Aristotle's answer on Russell's paradox

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An attempt of resolving Russell's paradox with the help of Aristotle's ideas is presented.

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Introduction

Quintessence of Aristotle's (384-322 B.C.) [1] ideas that are about to be discussed here is outlined elsewhere: "…Empedocles and Aristotle…were performing classification of elements based on elements' qualities. … Aristotle…claims there not only a single type of primary matter exists but two pairs of the opposites, or the primary qualities: dry and humid; cold and hot. Different combinations of primary qualities form four elements or *elementa*:

FIRE = hot + dry
WATER = cold + humid
AIR = hot + humid
EARTH = cold + dry

By combining the *elementa* or, by the same token, the primary properties, one could have given a birth to a vast variety of bodies. …The dream of medieval alchemists of making gold via transmutation of *elementa* was mostly inspired by Aristotle's thesis on possibility of combining the properties."[2]

Ancient Greeks were pretending their theories explain everything in the world, including other special theories about the world. In the spirit of this boldness let us try to apply aforementioned ideas of Aristotle to a set theory, introduced by Georg Cantor in 1880 [3].

Any researcher, who may have started one day studying set theory, could have experienced mixed feelings of limitless fascination along with an extreme irritation due to the ironic fact that two key entities of set theory – *set* and *element* are not defined within the theory. The situation is best described elsewhere:

"We give no technical definition of the concept of set, for any attempt to define *set* only uses words whose meaning is synonymous with *set* itself, such as *collection* or *family*. We say that a set is composed of elements, but we will not attempt to define this term either. We will soon clarify how these words are employed.” [4] Aristotle’s ideas could have provided one with a following definition of an object*:

Object is a carrier of the unique set of observable properties	(1)

Some clarifications need to be done here:

1. Carrier-Properties inseparability

Aristotle is not describing carrier and its properties as separate entities. Apparently, Aristotle did not think that an object, even hypothetically, can be stripped off its properties without changing the object’s identity. Furthermore, vice versa, properties, according to Aristotle, do not exist without their carriers. Truly, Aristotle is not discussing the *pure* properties, such as hot, dry, cold, and humid, but the *elementa*, each of which can be perceived as a

proto -
primary -
pre - object, characterized by two properties.

2. Uniqueness

If the sets of properties of some objects are identical (*i.e.* none of sets is unique) then the objects are indistinguishable.

3. Observability

Aristotle implicitly suggests an object must be defined by properties the object possesses. Indeed, none of Aristotle’s *elementa* is defined as to be:

- **not** dry
- **not** humid
- **not** cold
- **not** hot

* From now on a term *element* (of a set) will intentionally be replaced by an *object* to avoid its possible misidentification with any of Aristotle’s *elementa*.

Pushing the idea of the existence further, one may reasonably pose a question: "How can we make sure the object-defining property does exist?" The answer on this question had already been proposed by Robert Boyle in his seminal work "The Sceptical Chymist" in 1661 [1]. According to Boyle, existence must be testable, meaning it should experimentally (at least in principle) be detectable, meaning it should be *observable*.

N.B.: It is worth noting here that Boyle's experimental-testability-of-existence approach had not only made a foundation of what is known now as Analytical Chemistry but also has been successfully used as indispensable tool in shaping and developing the very structure of modern Physics. Here are some examples:

- Einstein's conscious avoidance in discussing the properties of unobservable aether had leaded him to creation of Special Relativity.
- Heisenberg's realization of sufficiency in dealing with only observable entities had brought him to formulation of groundbreaking uncertainty principle and the matrix version of Quantum Mechanics.
- Einstein *vs.* Bohr gedanken experiments' battles had resulted in formulation of Einstein-Podolsky-Rosen (EPR) paradox which still stimulates running the experiments in the field of correlated states.

4. Self-sufficiency of object-defining procedure

Requirement of object-defining property to be an observable property inevitably makes the object-defining procedure self-sufficient. In other words, any object is defined based on the existing properties of the object itself, with no relation to properties of other objects and, ultimately, with no relation to the very existence of other objects.

5. Inherent observability of objects

As long as we are dealing with objects, we are dealing with sets of observable properties that cannot be separated from their carriers = objects (see definition 1). Thus, carriers = objects are inherently observable.
As one can notice, definition (1) contains terms *set* and *property*, therefore they also need to be defined:

Set is a multitude of objects each of which has at least one property in common	(2)

Multitude here means the opposite to *one* or *single*.

Property is an observable attribute that allows one to conduct a comparison between objects or sets	(3)

Comparison is a procedure that allows one to determine whether two objects or sets are the same or different	(4)

It is easy to see that in a set of definitions 1-4 each of the terms *object*, *set*, and *property* is defined via the others. Even so, one should not become confused and recklessly jump to conclusion the definitions are fundamentally wrong because of the semantically vicious circle they seem to form. Indeed, the most elementary entities, apparently, do not come as single characteristics, but *pairs* of complementary attributes. For instance, in pairs:

war – peace

good – bad

none of the concepts can easily be defined independently from its corresponding counterpart. Most probably, right for this reason, Aristotle had *coupled* the elementary properties of opposite qualities. However, it is still questionable whether the most elementary entities objectively come as finite sets of complementary attributes or it is just a trick performed by human brain to make such a merciful cutoff in order to help us to cope with the complexity of our environment by stopping an endless and, otherwise, debilitating process of defining what is *elementary*.

N.B.: Idea that reality is naturally and most completely defined by its mutually complementary characteristics have found its elegant realization in 20th century in *complementarity principle*, originally introduced by Bohr for Quantum Mechanics. Heisenberg's uncertainty relation is the special case of the principle.

It is easy to see that definitions 1-4 itself are based only on one single dual concept: *same – different*, and, consequently, on a comparison as an ability of experimental observation of the duality. Final remark should be made about the absence of principal difference between *objects* and *sets* as long as we use definitions 1-4. Indeed, one can construct a combined (*i.e.* definition 1 + definition 2) definition for a *set*:

Set is a multitude of carriers of the unique sets of observable properties

(5)

That is why all clarifications 1-5 are applicable to sets as well.

Russell's Paradox

Now let us test the developments of Aristotle's ideas on Russell's Paradox, devised by Bertrand Russell to reveal the intrinsic contradictions of naïve set theory. For the sake of simplicity we will consider a more comprehensible version of the Paradox, the Barber's Paradox, presented by Russell in 1918:

"In a small village there was but one barber, and on his door was a sign which read: "I will shave anyone in this town who does not shave himself." In a small print the sign added: "Offer is void for those who shave themselves." To his dismay, the barber one day discovered that he cannot fulfill his promise, for the following reason: Should he shave himself? If he does, he would violate his own promise to shave only those in town who don't shave themselves. But if he does not shave himself, then again he would violate his promise, since according to the promise he must shave himself!" [3]

As one can see, the Paradox stems from the way the barber defines a set of his clients. Let us examine whether a *"…who does not shave himself"* can be used as an object/set-defining property or it cannot be. All we need to do is just to ask ourselves: Is it possible (at least in principle) to devise an experiment to observe the property *"…who does not shave himself"*? (The answer is:) Apparently not. Thus, the property *"…who does not shave himself"* is unobservable and, therefore, is illegitimate of being used as an object/set-defining property. On the other hand, the barber could have solved the problem by introducing an observable marker/attribute/property: "I will shave anyone in this town who wears a T-shirt, labeled: "Shave me!" " In this case if barber would like to shave himself all he would have to do is just to put the T-shirt on himself.

Concluding Remarks

Aristotle's ideas lead to some curious consequences:

(i) Empty sets are unobservable

Indeed, no one can deliver an observable set-defining property to form an

empty set. This conclusion is in excellent corroboration with another ancient concept: *Horror Vacui* [2, 3]. Anyone can easily find other confirmations to the thesis from modern Physics as well:
in Quantum Mechanics: Physical (*i.e. real*) vacuum is a very dense continuum of the filled states with negative energies.
in General Relativity: No empty space exists without the matter.

(ii) Definition of a Complement Set requires revision

Truly, according to a conventional definition: "…the complement $\overline{S}$ of S is that set consisting of all elements of U not belonging to S…" [4] It is easy to notice that a construction *"…not belonging to…"* is unobservable and is not self-sufficient, meaning that objects/sets are tried to be defined based not on their own, observable properties but on the *absence* of properties that other objects/sets possess.

(iii) Origin of superposition of states in Quantum Mechanics is explained

It is well-known that any object which is under consideration of Quantum Mechanics is treated as to be in a superposition of states. The superposition makes quantum mechanics *probabilistic*, concerning predictions of outcomes of the experiments. Idea of superposition have found its vivid demonstration in a glamorous story about Schrödinger's cat, an enigmatic animal who is simultaneously dead and alive until the experimenter actualizes one of the *cat-states* by observing the cat [5].

According to aforementioned developments of Aristotle's ideas, an experimenter (including his/her tool) along with the object of study are have to be able (at least in principle) to be united in one set. In other words, experimenter and the object of study have to be mutually suitable/compatible, meaning each of them has to have at least one common property. On the other hand, the object of study has to have at least one property which differs the object from the experimenter. Thus, the object *must* have at least *two* different properties, meaning the object should be in a superposition of states. Therefore, the very paradigm of experimental setup makes the objects that are in the superposition of states to be the only observable objects.

(iv) Set of all (sub)sets is unobservable

Set of all (sub)sets, or a Power set [3], is unobservable, because an experimenter's act of observation actualizes only *one single* partition. In other words, all possible subsets cannot be observable at once for the absence of duplicates, triplicates, …, n-plicates of each of the objects in the original set.

References

1. Isaac Asimov, *A Short History of Chemistry. An introduction to the ideas and concepts of chemistry*, Anchor Books Doubleday & Company, Inc., 1965

2. The citation is taken (and translated into English by author of the notes) from Russian edition:

   Фридьеш Каройхази, *Истинное Волшебство*, Москва, Атомиздат, 1980

   of the original book (in Hungarian):

   Karolyhazy Frigyes, *Igaz Varazslat*, Gondolat, Budapest, 1976

3. Eli Maor, *To Infinity and Beyond. A Cultural History of the Infinite*, Princeton University Press, 1991; ISBN 0-691-02511-8

4. Gary Chartrand, *Introductory Graph Theory*, Dover Publications, Inc., New York. Reprint. Originally published: Graphs as mathematical models. Boston: Prindle, Weber & Schmidt, c1977

5. Bryce S. DeWitt, *Quantum Mechanics and Reality*, Physics Today, Vol. 23, No. 9 (September 1970), p.155-165