $\alpha^- \le INMM^P(\alpha_1, \alpha_2, ..., \alpha_n) \le \alpha^+$.

In the following, we will explore some special cases of INMM operator with respect to the parameter vector *P*.

(1)When $P = (1, 0, \cdots, 0)$, the INMM reduces to the interval neutrosophic arithmetic averaging operator.

$$INMM^{(1,0,\cdots,0)}(\alpha_1, \alpha_2, ..., \alpha_n) = \frac{1}{n}\sum_{i=1}^{n}\alpha_i$$
$$= \left(\left[1 - \prod_{j=1}^{n}(1 - T_i^L)^{1/n}, 1 - \prod_{j=1}^{n}(1 - T_i^U)^{1/n}\right], \left[\prod_{j=1}^{n}\left(I_i^L\right)^{1/n}, \prod_{j=1}^{n}\left(I_i^U\right)^{1/n}\right], \left[\prod_{j=1}^{n}\left(F_i^L\right)^{1/n}, \prod_{j=1}^{n}\left(F_i^U\right)^{1/n}\right]\right) \quad (16)$$

(2)When $P = (\lambda, 0, \cdots, 0)$, the INMM reduces to the interval neutrosophic generalized arithmetic averaging operator.

$$INMM^{(\lambda,0,\cdots,0)}(\alpha_1,\alpha_2,...,\alpha_n)=\left(\frac{1}{n}\sum_{i=1}^{n}\alpha_i^{\lambda}\right)^{1/\lambda}=\left(\left[\left(1-\prod_{j=1}^{n}(1-\left(T_i^L\right)^{\lambda})^{1/n}\right)^{1/\lambda},\left(1-\prod_{j=1}^{n}(1-\left(T_i^U\right)^{\lambda})^{1/n}\right)^{1/\lambda}\right],\right.$$

$$\left[1-\left(1-\prod_{j=1}^{n}\left(1-(1-I_i^L)^{\lambda}\right)^{1/n}\right)^{1/\lambda},1-\left(1-\prod_{j=1}^{n}\left(1-(1-I_i^U)^{\lambda}\right)^{1/n}\right)^{1/\lambda}\right],$$

$$\left.\left[1-\left(1-\prod_{j=1}^{n}\left(1-(1-F_i^L)^{\lambda}\right)^{1/n}\right)^{1/\lambda},1-\left(1-\prod_{j=1}^{n}\left(1-(1-F_i^U)^{\lambda}\right)^{1/n}\right)^{1/\lambda}\right]\right) \tag{17}$$

(3)When $P=(1,1,0,0,\cdots,0)$, the INMM reduces to the interval neutrosophic BM operator.

$$INMM^{(1,1,0,0,\cdots,0)}(\alpha_1,\alpha_2,...,\alpha_n)=\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\i\neq j}}^{n}\alpha_i\alpha_j\right)^{\frac{1}{2}}=\left(\left[\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-T_i^LT_j^L\right)^{\frac{1}{n(n-1)}}\right)^{1/2},\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-T_i^UT_j^U\right)^{\frac{1}{n(n-1)}}\right)^{1/2}\right],\right.$$

$$\left[1-\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(I_i^L+I_j^L-I_i^LI_j^L\right)^{\frac{1}{n(n-1)}}\right)^{1/2},1-\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(I_i^U+I_j^U-I_i^UI_j^U\right)^{\frac{1}{n(n-1)}}\right)^{1/2}\right], \tag{18}$$

$$\left.\left[1-\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(F_i^L+F_j^L-F_i^LF_j^L\right)^{\frac{1}{n(n-1)}}\right)^{1/2},1-\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(F_i^U+F_j^U-F_i^UF_j^U\right)^{\frac{1}{n(n-1)}}\right)^{1/2}\right]\right)$$

(4)When $P=(\overbrace{1,1,\cdots,1}^{k},\overbrace{0,0,\cdots,0}^{n-k})$, the INMM reduces to the interval neutrosophic Maclaurin symmetric mean (MSM) operator.

$$INMM^{(\overbrace{1,1,\cdots,1}^{k},\overbrace{0,0,\cdots,0}^{n-k})}(\alpha_1,\alpha_2,...,\alpha_n)=\left(\frac{\bigoplus_{1\le i_1\prec\ldots\prec i_k\le n}\bigotimes_{j=1}^{k}\alpha_{i_j}}{C_n^k}\right)^{\frac{1}{k}}$$

$$=\left(\left[\left(1-\prod_{1\le i_1\prec\ldots\prec i_k\le n}\left(1-\prod_{j=1}^{k}T_{i_j}^L\right)^{1/C_n^k}\right)^{1/k},\left(1-\prod_{1\le i_1\prec\ldots\prec i_k\le n}\left(1-\prod_{j=1}^{k}T_{i_j}^U\right)^{1/C_n^k}\right)^{1/k}\right],\right.$$

$$\left[1-\left(1-\prod_{1\le i_1\prec\ldots\prec i_k\le n}\left(1-\prod_{j=1}^{k}\left(1-I_{i_j}^L\right)\right)^{1/C_n^k}\right)^{1/k},1-\left(1-\prod_{1\le i_1\prec\ldots\prec i_k\le n}\left(1-\prod_{j=1}^{k}\left(1-I_{i_j}^U\right)\right)^{1/C_n^k}\right)^{1/k}\right], \tag{19}$$

$$\left.\left[1-\left(1-\prod_{1\le i_1\prec\ldots\prec i_k\le n}\left(1-\prod_{j=1}^{k}\left(1-F_{i_j}^L\right)\right)^{1/C_n^k}\right)^{1/k},1-\left(1-\prod_{1\le i_1\prec\ldots\prec i_k\le n}\left(1-\prod_{j=1}^{k}\left(1-F_{i_j}^U\right)\right)^{1/C_n^k}\right)^{1/k}\right]\right)$$

(5)When $P=(1,1,\cdots,1)$, the INMM reduces to the interval neutrosophic geometric averaging operator.

$$INMM^{(1,1,\cdots,1)}(\alpha_1,\alpha_2,...,\alpha_n)=\left(\prod_{j=1}^{n}\alpha_j\right)^{1/n}$$

$$=\left(\left[\prod_{j=1}^{n}\left(T_j^L\right)^{1/n},\prod_{j=1}^{n}\left(T_j^U\right)^{1/n}\right],\left[1-\left(\prod_{j=1}^{n}(1-I_j^L)\right)^{1/n},1-\left(\prod_{j=1}^{n}(1-I_j^U)\right)^{1/n}\right],\left[1-\left(\prod_{j=1}^{n}(1-F_j^L)\right)^{1/n},1-\left(\prod_{j=1}^{n}(1-F_j^U)\right)^{1/n}\right]\right) \tag{20}$$

(6)When $P=(1/n,1/n,\cdots,1/n)$, the INMM reduces to the interval neutrosophic geometric averaging operator.

$$INMM^{(1/n,1/n,\cdots,1/n)}(\alpha_1,\alpha_2,...,\alpha_n)=\prod_{j=1}^{n}\alpha_j^{1/n}$$

$$=\left(\left[\prod_{j=1}^{n}\left(T_j^L\right)^{1/n},\prod_{j=1}^{n}\left(T_j^U\right)^{1/n}\right],\left[1-\left(\prod_{j=1}^{n}(1-I_j^L)\right)^{1/n},1-\left(\prod_{j=1}^{n}(1-I_j^U)\right)^{1/n}\right],\left[1-\left(\prod_{j=1}^{n}(1-F_j^L)\right)^{1/n},1-\left(\prod_{j=1}^{n}(1-F_j^U)\right)^{1/n}\right]\right) \quad (21)$$

Further, in order to discuss the monotonic of INMM operator about the parameter vector $P\in R^n$, we firstly cited a lemma.

Lemma 1. [20] Let $P=(p_1,p_2,\cdots,p_n)$ and $Q=(q_1,q_2,\cdots,q_n)$ be two the parameter vectors, if

$$\sum_{j=1}^{k}p_{[j]}\le\sum_{j=1}^{k}q_{[j]}\quad (j=1,2,\cdots,n-1)$$

$$\sum_{j=1}^{n}p_j=\sum_{j=1}^{n}q_j \quad (22)$$

where $([1],[2],\cdots,[n])$ is a permutation of $(i=1,2,...,n)$ and meets $p_{[j]}\ge p_{[j+1]},q_{[j]}\ge q_{[j+1]}$ for all $(i=1,2,...,n)$.

Then we can call that P is controlled by vector Q, expressed by $P\prec Q$.

Theorem 3.Let $\alpha_i=(\left[T_i^L,T_i^U\right],\left[I_i^L,I_i^U\right],\left[F_i^L,F_i^U\right])(i=1,2,...,n)$ be a collections of INNs, and $P=(p_1,p_2,\cdots,p_n)$, $Q=(q_1,q_2,\cdots,q_n)$ be two the parameter vectors, if $P\prec Q$, then

$$INMM^P(\alpha_1,\alpha_2,...,\alpha_n)\le INMM^Q(\alpha_1,\alpha_2,...,\alpha_n) \quad (23)$$

The proof this theorem is omitted, please refer to [23].

3.2. The interval neutrosophic weighted MM operator

In actual decision making, the weights of attributes will directly influence the decision-making results. However, INMM operator cannot consider the attribute weights, so it is very important to take into account the weights of attributes for information aggregation. In this subsection, we will propose a weighted INMM operator as follows.

Definition 9. Let $\alpha_i=\left(\left[T_i^L,T_i^U\right],\left[I_i^L,I_i^U\right],\left[F_i^L,F_i^U\right]\right)(i=1,2,...,n)$ be a collection of INNs, $w=\left(w_1,w_2,...,w_n\right)^T$ be the weight vector of α_i $(i=1,2,...,n)$, which satisfies $w_i\in[0,1]$ and $\sum_{i=1}^{n}w_i=1$, and let $P=(p_1,p_2,\cdots,p_n)\in R^n$ be a vector of parameters. If

$$INWMM^P(\alpha_1,\alpha_2,...,\alpha_n)=\left(\frac{1}{n!}\sum_{\vartheta\in S_n}\prod_{j=1}^{n}\left(nw_{\vartheta(j)}\alpha_{\vartheta(j)}\right)^{p_j}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}} \quad (24)$$

Then we call $INWMM^P$ the interval neutrosophic weighted MM (INWMM), where $\vartheta(j)(j=1,2,\cdots,n)$ is any a permutation of $(1,2,\cdots,n)$, and S_n is the collection of all permutations of $(1,2,\cdots,n)$.

Theorem 4. Let $\alpha_i=\left(\left[T_i^L,T_i^U\right],\left[I_i^L,I_i^U\right],\left[F_i^L,F_i^U\right]\right)(i=1,2,...,n)$ be a collection of INNs, then, the result from Definition 9 is an INN, even

$$INWMM^{P}\left(\alpha_1,\alpha_2,...,\alpha_n\right)=$$

$$\left(\left[\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-(1-T^{L}_{\vartheta(j)})^{nw_{\vartheta(j)}}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}},\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-(1-T^{U}_{\vartheta(j)})^{nw_{\vartheta(j)}}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}\right],\right.$$

$$\left[1-\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-\left(I^{L}_{\vartheta(j)}\right)^{nw_{\vartheta(j)}})^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}},1-\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-\left(I^{U}_{\vartheta(j)}\right)^{nw_{\vartheta(j)}})^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}\right], \quad (25)$$

$$\left.\left[1-\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-\left(F^{L}_{\vartheta(j)}\right)^{nw_{\vartheta(j)}})^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}},1-\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-\left(F^{U}_{\vartheta(j)}\right)^{nw_{\vartheta(j)}})^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}\right]\right)$$

Proof.

Because

$$nw_{\vartheta(j)}\alpha_{\vartheta(j)}=\left(\left[1-(1-T^{L}_{\vartheta(j)})^{nw_{\vartheta(j)}},1-(1-T^{U}_{\vartheta(j)})^{nw_{\vartheta(j)}}\right],\left[\left(I^{L}_{\vartheta(j)}\right)^{nw_{\vartheta(j)}},\left(I^{U}_{\vartheta(j)}\right)^{nw_{\vartheta(j)}}\right],\left[\left(F^{L}_{\vartheta(j)}\right)^{nw_{\vartheta(j)}},\left(F^{U}_{\vartheta(j)}\right)^{nw_{\vartheta(j)}}\right]\right),$$

we can replace $T^{L}_{\vartheta(j)}$ in Eq. (15) with $1-(1-T^{L}_{\vartheta(j)})^{nw_{\vartheta(j)}}$, $I^{L}_{\vartheta(j)}$ with $1-(1-I^{L}_{\vartheta(j)})^{nw_{\vartheta(j)}}$ and $F^{L}_{\vartheta(j)}$ with $1-(1-F^{L}_{\vartheta(j)})^{nw_{\vartheta(j)}}$, then we can get Eq.(25).

Because $\alpha_{\vartheta(j)}$ is an INN, $nw_{\vartheta(j)}\alpha_{\vartheta(j)}$ is also an INN. By Eq.(15), we know $INWMM^{P}\left(\alpha_1,\alpha_2,...,\alpha_n\right)$ is an INN.

In the following, we shall explore some desirable properties of INWMM operator.

Property 4 (Monotonicity). Let $\alpha_i=\left(\left[T_i^L,T_i^U\right],\left[I_i^L,I_i^U\right],\left[F_i^L,F_i^U\right]\right)$ and $\alpha_i'=\left(\left[T_i'^L,T_i'^U\right],\left[I_i'^L,I_i'^U\right],\left[F_i'^L,F_i'^U\right]\right)$ $(i=1,2,...,n)$ be two sets of INNs. If $T_i^L\ge T_i'^L, T_i^U\ge T_i'^U$, $I_i^L\le I_i'^L, I_i^U\le I_i'^U$, $F_i^L\le F_i'^L, F_i^U\le F_i'^U$ for all i, then

$$INWMM^{P}\left(\alpha_1,\alpha_2,...,\alpha_n\right)\ge INWMM^{P}\left(\alpha_1',\alpha_2',...,\alpha_n'\right).$$

The Proof is similar to that of INMM operator, it is omitted here.

Property 5 (Boundedness). Let $\alpha_i=\left(\left[T_i^L,T_i^U\right],\left[I_i^L,I_i^U\right],\left[F_i^L,F_i^U\right]\right)$ $(i=1,2,...,n)$ be a collections of INNs, and $\alpha^{-}=\left(\min(T_i),\max(I_i),\max(F_i)\right)$, $\alpha^{+}=\left(\max(T_i),\min(I_i),\min(F_i)\right)$, then

$$\left(\left[T^L_{\alpha^-},T^U_{\alpha^-}\right],\left[I^L_{\alpha^-},I^U_{\alpha^-}\right],\left[F^L_{\alpha^-},F^U_{\alpha^-}\right]\right)\le INWMM^{P}\left(\alpha_1,\alpha_2,...,\alpha_n\right)\le\left(\left[T^L_{\alpha^+},T^U_{\alpha^+}\right],\left[I^L_{\alpha^+},I^U_{\alpha^+}\right],\left[F^L_{\alpha^+},F^U_{\alpha^+}\right]\right),$$

where,

$$T^L_{\alpha^-}=\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-(1-\min(T_i^L))^{nw_{\vartheta(j)}}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}},T^U_{\alpha^-}=\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-(1-\min(T_i^U))^{nw_{\vartheta(j)}}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}},$$

$$I_{\alpha^-}^L = 1-\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-\max(I_i^L)^{nw_{\vartheta(j)}})^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}, I_{\alpha^-}^U = 1-\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-\max(I_i^U)^{nw_{\vartheta(j)}})^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}},$$

$$F_{\alpha^-}^L = 1-\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-\max(F_i^L)^{nw_{\vartheta(j)}})^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}, F_{\alpha^-}^U = 1-\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-\max(F_i^U)^{nw_{\vartheta(j)}})^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}},$$

$$T_{\alpha^+}^L = \left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-(1-\max(T_i^L))^{nw_{\vartheta(j)}}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}, T_{\alpha^+}^U = \left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-(1-\max(T_i^U))^{nw_{\vartheta(j)}}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}},$$

$$I_{\alpha^+}^L = 1-\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-\min(I_i^L)^{nw_{\vartheta(j)}})^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}, I_{\alpha^+}^U = 1-\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-\min(I_i^U)^{nw_{\vartheta(j)}})^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}},$$

$$F_{\alpha^+}^L = 1-\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-\min(F_i^L)^{nw_{\vartheta(j)}})^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}, F_{\alpha^+}^U = 1-\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-\min(F_i^U)^{nw_{\vartheta(j)}})^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}.$$

Proof.

According to Property 4, we have

$$INWMM^{P}\left(\alpha^-,\alpha^-,...,\alpha^-\right) \le INWMM^{P}\left(\alpha_1,\alpha_2,...,\alpha_n\right) \le INWMM^{P}\left(\alpha^+,\alpha^+,...,\alpha^+\right),$$

According to Eq. (27), we have

$$INWMM^{P}\left(\alpha^-,\alpha^-,...,\alpha^-\right) =$$

$$\left(\left[\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-(1-\min(T_i^L))^{nw_{\vartheta(j)}}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}, \left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-(1-\min(T_i^U))^{nw_{\vartheta(j)}}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}\right],\right.$$

$$\left[1-\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-\max(I_i^L)^{nw_{\vartheta(j)}})^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}, 1-\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-\max(I_i^U)^{nw_{\vartheta(j)}})^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}\right],$$

$$\left.\left[1-\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-\max(F_i^L)^{nw_{\vartheta(j)}})^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}, 1-\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-\max(F_i^U)^{nw_{\vartheta(j)}})^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}\right]\right)$$

and

$$INWMM^{P}\left(\alpha^+,\alpha^+,...,\alpha^+\right) =$$

$$\left(\left[\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-(1-\max(T_i^L))^{nw_{\vartheta(j)}}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}, \left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-(1-\max(T_i^U))^{nw_{\vartheta(j)}}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}\right],\right.$$

$$\left[1-\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-\min(I_i^L)^{nw_{\vartheta(j)}})^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}},1-\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-\min(I_i^U)^{nw_{\vartheta(j)}})^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}\right],$$

$$\left.\left[1-\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-\min(F_i^L)^{nw_{\vartheta(j)}})^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}},1-\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-\min(F_i^U)^{nw_{\vartheta(j)}})^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}\right]\right).$$

So, $INWMM^P\left(\alpha^-,\alpha^-,...,\alpha^-\right)\leq INWMM^P\left(\alpha_1,\alpha_2,...,\alpha_n\right)\leq INWMM^P\left(\alpha^+,\alpha^+,...,\alpha^+\right)$.

Theorem 5.The INMM operator is a special case of the INWMM operator.

Proof.

When $w=\left(\frac{1}{n},\frac{1}{n},\cdots,\frac{1}{n}\right)$

$$INWMM^P\left(\alpha_1,\alpha_2,...,\alpha_n\right)=\left(\left[\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-(1-T_{\vartheta(j)}^L)^{nw_{\vartheta(j)}}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}},\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-(1-T_{\vartheta(j)}^U)^{nw_{\vartheta(j)}}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}\right],\right.$$

$$\left[1-\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-\left(I_{\vartheta(j)}^L\right)^{nw_{\vartheta(j)}})^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}},1-\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-\left(I_{\vartheta(j)}^U\right)^{nw_{\vartheta(j)}})^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}\right],$$

$$\left.\left[1-\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-\left(F_{\vartheta(j)}^L\right)^{nw_{\vartheta(j)}})^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}},1-\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-\left(F_{\vartheta(j)}^U\right)^{nw_{\vartheta(j)}})^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}\right]\right)$$

$$=\left(\left[\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-(1-T_{\vartheta(j)}^L)^{n\times\frac{1}{n}}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}},\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-(1-T_{\vartheta(j)}^U)^{n\times\frac{1}{n}}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}\right],\right.$$

$$\left[1-\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-\left(I_{\vartheta(j)}^L\right)^{n\times\frac{1}{n}})^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}},1-\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-\left(I_{\vartheta(j)}^U\right)^{n\times\frac{1}{n}})^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}\right],$$

$$\left.\left[1-\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-\left(F_{\vartheta(j)}^L\right)^{n\times\frac{1}{n}})^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}},1-\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-\left(F_{\vartheta(j)}^U\right)^{n\times\frac{1}{n}})^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}\right]\right)$$

$$=\left(\left[\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(T_{\vartheta(j)}^L\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}},\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(T_{\vartheta(j)}^U\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}\right],\right.$$

$$\left[1-\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-\left(I_{\vartheta(j)}^L\right))^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}},1-\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-\left(I_{\vartheta(j)}^U\right))^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}\right],$$

$$\left.\left[1-\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-\left(F_{\vartheta(j)}^L\right))^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}},1-\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-\left(F_{\vartheta(j)}^U\right))^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}\right]\right)$$

$$=INMM^P\left(\alpha_1,\alpha_2,...,\alpha_n\right).$$

3.3. The interval neutrosophic weighted dual MM operator

In the theory of aggregation operator, there exist two types, i.e., original operator and its dual operator, for example, arithmetic average operator and geometric average operator. In this section, we will propose the dual MM operator for interval neutrosophic numbers based on the INMM operator as follows.

Definition 10. Let $\alpha_i = \left(\left[T_i^L, T_i^U\right], \left[I_i^L, I_i^U\right], \left[F_i^L, F_i^U\right]\right) (i = 1, 2, ..., n)$ be a collection of INNs, and $P = (p_1, p_2, \cdots, p_n) \in R^n$ be a vector of parameters. If

$$INDMM^P(\alpha_1, \alpha_2, ..., \alpha_n) = \frac{1}{\sum_{j=1}^{n} p_j} \left(\prod_{\vartheta \in S_n} \sum_{j=1}^{n} \left(p_j \alpha_{\vartheta(j)} \right) \right)^{\frac{1}{n!}}. \tag{26}$$

Then we call $INDMM^P$ the interval neutrosophic dual MM (INDMM), where $\vartheta(j) (j = 1, 2, \cdots, n)$ is any a permutation of $(1, 2, \cdots, n)$, and S_n is the collection of all permutations of $(1, 2, \cdots, n)$.

Theorem 6. Let $\alpha_i = \left(\left[T_i^L, T_i^U\right], \left[I_i^L, I_i^U\right], \left[F_i^L, F_i^U\right]\right) (i = 1, 2, ..., n)$ be a collection of INNs, then, the result from Definition 10 is an INN, too, even

$$INDMM^P(\alpha_1, \alpha_2, ..., \alpha_n) =$$

$$\left(\left[1 - \left(1 - \left(\prod_{\vartheta \in S_n} \left(1 - \prod_{j=1}^{n} (1 - T_{\vartheta(j)}^L)^{p_j} \right) \right)^{\frac{1}{n!}} \right)^{\frac{1}{\sum_{j=1}^{n} p_j}}, 1 - \left(1 - \left(\prod_{\vartheta \in S_n} \left(1 - \prod_{j=1}^{n} (1 - T_{\vartheta(j)}^U)^{p_j} \right) \right)^{\frac{1}{n!}} \right)^{\frac{1}{\sum_{j=1}^{n} p_j}} \right], \right.$$

$$\left[\left(1 - \left(\prod_{\vartheta \in S_n} \left(1 - \prod_{j=1}^{n} \left(I_{\vartheta(j)}^L \right)^{p_j} \right) \right)^{\frac{1}{n!}} \right)^{\frac{1}{\sum_{j=1}^{n} p_j}}, \left(1 - \left(\prod_{\vartheta \in S_n} \left(1 - \prod_{j=1}^{n} \left(I_{\vartheta(j)}^U \right)^{p_j} \right) \right)^{\frac{1}{n!}} \right)^{\frac{1}{\sum_{j=1}^{n} p_j}} \right], \tag{27}$$

$$\left. \left[\left(1 - \left(\prod_{\vartheta \in S_n} \left(1 - \prod_{j=1}^{n} \left(F_{\vartheta(j)}^L \right)^{p_j} \right) \right)^{\frac{1}{n!}} \right)^{\frac{1}{\sum_{j=1}^{n} p_j}}, \left(1 - \left(\prod_{\vartheta \in S_n} \left(1 - \prod_{j=1}^{n} \left(F_{\vartheta(j)}^U \right)^{p_j} \right) \right)^{\frac{1}{n!}} \right)^{\frac{1}{\sum_{j=1}^{n} p_j}} \right] \right)$$

Proof.

We need to prove (1) Eq. (27) is kept; (2) Eq. (27) is an INN.

(1) Firstly, we prove the Eq. (27) is kept.

According to the operational laws of INNs, we get

$$p_j \alpha_{\vartheta(j)} = \left(\left[1 - (1 - T_{\vartheta(j)}^L)^{p_j}, 1 - (1 - T_{\vartheta(j)}^U)^{p_j} \right], \left[\left(I_{\vartheta(j)}^L \right)^{p_j}, \left(I_{\vartheta(j)}^U \right)^{p_j} \right], \left[\left(F_{\vartheta(j)}^L \right)^{p_j}, \left(F_{\vartheta(j)}^U \right)^{p_j} \right] \right)$$ and

$$\sum_{j=1}^{n} \left(p_j \alpha_{\vartheta(j)} \right) = \left(\left[1 - \prod_{j=1}^{n} (1 - T_{\vartheta(j)}^L)^{p_j}, 1 - \prod_{j=1}^{n} (1 - T_{\vartheta(j)}^U)^{p_j} \right], \left[\prod_{j=1}^{n} \left(I_{\vartheta(j)}^L \right)^{p_j}, \prod_{j=1}^{n} \left(I_{\vartheta(j)}^U \right)^{p_j} \right], \right.$$

$$\left. \left[\prod_{j=1}^{n} \left(F_{\vartheta(j)}^L \right)^{p_j}, \prod_{j=1}^{n} \left(F_{\vartheta(j)}^U \right)^{p_j} \right] \right),$$

then

$$\prod_{\vartheta\in S_n}\sum_{j=1}^{n}\left(p_j\alpha_{\vartheta(j)}\right)=$$

$$\left(\left[\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-T_{\vartheta(j)}^{L})^{p_j}\right),\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-T_{\vartheta(j)}^{U})^{p_j}\right)\right],\left[1-\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(I_{\vartheta(j)}^{L}\right)^{p_j}\right),1-\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(I_{\vartheta(j)}^{U}\right)^{p_j}\right)\right],,\right.$$

$$\left.\left[1-\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(F_{\vartheta(j)}^{L}\right)^{p_j}\right),1-\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(F_{\vartheta(j)}^{U}\right)^{p_j}\right)\right]\right)$$

$$\left(\prod_{\vartheta\in S_n}\sum_{j=1}^{n}\left(p_j\alpha_{\vartheta(j)}\right)\right)^{\frac{1}{n!}}=\left(\left[\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-T_{\vartheta(j)}^{L})^{p_j}\right)^{\frac{1}{n!}},\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-T_{\vartheta(j)}^{U})^{p_j}\right)^{\frac{1}{n!}}\right],\right.$$

further, $\left[1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(I_{\vartheta(j)}^{L}\right)^{p_j}\right)\right)^{\frac{1}{n!}},1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(I_{\vartheta(j)}^{U}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right],$,

$$\left.\left[1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(F_{\vartheta(j)}^{L}\right)^{p_j}\right)\right)^{\frac{1}{n!}},1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(F_{\vartheta(j)}^{U}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right]\right)$$

so,

$$\frac{1}{\sum_{j=1}^{n}p_j}\left(\prod_{\vartheta\in S_n}\sum_{j=1}^{n}\left(p_j\alpha_{\vartheta(j)}\right)\right)^{\frac{1}{n!}}=$$

$$\left(\left[1-\left(1-\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-T_{\vartheta(j)}^{L})^{p_j}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}},1-\left(1-\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-T_{\vartheta(j)}^{U})^{p_j}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}\right],\right.$$

$$\left[\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(I_{\vartheta(j)}^{L}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}},\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(I_{\vartheta(j)}^{U}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}\right],$$

$$\left.\left[\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(F_{\vartheta(j)}^{L}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}},\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(F_{\vartheta(j)}^{U}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}\right]\right).$$

i.e., (27) is kept.

(2) Then we will prove that (27) is an INN.

Let

$$T^{L}=1-\left(1-\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-T_{\vartheta(j)}^{L})^{p_j}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}},T^{U}=1-\left(1-\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-T_{\vartheta(j)}^{U})^{p_j}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}},$$

$$I^{L}=\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(I_{\vartheta(j)}^{L}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}},I^{U}=\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(I_{\vartheta(j)}^{U}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}},$$

$$F^L = \left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(F^L_{\vartheta(j)}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}, F^U = \left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(F^U_{\vartheta(j)}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}},$$

Then we need prove the following two conditions.

(i) $\left[T^L, T^U\right]\subseteq[0,1], \left[I^L, I^U\right]\subseteq[0,1], \left[F^L, F^U\right]\subseteq[0,1]$;

(ii) $0 \le T^U + I^U + F^U \le 3$.

(i) Since $T^L_{\vartheta(j)} \in [0,1]$, we can get

$$\left(1-T^L_{\vartheta(j)}\right)\in[0,1], (1-T^L_{\vartheta(j)})^{p_j} \in [0,1] \text{ and } \prod_{j}^{n}(1-T^L_{\vartheta(j)})^{p_j} \in [0,1],$$

then

$$\left(1-\prod_{j=1}^{n}(1-T^L_{\vartheta(j)})^{p_j}\right)\in[0,1], \left(1-\prod_{j=1}^{n}(1-T^L_{\vartheta(j)})^{p_j}\right)^{\frac{1}{n!}}\in[0,1], \text{ and } \prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-T^L_{\vartheta(j)})^{p_j}\right)^{\frac{1}{n!}}\in[0,1],$$

further,

$$\left(1-\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-T^L_{\vartheta(j)})^{p_j}\right)^{\frac{1}{n!}}\right)\in[0,1], \left(1-\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-T^L_{\vartheta(j)})^{p_j}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}\in[0,1],$$

and $1-\left(1-\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-T^L_{\vartheta(j)})^{p_j}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}\in[0,1]$.

i.e., $0 \le T^L \le 1$.

Similarly, we can get $0 \le T^U \le 1, 0 \le I^L \le 1, 0 \le I^U \le 1, 0 \le F^L \le 1, 0 \le F^U \le 1$.

So, condition (i) is met.

(ii) Since $0 \le T^U \le 1$, $0 \le I^U \le 1$, $0 \le F^U \le 1$, then we can get $0 \le T^U + I^U + F^U \le 3$.

According to (i) and (ii), we can know the aggregation result from (27) is still an INN.

Then According to (1) and (2), theorem 6 is kept.

Example 3. Let $x = ([0.3, 0.4], [0.1, 0.3], [0.4, 0.5])$, $y = ([0.5, 0.6], [0.1, 0.4], [0.1, 0.3])$ and $z = ([0.4, 0.5], [0.1, 0.3], [0.3, 0.4])$ be three INNs, and $P = (1.0, 0.5, 0.4)$, then according to (27), we have

$$INDMM^{(1.0,0.5,0.4)}(x,y,z) =$$

$$\left(\left[1-\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-T^L_{\vartheta(j)})^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}, 1-\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-T^U_{\vartheta(j)})^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}\right],\right.$$

$$\left[\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(I_{\vartheta(j)}^{L}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}},\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(I_{\vartheta(j)}^{U}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}\right],$$

$$\left.\left[\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(F_{\vartheta(j)}^{L}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}},\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(F_{\vartheta(j)}^{U}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}\right]\right)$$

$$=\left(\left[\left(1-\left(1-\left(\begin{array}{l}\left(1-(1-0.3)^{1.0}\times(1-0.5)^{0.5}\times(1-0.4)^{0.4}\right)\times\left(1-(1-0.3)^{1.0}\times(1-0.4)^{0.5}\times(1-0.5)^{0.4}\right)\times\\\left(1-(1-0.5)^{1.0}\times(1-0.4)^{0.5}\times(1-0.3)^{0.4}\right)\times\left(1-(1-0.5)^{1.0}\times(1-0.3)^{0.5}\times(1-0.4)^{0.4}\right)\times\\\left(1-(1-0.4)^{1.0}\times(1-0.3)^{0.5}\times(1-0.5)^{0.4}\right)\times\left(1-(1-0.4)^{1.0}\times(1-0.5)^{0.5}\times(1-0.3)^{0.4}\right)\end{array}\right)^{\frac{1}{3!}}\right)^{\frac{1}{1.0+0.5+0.4}}\right),\right.\right.$$

$$\left.1-\left(1-\left(\begin{array}{l}\left(1-(1-0.4)^{1.0}\times(1-0.6)^{0.5}\times(1-0.5)^{0.4}\right)\times\left(1-(1-0.4)^{1.0}\times(1-0.5)^{0.5}\times(1-0.6)^{0.4}\right)\times\\\left(1-(1-0.6)^{1.0}\times(1-0.4)^{0.5}\times(1-0.5)^{0.4}\right)\times\left(1-(1-0.6)^{1.0}\times(1-0.5)^{0.5}\times(1-0.4)^{0.4}\right)\times\\\left(1-(1-0.5)^{1.0}\times(1-0.4)^{0.5}\times(1-0.6)^{0.4}\right)\times\left(1-(1-0.5)^{1.0}\times(1-0.6)^{0.5}\times(1-0.4)^{0.4}\right)\end{array}\right)^{\frac{1}{3!}}\right)^{\frac{1}{1.0+0.5+0.4}}\right],$$

$$\left[\left(1-\left((1-0.1^{1.0}\times0.1^{0.5}\times0.1^{0.4})\wedge6\right)^{\frac{1}{3!}}\right)^{\frac{1}{1.0+0.5+0.4}},\right.$$

$$\left.\left(1-\left(\left(1-0.3^{1.0}\times0.4^{0.5}\times0.3^{0.4}\right)\wedge2\times\left(1-0.3^{1.0}\times0.3^{0.5}\times0.4^{0.4}\right)\wedge2\times\left(1-0.4^{1.0}\times0.3^{0.5}\times0.3^{0.4}\right)\wedge2\right)^{\frac{1}{3!}}\right)^{\frac{1}{1.0+0.5+0.4}}\right],$$

$$\left[\left(1-\left(\begin{array}{l}\left(1-0.4^{1.0}\times0.1^{0.5}\times0.3^{0.4}\right)\times\left(1-0.4^{1.0}\times0.3^{0.5}\times0.1^{0.4}\right)\times\left(1-0.1^{1.0}\times0.3^{0.5}\times0.4^{0.4}\right)\times\\\left(1-0.1^{1.0}\times0.4^{0.5}\times0.3^{0.4}\right)\times\left(1-0.3^{1.0}\times0.4^{0.5}\times0.1^{0.4}\right)\times\left(1-0.3^{1.0}\times0.1^{0.5}\times0.4^{0.4}\right)\end{array}\right)^{\frac{1}{3!}}\right)^{\frac{1}{1.0+0.5+0.4}},\right.$$

$$\left.\left.\left(1-\left(\begin{array}{l}\left(1-0.5^{1.0}\times0.4^{0.5}\times0.3^{0.4}\right)\times\left(1-0.5^{1.0}\times0.3^{0.5}\times0.4^{0.4}\right)\times\left(1-0.4^{1.0}\times0.3^{0.5}\times0.5^{0.4}\right)\times\\\left(1-0.4^{1.0}\times0.5^{0.5}\times0.3^{0.4}\right)\times\left(1-0.3^{1.0}\times0.4^{0.5}\times0.5^{0.4}\right)\times\left(1-0.3^{1.0}\times0.5^{0.5}\times0.4^{0.4}\right)\end{array}\right)^{\frac{1}{3!}}\right)^{\frac{1}{1.0+0.5+0.4}}\right]\right)$$

$$=\left([0.404,0.505],[0.1,0.343],[0.236,0.393]\right)$$

Next, we will discuss some properties of INDMM operator.

Property 7 (Idempotency). If all $\alpha_i\ (i=1,2,...,n)$ are equal, i.e., $\alpha_i=\alpha=(T,I,F)$, then

$$INDMM^{P}\left(\alpha_1,\alpha_2,...,\alpha_n\right)=\alpha\ .$$

Property 8 (Monotonicity). Let $\alpha_i=\left(\left[T_i^L,T_i^U\right],\left[I_i^L,I_i^U\right],\left[F_i^L,F_i^U\right]\right)$ and $\alpha_i'=\left(\left[T_i'^L,T_i'^U\right],\left[I_i'^L,I_i'^U\right],\left[F_i'^L,F_i'^U\right]\right)$ $(i=1,2,...,n)$ be two sets of INNs. If $T_i^L\geq T_i'^L, T_i^U\geq T_i'^U$, $I_i^L\leq I_i'^L, I_i^U\leq I_i'^U$, $F_i^L\leq F_i'^L, F_i^U\leq F_i'^U$ for all i, then

$$INDMM^{P}\left(\alpha_1,\alpha_2,...,\alpha_n\right)\geq INDMM^{P}\left(\alpha_1',\alpha_2',...,\alpha_n'\right).$$

Property 9 (Boundedness). Let $\alpha_i = \left(\left[T_i^L, T_i^U\right], \left[I_i^L, I_i^U\right], \left[F_i^L, F_i^U\right]\right)$ $(i = 1, 2, ..., n)$ be a collections of INNs, and $\alpha^- = \left(\min(T_i), \max(I_i), \max(F_i)\right)$, $\alpha^+ = \left(\max(T_i), \min(I_i), \min(F_i)\right)$, then

$$\alpha^- \le INDMM^P\left(\alpha_1, \alpha_2, ..., \alpha_n\right) \le \alpha^+.$$

In the following, we will explore some special cases of INDMM operator with respect to the parameter vector.

(1)When $P = (1, 0, \cdots, 0)$, the INDMM reduces to the interval neutrosophic geometric averaging operator.

$$INDMM^{(1,0,\cdots,0)}\left(\alpha_1, \alpha_2, ..., \alpha_n\right) = \left(\left[\prod_{j=1}^{n}\left(T_i^L\right)^{1/n}, \prod_{j=1}^{n}\left(T_i^U\right)^{1/n}\right], \left[1-\prod_{j=1}^{n}(1-I_i^L)^{1/n}, 1-\prod_{j=1}^{n}(1-I_i^U)^{1/n}\right], \left[1-\prod_{j=1}^{n}(1-T_i^L)^{1/n}, 1-\prod_{j=1}^{n}(1-T_i^U)^{1/n}\right]\right) \quad (28)$$

(2)When $P = (\lambda, 0, \cdots, 0)$, the INDMM reduces to the interval neutrosophic generalized geometric averaging operator.

$$INDMM^{(\lambda,0,\cdots,0)}\left(\alpha_1, \alpha_2, ..., \alpha_n\right) = \left(\left[1-\left(1-\prod_{j=1}^{n}\left(1-(1-T_i^L)^{\lambda}\right)^{1/n}\right)^{1/\lambda}, 1-\left(1-\prod_{j=1}^{n}\left(1-(1-T_i^U)^{\lambda}\right)^{1/n}\right)^{1/\lambda}\right],\right.$$
$$\left.\left[\left(1-\prod_{j=1}^{n}(1-\left(I_i^L\right)^{\lambda})^{1/n}\right)^{1/\lambda}, \left(1-\prod_{j=1}^{n}(1-\left(I_i^U\right)^{\lambda})^{1/n}\right)^{1/\lambda}\right], \left[\left(1-\prod_{j=1}^{n}(1-\left(F_i^L\right)^{\lambda})^{1/n}\right)^{1/\lambda}, \left(1-\prod_{j=1}^{n}(1-\left(F_i^U\right)^{\lambda})^{1/n}\right)^{1/\lambda}\right]\right) \quad (29)$$

(3)When $P = (1, 1, 0, 0, \cdots, 0)$, the INDMM reduces to the interval neutrosophic geometric BM operator.

$$INDMM^{(1,1,0,0,\cdots,0)}\left(\alpha_1, \alpha_2, ..., \alpha_n\right) =$$
$$\left(\left[1-\left(1-\prod_{\substack{i,j=1\\ i\ne j}}^{n}\left(T_i^L + T_j^L - T_i^L T_j^L\right)^{\frac{1}{n(n-1)}}\right)^{1/2}, 1-\left(1-\prod_{\substack{i,j=1\\ i\ne j}}^{n}\left(T_i^U + T_j^U - T_i^U T_j^U\right)^{\frac{1}{n(n-1)}}\right)^{1/2}\right],\right.$$
$$\left[\left(1-\prod_{\substack{i,j=1\\ i\ne j}}^{n}\left(1-I_i^L I_j^L\right)^{\frac{1}{n(n-1)}}\right)^{1/2}, \left(1-\prod_{\substack{i,j=1\\ i\ne j}}^{n}\left(1-I_i^L I_j^L\right)^{\frac{1}{n(n-1)}}\right)^{1/2}\right], \quad (30)$$
$$\left.\left[\left(1-\prod_{\substack{i,j=1\\ i\ne j}}^{n}\left(1-F_i^L F_j^L\right)^{\frac{1}{n(n-1)}}\right)^{1/2}, \left(1-\prod_{\substack{i,j=1\\ i\ne j}}^{n}\left(1-F_i^L IF_j^L\right)^{\frac{1}{n(n-1)}}\right)^{1/2}\right]\right)$$

(4)When $P = (\overbrace{1, 1, \cdots, 1}^{k}, \overbrace{0, 0, \cdots, 0}^{n-k})$, the INDMM reduces to the interval neutrosophic geometric Maclaurin symmetric mean (MSM) operator.

$$INDMM^{(\overbrace{1,1,\cdots,1}^{k},\overbrace{0,0,\cdots,0}^{n-k})}\left(\alpha_1, \alpha_2, ..., \alpha_n\right) =$$
$$\left(\left[1-\left(1-\prod_{1\le i_1 \prec ... \prec i_k \le n}\left(1-\prod_{j=1}^{k}\left(1-T_{i_j}^L\right)\right)^{1/C_n^k}\right)^{1/k}, 1-\left(1-\prod_{1\le i_1 \prec ... \prec i_k \le n}\left(1-\prod_{j=1}^{k}\left(1-T_{i_j}^U\right)\right)^{1/C_n^k}\right)^{1/k}\right],\right.$$

$$\left[\left(1-\prod_{1\le i_1\prec\ldots\prec i_k\le n}\left(1-\prod_{j=1}^{k} I_{i_j}^{L}\right)^{1/C_n^k}\right)^{1/k},\left(1-\prod_{1\le i_1\prec\ldots\prec i_k\le n}\left(1-\prod_{j=1}^{k} I_{i_j}^{U}\right)^{1/C_n^k}\right)^{1/k}\right],$$
$$\left.\left[\left(1-\prod_{1\le i_1\prec\ldots\prec i_k\le n}\left(1-\prod_{j=1}^{k} F_{i_j}^{L}\right)^{1/C_n^k}\right)^{1/k},\left(1-\prod_{1\le i_1\prec\ldots\prec i_k\le n}\left(1-\prod_{j=1}^{k} F_{i_j}^{U}\right)^{1/C_n^k}\right)^{1/k}\right]\right) \quad (31)$$

(5)When $P=(1,1,\cdots,1)$, the IFDMM reduces to the interval neutrosophic arithmetic averaging operator.

$$INDMM^{(1,1,\cdots,1)}(\alpha_1,\alpha_2,...,\alpha_n)=$$
$$\left(\left[1-\left(\prod_{j=1}^{n}(1-T_j^L)\right)^{1/n},1-\left(\prod_{j=1}^{n}(1-T_j^U)\right)^{1/n}\right],\left[\prod_{j=1}^{n}\left(I_j^L\right)^{1/n},\prod_{j=1}^{n}\left(I_j^U\right)^{1/n}\right],\left[\prod_{j=1}^{n}\left(F_j^L\right)^{1/n},\prod_{j=1}^{n}\left(F_j^U\right)^{1/n}\right]\right) \quad (32)$$

(6)When $P=\left(\frac{1}{n},\frac{1}{n},\cdots,\frac{1}{n}\right)$, the IFMM reduces to the interval neutrosophic the arithmetic averaging operator.

$$INDMM^{\left(\frac{1}{n},\frac{1}{n},\cdots,\frac{1}{n}\right)}(\alpha_1,\alpha_2,...,\alpha_n)=$$
$$\left(\left[1-\left(\prod_{j=1}^{n}(1-T_j^L)\right)^{1/n},1-\left(\prod_{j=1}^{n}(1-T_j^U)\right)^{1/n}\right],\left[\prod_{j=1}^{n}\left(I_j^L\right)^{1/n},\prod_{j=1}^{n}\left(I_j^U\right)^{1/n}\right],\left[\prod_{j=1}^{n}\left(F_j^L\right)^{1/n},\prod_{j=1}^{n}\left(F_j^U\right)^{1/n}\right]\right) \quad (33)$$

Theorem 7. Let $\alpha_i=(\left[T_i^L,T_i^U\right],\left[I_i^L,I_i^U\right],\left[F_i^L,F_i^U\right])(i=1,2,...,n)$ be a collections of INNs, and $P=(p_1,p_2,\cdots,p_n)$, $Q=(q_1,q_2,\cdots,q_n)$ be two the parameter vectors, if $P\prec Q$, then

$$INDMM^P(\alpha_1,\alpha_2,...,\alpha_n)\ge INDMM^Q(\alpha_1,\alpha_2,...,\alpha_n) \quad (34)$$

3.4. The interval neutrosophic dual weighted MM operator

Similar to INWMM operator, we will propose interval neutrosophic dual weighted MM (INDWMM) operator so as to consider the attribute weights, which is defined as follows.

Definition 11. Let $\alpha_i=\left(\left[T_i^L,T_i^U\right],\left[I_i^L,I_i^U\right],\left[F_i^L,F_i^U\right]\right)(i=1,2,...,n)$ be a collection of INNs, $w=(w_1,w_2,...,w_n)^T$ be the weight vector of $\alpha_i\ (i=1,2,...,n)$, which satisfies $w_i\in[0,1]$ and $\sum_{i=1}^{n}w_i=1$, and let $P=(p_1,p_2,\cdots,p_n)\in R^n$ be a vector of parameters. If

$$INDWMM^P(\alpha_1,\alpha_2,...,\alpha_n)=\frac{1}{\sum_{j=1}^{n}p_j}\left(\prod_{\vartheta\in S_n}\sum_{j=1}^{n}\left(p_j\alpha_{\vartheta(j)}^{nw_{\vartheta(j)}}\right)\right)^{\frac{1}{n!}} \quad (35)$$

Then we call $INDWMM^P$ the interval neutrosophic dual weighted MM (INDWMM), where $\vartheta(j)(j=1,2,\cdots,n)$ is any a permutation of $(1,2,\cdots,n)$, and S_n is the collection of all permutations of $(1,2,\cdots,n)$.

Theorem 8. Let $\alpha_i=\left(\left[T_i^L,T_i^U\right],\left[I_i^L,I_i^U\right],\left[F_i^L,F_i^U\right]\right)(i=1,2,...,n)$ be a collection of INNs, then, the result from

Definition 11 is an INN, too, even

$$INDWMM^{p}\left(\alpha_1,\alpha_2,...,\alpha_n\right)=$$

$$\left(\left[1-\left(1-\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-\left(T_{\vartheta(j)}^{L}\right)^{nw_{\vartheta(j)}})^{p_j}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}},1-\left(1-\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-\left(T_{\vartheta(j)}^{U}\right)^{nw_{\vartheta(j)}})^{p_j}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}\right],\right.$$

$$\left[\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-(1-I_{\vartheta(j)}^{L})^{nw_{\vartheta(j)}}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}},\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-(1-I_{\vartheta(j)}^{U})^{nw_{\vartheta(j)}}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}\right], \tag{36}$$

$$\left.\left[\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-(1-F_{\vartheta(j)}^{L})^{nw_{\vartheta(j)}}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}},\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-(1-F_{\vartheta(j)}^{U})^{nw_{\vartheta(j)}}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}\right]\right)$$

Proof.

Because

$$\alpha_{\vartheta(j)}^{nw_{\vartheta(j)}}=\left(\left[\left(T_{\vartheta(j)}^{L}\right)^{nw_{\vartheta(j)}},\left(T_{\vartheta(j)}^{U}\right)^{nw_{\vartheta(j)}}\right],\left[1-(1-I_{\vartheta(j)}^{L})^{nw_{\vartheta(j)}},1-(1-I_{\vartheta(j)}^{U})^{nw_{\vartheta(j)}}\right],\left[1-(1-F_{\vartheta(j)}^{L})^{nw_{\vartheta(j)}},1-(1-F_{\vartheta(j)}^{U})^{nw_{\vartheta(j)}}\right]\right)$$

,we can replace $T_{\vartheta(j)}^{L},T_{\vartheta(j)}^{U}$ in Eq. (27) with $\left(T_{\vartheta(j)}^{L}\right)^{nw_{\vartheta(j)}},\left(T_{\vartheta(j)}^{U}\right)^{nw_{\vartheta(j)}}$, $I_{\vartheta(j)}^{L},I_{\vartheta(j)}^{U}$ with $1-(1-I_{\vartheta(j)}^{L})^{nw_{\vartheta(j)}},1-(1-I_{\vartheta(j)}^{U})^{nw_{\vartheta(j)}}$, and $F_{\vartheta(j)}^{L},F_{\vartheta(j)}^{U}$ with $1-(1-F_{\vartheta(j)}^{L})^{nw_{\vartheta(j)}},1-(1-F_{\vartheta(j)}^{U})^{nw_{\vartheta(j)}}$, then we can get Eq. (36).

Because $\alpha_{\vartheta(j)}$ is an INN, $\alpha_{\vartheta(j)}^{nw_{\vartheta(j)}}$ is also an INN. By Eq. (27), we know $INDWMM^{p}\left(\alpha_1,\alpha_2,...,\alpha_n\right)$ is an INN.

In the following, we shall explore some desirable properties of INDWMM operator.

Property 10 (Monotonicity). Let $\alpha_i=\left(\left[T_i^L,T_i^U\right],\left[I_i^L,I_i^U\right],\left[F_i^L,F_i^U\right]\right)$ and $\alpha_i'=\left(\left[T_i'^L,T_i'^U\right],\left[I_i'^L,I_i'^U\right],\left[F_i'^L,F_i'^U\right]\right)$ $\left(i=1,2,...,n\right)$ be two sets of INNs. If $T_i^L\geq T_i'^L,T_i^U\geq T_i'^U$, $I_i^L\leq I_i'^L,I_i^U\leq I_i'^U$, $F_i^L\leq F_i'^L,F_i^U\leq F_i'^U$ for all i , then

$$INDWMM^{p}\left(\alpha_1,\alpha_2,...,\alpha_n\right)\geq INDWMM^{p}\left(\alpha_1',\alpha_2',...,\alpha_n'\right).$$

Property 11 (Boundedness). Let $\alpha_i=(\left[T_i^L,T_i^U\right],\left[I_i^L,I_i^U\right],\left[F_i^L,F_i^U\right])$ $\left(i=1,2,...,n\right)$ be a collections of INNs, and $\alpha^{-}=\left(\min(T_i),\max(I_i),\max(F_i)\right)$, $\alpha^{+}=\left(\max(T_i),\min(I_i),\min(F_i)\right)$, Then

$$\left(\left[T_{\alpha^-}^{L},T_{\alpha^-}^{U}\right],\left[I_{\alpha^-}^{L},I_{\alpha^-}^{U}\right],\left[F_{\alpha^-}^{L},F_{\alpha^-}^{U}\right]\right)\leq INDWMM^{p}\left(\alpha_1,\alpha_2,...,\alpha_n\right)\leq\left(\left[T_{\alpha^+}^{L},T_{\alpha^+}^{U}\right],\left[I_{\alpha^+}^{L},I_{\alpha^+}^{U}\right],\left[F_{\alpha^+}^{L},F_{\alpha^+}^{U}\right]\right).$$

where,

$$T_{\alpha^-}^{L}=1-\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-\max(T_i^L)^{nw_{\vartheta(j)}})^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}},T_{\alpha^-}^{U}=1-\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-\max(T_i^U)^{nw_{\vartheta(j)}})^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}},$$

$$I^L_{\alpha^-}=\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-(1-\min(I_i^L))^{nw_{\vartheta(j)}}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}},I^U_{\alpha^-}=\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-(1-\min(I_i^U))^{nw_{\vartheta(j)}}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}},$$

$$F^L_{\alpha^-}=\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-(1-\min(F_i^L))^{nw_{\vartheta(j)}}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}},F^U_{\alpha^-}=\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-(1-\min(F_i^U))^{nw_{\vartheta(j)}}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}},$$

$$T^L_{\alpha^+}=1-\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-\min(T_i^L)^{nw_{\vartheta(j)}})^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}},T^U_{\alpha^+}=1-\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-\min(T_i^U)^{nw_{\vartheta(j)}})^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}},$$

$$I^L_{\alpha^+}=\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-(1-\max(I_i^L))^{nw_{\vartheta(j)}}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}},I^U_{\alpha^+}=\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-(1-\max(I_i^U))^{nw_{\vartheta(j)}}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}},$$

$$F^L_{\alpha^+}=\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-(1-\max(F_i^L))^{nw_{\vartheta(j)}}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}},F^U_{\alpha^+}=\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-(1-\max(F_i^U))^{nw_{\vartheta(j)}}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}.$$

Theorem 9.The INDMM operator is a special case of the INDWMM operator.

Proof.

When $w=\left(\frac{1}{n},\frac{1}{n},\cdots,\frac{1}{n}\right)$,

$$INDWMM^P(\alpha_1,\alpha_2,...,\alpha_n)=$$

$$\left(\left[1-\left(1-\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-\left(T^L_{\vartheta(j)}\right)^{nw_{\vartheta(j)}})^{p_j}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}},1-\left(1-\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-\left(T^U_{\vartheta(j)}\right)^{nw_{\vartheta(j)}})^{p_j}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}\right],\right.$$

$$\left[\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-(1-I^L_{\vartheta(j)})^{nw_{\vartheta(j)}}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}},\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-(1-I^U_{\vartheta(j)})^{nw_{\vartheta(j)}}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}\right],$$

$$\left.\left[\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-(1-F^L_{\vartheta(j)})^{nw_{\vartheta(j)}}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}},\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-(1-F^U_{\vartheta(j)})^{nw_{\vartheta(j)}}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}\right]\right)$$

$$=\left(\left[1-\left(1-\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-\left(T^L_{\vartheta(j)}\right)^{n\times\frac{1}{n}})^{p_j}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}},1-\left(1-\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-\left(T^U_{\vartheta(j)}\right)^{n\times\frac{1}{n}})^{p_j}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}\right],\right.$$

$$\left[\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-(1-I^L_{\vartheta(j)})^{n\times\frac{1}{n}}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}},\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-(1-I^U_{\vartheta(j)})^{n\times\frac{1}{n}}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}\right],$$

$$\left[\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-(1-F^{L}_{\vartheta(j)})^{n\times\frac{1}{n}}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}},\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-(1-F^{U}_{\vartheta(j)})^{n\times\frac{1}{n}}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}\right]\Bigg)$$

$$=\left(\left[1-\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-T^{L}_{\vartheta(j)})^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}},1-\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-T^{U}_{\vartheta(j)})^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}\right],\right.$$

$$\left[\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(I^{L}_{\vartheta(j)}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}},\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(I^{U}_{\vartheta(j)}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}\right],$$

$$\left.\left[\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(F^{L}_{\vartheta(j)}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}},\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(F^{U}_{\vartheta(j)}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}\right]\right)$$

$$= INDMM^{P}\left(\alpha_1,\alpha_2,...,\alpha_n\right).$$

4. The decision making approach based on the proposed operators

In this section, based on the proposed INWMM or INDWMM operators, we will develop a novel MADM method, which is described as follows.

Suppose we need evaluate m alternatives $\{A_1, A_2, \cdots, A_m\}$ with respect to n attributes $\{C_1, C_2, \cdots, C_n\}$ in a MADM problem, where, the weight vector of the attributes is $\omega = (\omega_1, \omega_2, \cdots, \omega_n)$ satisfying $\omega_j \geq 0 (j = 1, 2, \cdots, n), \sum_{j=1}^{n} \omega_j = 1$. $R = [r_{ij}]_{m\times n}$ is the given decision matrix of this decision problem, where $r_{ij} = \left(T_{ij}, I_{ij}, F_{ij}\right)$ is an INN given by the decision maker with respect to alternative A_i for attribute C_j. Then the goal is to rank the alternatives.

In the following, we will use the proposed INWMM or INDWMM operators to solve this MADM problem and the detailed decision steps are shown as follows:

Step 1: Normalizing the attribute values. In real decision, there exist two types of the attributes which are cost type and benefit type. It is necessary to convert them to the same type so as to give the right decision making. Usually we convert cost type to benefit one by the following formula (Note: The converted attribute value is still expressed by r_{ij}):

$$r_{ij} = \left(\left[F^{L}_{ij}, F^{U}_{ij}\right], \left[1-I^{L}_{ij}, 1-I^{U}_{ij}\right], \left[T^{L}_{ij}, T^{U}_{ij}\right]\right) \quad (37)$$

Step 2: Aggregating all attribute values r_{ij} $(j = 1, 2, \cdots, n)$ to the comprehensive value Z_i by INWMM or INDWMM operators shown as follows:

$$z_i = INWMM(r_{i1}, r_{i2}, \cdots, r_{im}), \quad (38)$$

$$\text{or } z_i = INDWMM(r_{i1}, r_{i2}, \cdots, r_{im}). \quad (39)$$

Step 3: Ranking z_i $(i = 1, 2, \cdots, m)$ based on the score function and accuracy function by Definition 6.

Step 4: Ranking all the alternatives. The bigger the INN z_i is, the better the alternative A_i is.

5. An illustrative example

In this section, an example for the multicriteria decision making is used to demonstrate of the application of the proposed decision making method, as well as the effectiveness of the proposed method.

Let us consider the decision making problem adapted from [32]. There is an investment company, which wants to invest a sum of money in the best option. There are four possible alternatives: (1) A_1 is a car company; (2) A_2 is a food company; (3) A_3 is a computer company; (4) A_4 is an arms company. The investment company must make a decision according to the following three criteria: (1) C_1 is the risk analysis; (2) C_2 is the growth analysis; (3) C_3 is the environmental impact analysis, where C_1 and C_2 are benefit criteria and C_3 is a cost criterion. The weight vector of the criteria is $w = (0.35, 0.40, 0.25)^T$. The four possible alternatives are evaluated with respect to the above three criteria by the form of INNs, and interval neutrosophic decision matrix D is listed in table 1. The goal is to rank alternatives.

Table 1 interval neutrosophic decision matrix D

	C_1	C_2	C_3
A_1	$([0.4,0.5],[0.2,0.3],[0.3,0.4])$	$([0.4,0.6],[0.1,0.3],[0.2,0.4])$	$([0.7,0.9],[0.2,0.3],[0.4,0.5])$
A_2	$([0.6,0.7],[0.1,0.2],[0.2,0.3])$	$([0.6,0.7],[0.1,0.2],[0.2,0.3])$	$([0.3,0.6],[0.3,0.5],[0.8,0.9])$
A_3	$([0.3,0.6],[0.2,0.3],[0.3,0.4])$	$([0.5,0.6],[0.2,0.3],[0.3,0.4])$	$([0.4,0.5],[0.2,0.4],[0.7,0.9])$
A_4	$([0.7,0.8],[0.0,0.1],[0.1,0.2])$	$([0.6,0.7],[0.1,0.2],[0.1,0.3])$	$([0.6,0.7],[0.3,0.4],[0.8,0.9])$

5.1 The decision making steps

To get the best alternative(s), the steps are shown as follows:

Step 1: Normalizing the attribute values.

Since C_1 and C_2 are benefit attributes, and C_3 is a cost criterion, we use the formulas (37) to get the standardized decision matrix, which is shown in Table 2.

Table 2 standardized decision matrix D

	C_1	C_2	C_3
A_1	$([0.4,0.5],[0.2,0.3],[0.3,0.4])$	$([0.4,0.6],[0.1,0.3],[0.2,0.4])$	$([0.4,0.5],[0.8,0.7],[0.7,0.9])$
A_2	$([0.6,0.7],[0.1,0.2],[0.2,0.3])$	$([0.6,0.7],[0.1,0.2],[0.2,0.3])$	$([0.8,0.9],[0.7,0.5],[0.3,0.6])$
A_3	$([0.3,0.6],[0.2,0.3],[0.3,0.4])$	$([0.5,0.6],[0.2,0.3],[0.3,0.4])$	$([0.7,0.9],[0.8,0.6],[0.4,0.5])$
A_4	$([0.7,0.8],[0.0,0.1],[0.1,0.2])$	$([0.6,0.7],[0.1,0.2],[0.1,0.3])$	$([0.8,0.9],[0.7,0.6],[0.6,0.7])$

Step 2: Aggregating all attribute values r_{ij} $(j = 1, 2, \cdots, n)$ to the comprehensive value Z_i by INWMM or INDWMM operators shown as follows (suppose $P = (1,1,1)$)

(1) For the INWMM operator, we have

$z_1 = ([0.393, 0.526], [0.450, 0.470], [0.441, 0.668])$, $z_2 = ([0.640, 0.737], [0.361, 0.314], [0.241, 0.420])$,
$z_3 = ([0.459, 0.660], [0.488, 0.420], [0.342, 0.442])$, $z_4 = ([0.673, 0.769], [0.341, 0.336], [0.230, 0.447])$

(2) For the INDWMM operator, we have
$z_1 = ([0.411, 0.547], [0.239, 0.383], [0.333, 0.498])$, $z_2 = ([0.686, 0.794], [0.183, 0.263], [0.224, 0.366])$,
$z_3 = ([0.534, 0.750], [0.302, 0.366], [0.323, 0.421])$, $z_4 = ([0.714, 0.820], [0.002, 0.221], [0.174, 0.335])$.

Step 3: Calculate the score function $S(z_i)(i = 1, 2, 3, 4)$ of the collective overall values $z_i (i = 1, 2, 3, 4)$.

(1) For the INWMM operator, we have

$S(z_1) = 1.439$, $S(z_2) = 2.009$, $S(z_3) = 1.714$, $S(z_4) = 2.021$

(2) For the INDWMM operator, we have

$S(z_1) = 1.753$, $S(z_2) = 2.223$, $S(z_3) = 1.936$, $S(z_4) = 2.402$.

Step 4: Ranking all the alternatives.

According to the score functions $S(z_i)(i = 1, 2, 3, 4)$, we can rank the alternatives $\{A_1, A_2, A_3, A_4\}$ shown as follows

$A_4 \succ A_2 \succ A_3 \succ A_1$

So, the best alternative is A_4.

5.2 The influence of the parameter vector P on decision making result of this example

In order to illustrate the influence of the parameter vector P on decision making of this example, we set different parameters vector P to show the ranking results of this example. The results are shown in Table 3 and Table 4.

Table 3 Ranking by utilizing the different parameter vector P of the INWMM operator

Parameter vector P	The score function $S(z_i)$	Ranking
$P = (1, 0, 0)$	$S(z_1) = 1.640, S(z_2) = 2.183$ $S(z_3) = 1.897, S(z_4) = 2.372$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$P = (1,1, 0)$	$S(z_1) = 1.542, S(z_2) = 2.078$ $S(z_3) = 1.773, S(z_4) = 2.121$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$P = (1,1,1)$	$S(z_1) = 1.439, S(z_2) = 2.009$ $S(z_3) = 1.714, S(z_4) = 2.021$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$P = (0.5, 0.5, 0.5)$	$S(z_1) = 1.439, S(z_2) = 2.009$ $S(z_3) = 1.714, S(z_4) = 2.021$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$P = (2, 0, 0)$	$S(z_1) = 1.680, S(z_2) = 2.205$ $S(z_3) = 1.932, S(z_4) = 2.395$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$P = (3, 0, 0)$	$S(z_1) = 1.713, S(z_2) = 2.225$ $S(z_3) = 1.964, S(z_4) = 2.420$	$A_4 \succ A_2 \succ A_3 \succ A_1$

Table 4 Ranking by utilizing the different parameter vector P of the INDWMM operator

Parameter vector P	The score function $S(z_i)$	Ranking
$P=(1,0,0)$	$S(z_1)=1.352, S(z_2)=2.004$ $S(z_3)=1.710, S(z_4)=1.964$	$A_2 \succ A_4 \succ A_3 \succ A_1$
$P=(1,1,0)$	$S(z_1)=1.668, S(z_2)=2.205$ $S(z_3)=1.932, S(z_4)=2.395$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$P=(1,1,1)$	$S(z_1)=1.753, S(z_2)=2.223$ $S(z_3)=1.936, S(z_4)=2.402$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$P=(0.5,0.5,0.5)$	$S(z_1)=1.753, S(z_2)=2.223$ $S(z_3)=1.936, S(z_4)=2.402$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$P=(2,0,0)$	$S(z_1)=1.247, S(z_2)=1.913$ $S(z_3)=1.634, S(z_4)=1.811$	$A_2 \succ A_4 \succ A_3 \succ A_1$
$P=(3,0,0)$	$S(z_1)=1.164, S(z_2)=1.839$ $S(z_3)=1.569, S(z_4)=1.705$	$A_2 \succ A_4 \succ A_3 \succ A_1$

As we can see from Table 3, the score functions using the different parameter vector P are different, but the ranking results are the same. From Table 4, we can know the ranking results may be different for the different parameters vector P , when the parameter vector P has only one real number and the rest are 0, that is, when the INDWMM operator reduce to the interval neutrosophic generalized geometric averaging operator, its ranking order is $A_2 \succ A_4 \succ A_3 \succ A_1$; whereas in other case, the ranking results are the same as Table 3. In other words, we consider the interrelationship of attributes, the best alternative is A_4 , otherwise is A_2 . In general, for the INWMM operator, we can find that the more interrelationships of attributes we consider, that is to say, there are fewer 0 in the parameter vector P , the smaller value of score functions will become. The parameter vector P have greater control ability, the values of score function will become greater. However, for the INDWMM operator, the result is just the opposite, the more interrelationships of attributes we consider, the greater value of score functions will become. The parameter vector P have greater control ability, the values of score function will become small. So, different decision makers can set different parameter vector P according to different risk preference.

5.3 Comparing with the other methods

To further prove the effectiveness of the developed methods in this paper, we solve the same illustrative example by two existing MADM methods including the similarity measure proposed by Ye [32], the interval neutrosophic weighted Bonferroni mean (INWBM) operator extended from the normal neutrosophic weighted Bonferroni mean (NNWBM) operator [11]. The ranking results by these methods are shown in Table 5 (for the INDWMM operator, there are the same results as the INWMM operator, and they are omitted) .

From Table 5, we can see that these methods produced the same ranking results. This shows that the new methods proposed in this paper are effective and feasible. Then, we give further analysis, when $P=(1,0,0)$, the INWMM reduces to the interval neutrosophic arithmetic weighted averaging operator. In other words, when $P=(1,0,0)$, we can think that the input arguments are independent and the interrelationship among input arguments is not considered, just as the method in [32] is based on a similarity measure. When $P=(1,1,0)$ or $P=(2,2,0)$,the INWMM operator reduces to the interval neutrosophic weighted Bonferroni mean operator, which can captures interrelationship of two arguments. So we can get that the proposed methods in this paper are generalization of some existing methods.

Table 5 Ranking results compared with similarity measure method

Aggregation operator	Parameter value	Ranking
similarity measure [32]	No	$A_4 \succ A_2 \succ A_3 \succ A_1$
INWBM [11]	$p = q = 1$	$A_4 \succ A_2 \succ A_3 \succ A_1$
INWMM in this paper	$P = (1, 0, 0)$	$A_4 \succ A_2 \succ A_3 \succ A_1$
INWMM in this paper	$P = (1,1, 0)$	$A_4 \succ A_2 \succ A_3 \succ A_1$
INWMM in this paper	$P = (2, 2, 0)$	$A_4 \succ A_2 \succ A_3 \succ A_1$
INWMM in this paper	$P = (1,1,1)$	$A_4 \succ A_2 \succ A_3 \succ A_1$
INWMM in this paper	$P = (2, 2, 2)$	$A_4 \succ A_2 \succ A_3 \succ A_1$

However, there usually exist the interrelationships among more than two attributes in real decision making, BM operator can only consider the interrelationship between any two input arguments. In order to compare the performance and advantage of the new proposed method with the above existing methods, we revise the truth-membership values of the alternative A_4 which are listed in Table 6 and the final ranking results of the alternatives are shown in Table 7.

Table 6 modified decision matrix D

	C_1	C_2	C_3
A_1	$([0.4, 0.5],[0.2, 0.3],[0.3, 0.4])$	$([0.4, 0.6],[0.1, 0.3],[0.2, 0.4])$	$([0.4, 0.5],[0.8, 0.7],[0.7, 0.9])$
A_2	$([0.6, 0.7],[0.1, 0.2],[0.2, 0.3])$	$([0.6, 0.7],[0.1, 0.2],[0.2, 0.3])$	$([0.8, 0.9],[0.7, 0.5],[0.3, 0.6])$
A_3	$([0.3, 0.6],[0.2, 0.3],[0.3, 0.4])$	$([0.5, 0.6],[0.2, 0.3],[0.3, 0.4])$	$([0.7, 0.9],[0.8, 0.6],[0.4, 0.5])$
A_4	$([0.4, 0.8],[0.0, 0.1],[0.1, 0.2])$	$([0.5, 0.7],[0.1, 0.2],[0.1, 0.3])$	$([0.8, 0.9],[0.7, 0.6],[0.6, 0.7])$

Table 7 Ranking results by different methods

Aggregation operator	Parameter value	Ranking
INWBM [11]	$p = q = 1$	$A_4 \succ A_2 \succ A_3 \succ A_1$
INWMM in this paper	$P = (1,1,1)$	$A_2 \succ A_3 \succ A_4 \succ A_1$
INWDMM in this paper	$P = (1,1,1)$	$A_2 \succ A_4 \succ A_3 \succ A_1$

From table 7, we can know when we consider interrelationship of three arguments, the ranking results are different with that produced by considering interrelationship of two arguments. In a realistic decision making environment, we need consider the interrelationship for two arguments or multiple arguments according to the actual decision need, and the proposed methods in this paper can capture interrelationship of any multiple arguments, even don't consider the interrelationship by parameter vector P .

In a word, according to the comparisons and analysis above, the proposed methods based on INWMM operator and the INDWMM operator in this paper is better and more convenient than the existing other methods in considering interrelationship of attributes.

6. Conclusion

NS is a generalization of fuzzy set, paraconsistent set, intuitionistic fuzzy set, paradoxist set etc. and the MM operator has a prominent characteristic that it can consider the interaction relationships among any multiple attributes by a parameter vector P . In this paper, we proposed some new MM aggregation operators to deal with MADM problems under the interval neutrosophic environment, include the interval neutrosophic MM (INMM) operator, the interval neutrosophic weighted MM (INWMM) operator, the interval neutrosophic dual MM (INWMM) operator and the interval neutrosophic dual weighted MM (INDWMM) operator. Then, the desirable properties and some special cases were discussed in detail. Moreover, we presented two new methods based on the proposed aggregation operators. Finally, we used an example to illustrate the feasibility and validity of the new methods by comparing with the other existing methods. In the future, we will research some applications of proposed methods in real decision making.

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