OPERATIONS ON COMPLEX MULTI-FUZZY SETS

Yousef Al-Qudah and Nasruddin Hassan [1]
School of Mathematical Sciences, Faculty of Science and Technology, Universiti Kebangsaan Malaysia, 43600 UKM Bangi Selangor DE, MALAYSIA

Abstract. In this paper, we introduce the concept of complex multi-fuzzy sets (CM^kFSs) as a generalization of the concept of multi-fuzzy sets by adding the phase term to the definition of multi-fuzzy sets. In other words, we extend the range of multi-membership function from the interval [0,1] to unit circle in the complex plane. The novelty of CM^kFSs lies in the ability of complex multi- membership functions to achieve more range of values while handling uncertainty of data that is periodic in nature. The basic operations on CM^kFSs, namely complement, union, intersection, product and Cartesian product are studied along with accompanying examples. Properties of these operations are derived. Finally, we introduce the intuitive definition of the distance measure between two complex multi-fuzzy sets which are used to define δ-equalities of complex multi-fuzzy sets.

Keywords. Complex multi-fuzzy set, multi-fuzzy sets, fuzzy set, distance measure

1. Introduction

The concept of fuzzy set (FS) was used for the first time by Zadeh [1] to handle uncertainty in many fields of everyday life. Fuzzy set theory generalised the range values of classical set theory from the integer 0 and 1 to the interval [0, 1]. In addition, this concept has proven to be very useful in many different fields [2-4].

The theory of multi-fuzzy sets [5,6] is a generalisation of Zadeh's fuzzy sets [1] and Atanassov's intuitionistic fuzzy sets [7] in terms of multi-dimensional membership functions and multi-level fuzziness.

Ramot et al. [8] presented a new innovative concept called complex fuzzy sets (CFSs). The complex fuzzy sets is characterized by a membership function whose range is not limited to $[0,1]$ but extended to the unit circle in the complex plane, where the degree of membership function μ is traded by a complex-valued function of the form $r_{\tilde{s}}(x)e^{i\omega_{\tilde{s}}(x)}$ $\left(i=\sqrt{-1}\right)$, where $r_{\tilde{s}}(x)$ and $\omega_{\tilde{s}}(x)$ are both real-valued functions and $r_{\tilde{s}}(x)e^{i\omega_{\tilde{s}}(x)}$ the range are in a complex unit circle. They also added an additional term

[1]Corresponding author. Nasruddin Hassan, School of Mathematical Sciences, Faculty of Science and Technology, Universiti Kebangsaan Malaysia 43600 UKM Bangi Selangor DE, Malaysia. Tel.: +603 89213710; E-mail: nas@ukm.edu.my

called the phase term to solve the enigma in translating some complex-valued functions on physical terms to human language and vice versa. Ramot et al. [8,9] discussed several important operations such as complement, union, and intersection and discussed fuzzy relations for such complex fuzzy sets. The complex fuzzy sets are used to represent the information with uncertainty and periodicity simultaneously.

Alkouri and Salleh [10] introduced the concept of complex intuitionistic fuzzy set (CIFS) which is generalized from the innovative concept of a complex fuzzy set . The complex fuzzy sets and complex intuitionistic fuzzy sets cannot handle imprecise, indeterminate, inconsistent, and incomplete information of periodic nature. Ali and Smarandache [11] presented the concept of complex neutrosophic set. The complex neutrosophic set can handle the redundant nature of uncertainty, incompleteness, indeterminacy and inconsistency among others.

The concept of "proximity measure" was used for the first time by Pappis [12], with an attempt to show that precise membership values should normally be of no practical significance. Hong and Hwang [13] then proposed an important generalization. Moreover, Cai [14,15] introduced and discussed δ-equalities of fuzzy sets and proposed that if two fuzzy sets are equal to an extent of δ, then they are said to be δ-equal. Zhang et al.[16] introduced the concept of δ-equalities of complex fuzzy set and applied it to signal processing.

We will extend the discussion on multi-fuzzy sets further by proposing a new concept of complex multi-fuzzy sets (CM^{k}FSs) by adding a second dimension (phase term) to multi-membership function of multi-fuzzy sets. Complete characterization of many real life problems that have a periodic nature can then be handled by complex multi-fuzzy membership functions of the objects involved in these periodic problems. Then, we define its basic operations, namely complement, union, intersection, complex multi-fuzzy product, Cartesian product, distance measure and study their properties. Finally, we introduce the δ-equalities of complex multi-fuzzy sets.

2. Preliminaries

In the current section we will briefly recall the notions of fuzzy sets, multi-fuzzy sets and complex fuzzy sets which are relevant to this paper.

First we shall recall the basic definitions of fuzzy sets. The theory of fuzzy sets, first developed by Zadeh in 1965 [1] is as follows.

DEFINITION 2.1. (see [1]) A fuzzy set $\tilde{S}$ in a universe of discourse X is characterised by a membership function $\mu_{\tilde{S}}(x)$ that takes values in the interval $[0,1]$ for all $x \in X$.

Sebastian and Ramakrishnan [6] introduced the following definition of multi-fuzzy sets.

DEFINITION 2.2. Let k be a positive integer and X be a non-empty set. A multi-fuzzy set $\tilde{S}$ in X is a set of ordered sequences $\tilde{S} = \{\langle x, \mu_1(x), ..., \mu_k(x)\rangle : x \in X\}$, where $\mu_i : X \longrightarrow L_i = [0,1]$, $i = 1,2,...,k$.

The function $\mu_{\tilde{S}}(x) = (\mu_1(x), ..., \mu_k(x))$ is called the multi-membership function of multi-fuzzy sets $\tilde{S}$, k is called a dimension of $\tilde{S}$. The set of all multi-fuzzy sets of dimension k in X is denoted by M^kFS(x).

In the following, we give some basic definitions and set theoretic operations of complex fuzzy sets.

DEFINITION 2.3. (see [8]) A complex fuzzy set (CFS) $\tilde{S}$, defined on a universe of discourse X, is characterised by a membership function $\mu_{\tilde{s}}(x)$ that assigns to any element $x \in X$ a complex-valued grade of membership in $\tilde{S}$. By definition, the values of $\mu_{\tilde{s}}(x)$ may receive all lying within the unit circle in the complex plane and are thus of the form $\mu_{\tilde{s}}(x) = r_{\tilde{s}}(x).e^{iArg_{\tilde{s}}(x)}$, where $i = \sqrt{-1}$, $r_{\tilde{s}}(x)$ is a real-valued function such that $r_{\tilde{s}}(x) \in [0,1]$ and $e^{iArg_{\tilde{s}}(x)}$ is a periodic function whose periodic law and principal period are, respectively , 2π and $0 < arg_{\tilde{s}}(x) \leq 2\pi$, i.e., $Arg_{\tilde{s}}(x) = arg_{\tilde{s}}(x) + 2k\pi$, $k = 0, \pm1, \pm2, ...$, where $arg_{\tilde{s}}(x)$ is the principal argument. The principal argument $arg_{\tilde{S}}(x)$ will be used in the following text. The CFS $\tilde{S}$ may be represented as the set of ordered pairs

$$\tilde{S} = \{(x, \mu_{\tilde{s}}(x)) : x \in X\} = \{(x, r_{\tilde{s}}(x).e^{iarg_{\tilde{s}}(x)}) : x \in X\}.$$

DEFINITION 2.4. (see [8]) Let $\tilde{A}$ and $\tilde{B}$ be two complex fuzzy sets on X and, let $\mu_{\tilde{A}}(x) = r_{\tilde{A}}(x).e^{iarg_{\tilde{A}}(x)}$ and $\mu_{\tilde{B}}(x) = r_{\tilde{B}}(x).e^{iarg_{\tilde{B}}(x)}$ be their membership functions, respectively.

(a)The complex fuzzy complement of $\tilde{A}$, denoted by $\tilde{A}^c$, is specified by a function

$$\tilde{A}^c = \{(x, r_{\tilde{A}^c}(x).e^{i\ arg_{\tilde{A}^c}(x)}) : x \in X\} = \{(x, (1 - r_{\tilde{A}}(x))\ .e^{i(2\pi - arg_{\tilde{A}^c})(x)}) : x \in X\},$$

(b) The complex fuzzy union of $\tilde{A}$ and $\tilde{B}$, denoted by $\tilde{A} \cup \tilde{B}$, is specified by a function

$$\mu_{\tilde{A}\cup\tilde{B}}(x) = r_{\tilde{A}\cup\tilde{B}}(x).e^{i\ arg_{\tilde{A}\cup\tilde{B}}(x)} = max(r_{\tilde{A}}(x), r_{\tilde{B}}(x)).e^{i\ max(arg_{\tilde{A}}(x), arg_{\tilde{B}}(x))},$$

(c) The complex fuzzy intersection of $\tilde{A}$ and $\tilde{B}$, denoted by $\tilde{A} \cap \tilde{B}$, is specified by a function

$$\mu_{\tilde{A}\cap\tilde{B}}(x) = r_{\tilde{A}\cap\tilde{B}}(x).e^{i\ arg_{\tilde{A}\cap\tilde{B}}(x)} = min(r_{\tilde{A}}(x), r_{\tilde{B}}(x)).e^{i\ min(arg_{\tilde{A}}(x), arg_{\tilde{B}}(x))},$$

(d) We say that $\tilde{A}$ is greater than $\tilde{B}$, denoted by $\tilde{A} \supseteq \tilde{B}$ or $\tilde{B} \subseteq \tilde{A}$, if for any $x \in X$,

$$r_{\tilde{A}}(x) \leq r_{\tilde{B}}(x) \ \ and \ \ arg_{\tilde{A}}(x) \leq arg_{\tilde{B}}(x).$$

DEFINITION 2.5. (see [16]) Let $\tilde{A}$ and $\tilde{B}$ be two complex fuzzy sets on X and, let $\mu_{\tilde{A}}(x) = r_{\tilde{A}}(x).e^{iarg_{\tilde{A}}(x)}$ and $\mu_B(x) = r_{\tilde{B}}(x).e^{iarg_{\tilde{B}}(x)}$ be their membership functions, respectively. The complex fuzzy product of $\tilde{A}$ and $\tilde{B}$, denoted $\tilde{A} \circ \tilde{B}$, is specified by a function

$$\mu_{\tilde{A}\circ\tilde{B}} = r_{\tilde{A}\circ\tilde{B}}(x).e^{i\,arg_{\tilde{A}\circ\tilde{B}}(x)} = (r_{\tilde{A}}(x)\cdot r_{\tilde{B}}(x).e^{i2\pi\left(\frac{arg_{\tilde{A}}(x)}{2\pi}\cdot\frac{arg_{\tilde{B}}(x)}{2\pi}\right)}.$$

DEFINITION 2.6. (see [16]) Let $\tilde{A}_n$ be N complex fuzzy sets on $\mu_{\tilde{A}}(x) = r_{\tilde{A}}(x).e^{iarg_{\tilde{A}}(x)}$ their membership functions, respectively. The complex fuzzy Cartesian product of $\tilde{A}_n$, $n = 1,\ldots,N$, denoted $\tilde{A}_1 \times \ldots \times \tilde{A}_N$, is specified by a function

$$\mu_{\tilde{A}_1\times\ldots\times\tilde{A}_N} = r_{\tilde{A}_1\times\ldots\times\tilde{A}_N}(x).e^{iarg_{\tilde{A}_1\times\ldots\times\tilde{A}_N}(x)}$$

$$= min\left(r_{\tilde{A}_1}(x_1),\ldots,r_{\tilde{A}_N}(x_N)\right).e^{i\,\min\left(arg_{\tilde{A}_1}(x_1),\ldots,arg_{\tilde{A}_N}(x_N)\right)},$$

Where $x = (x_1,\ldots,x_N) \in \underbrace{X\times\ldots\times X}_{N}$.

DEFINITION 2.7. (see [16]) Let $\tilde{A}$ and $\tilde{B}$ be two complex fuzzy sets on X and, let $\mu_{\tilde{A}}(x) = r_{\tilde{A}}(x).e^{iarg_{\tilde{A}}(x)}$ and $\mu_{\tilde{B}}(x) = r_{\tilde{B}}(x).e^{iarg_{\tilde{B}}(x)}$ be their membership functions, respectively. Then $\tilde{A}$ and $\tilde{B}$ are said to be δ-equal, denoted $\tilde{A} = (\delta)\tilde{B}$, if and only if $d(\tilde{A},\tilde{B}) \le 1-\delta$, where $0 \le \delta \le 1$.

3. Complex multi-fuzzy sets

In this section, we introduce the definition of complex multi-fuzzy sets which are a generalisation of multi-fuzzy sets [5], by extending the range of multi-membership functions from the interval $[0,1]$ to the unit circle in the complex plane.

We begin by proposing the definition of complex multi-fuzzy sets based on multi-fuzzy sets and complex fuzzy sets, and give an illustrative example of it.

DEFINITION 3.1. (see [17]) Let X be a non-empty set, N the set of natural numbers and $\{L_j : j \in N\}$ a family of complete lattices. A complex multi-fuzzy set (CMFS) $\tilde{S}$, defined on a universe of discourse X, is characterised by a multi-membership function $\mu_{\tilde{S}}(x) = \left(\mu_{\tilde{s}_j}(x)\right)_{j\in N}$, that assigns to any element $x \in X$ a complex-valued grade of membership in $\tilde{S}$. By definition, the values of $\mu_{\tilde{s}}(x)$ may receive all lying within the unit circle in the complex plane, and are thus of the form $\mu_{\tilde{s}}(x) = (r_{\tilde{s}_j}(x).e^{iArg_{\tilde{s}_j}(x)})_{j\in N}$, $(i = \sqrt{-1}), \left(r_{\tilde{s}_j}(x)\right)_{j\in N}$ are real-valued function such that $\left(r_{\tilde{s}_j}(x)\right)_{j\in N} \in [0,1]$ and $(e^{iArg_{\tilde{s}_j}(x)})_{j\in N}$ are periodic function whose periodic law and principal period are, respectively, 2π and $0 < \left(arg_{\tilde{s}_j}(x)\right)_{j\in N} \le 2\pi$, *i.e.*, $\left(Arg_{\tilde{s}_j}(x)\right)_{j\in N} = arg_{\tilde{s}_j}(x) + 2k\pi$, $K = 0,\pm1,\ldots$,where $arg_{\tilde{s}_j}(x)$ is the principal argument. The principal argument $\left(arg_{\tilde{s}_j}(x)\right)_{j\in N}$ will be used on the following text. The CMFS $\tilde{S}$ may be represented as the set of ordered sequence

$$\tilde{S} = \{(x, \left(\mu_{\tilde{s}_j}(x) = a_j\right)_{j\in N}) : x \in X\} = \{x, ((r_{\tilde{s}_j}(x).e^{iarg_{\tilde{s}_j}(x)})_{j\in N}) : x \in X\}.$$

where $\mu_{\tilde{s}_j} : X \to L_j = \{a_j : a_j \in C, |a_j| \le 1\}$ for all j.

The function $(\mu_{\tilde{S}}(x) = r_{\tilde{s}_j}(x).e^{iArg_{\tilde{s}_j}(x)})_{j\in N}$ is called the complex multi-membership function of complex multi-fuzzy set $\tilde{S}$. If $|N| = k$, then k is called the dimension of $\tilde{S}$. The set of all complex multi-fuzzy sets of dimension k in X is denoted by CM^{k}FS(X). In this paper $L_j = \{a_j : a_j \in C, |a_j| \leq 1\}$ and we will study some properties of complex multi-fuzzy sets of dimension k.

Remark 3.2.

i. A complex multi-fuzzy set is a generalization of multi-fuzzy set. Assume the multi-fuzzy set $\tilde{S}$ is characterized by the real-valued multi- membership function $(\gamma_j(x))_{j\in N}$. Transforming $\tilde{S}$ into complex multi-fuzzy sets is achieved by setting the amplitude terms $\left(r_{\tilde{s}_j}(x)\right)_{j\in N}$ to be equal to $(\gamma_j(x))_{j\in N}$ and the phase terms $\left(arg_{\tilde{s}_j}(x)\right)_{j\in N}$ to equal zero for all x. In other words, without a phase term, the complex multi-fuzzy sets effectively reduces to conventional multi-fuzzy set.This interpretation is supported by the fact that the range of $\left(r_{\tilde{s}_j}(x)\right)_{j\in N}$ is $[0,1]$, as for real-valued grade of multi- membership function.

ii. A complex multi-fuzzy set of dimension one is reduced to a complex fuzzy set, while a complex multi-fuzzy set of dimension two with $|\mu_{\tilde{s}_1}(x)| + |\mu_{\tilde{s}_2}(x)| \leq 1$ is reduced to a complex Atanassov's intuitionistic fuzzy set.

iii. If $\sum_{j=1}^{k} \left|\mu_{\tilde{s}_j}(x)\right| \leq 1$, where $\mu_{\tilde{s}_j}(x) = r_{\tilde{s}_j}(x).e^{iarg_{\tilde{s}_j}(x)}$ for all $x \in X$, then the complex multi-fuzzy set of dimension k is called a normalized complex multi-fuzzy set. If $\sum_{j=1}^{k} \left|\mu_{\tilde{s}_j}(x)\right| \leq l > 1$, for some $x \in X$, we redefine the complex multi-membership degree $\left(|\mu_{\tilde{s}_1}(x)|, \ldots, \left|\mu_{\tilde{s}_k}(x)\right|\right)$ as $\frac{1}{l}\left(|\mu_{\tilde{s}_1}(x)|, \ldots, \left|\mu_{\tilde{s}_k}(x)\right|\right)$. Hence the non-normalized complex multi-fuzzy set can be changed into a normalized multi-fuzzy set.

Example 3.3. Let $X = \{x_1, x_2, x_3\}$ be a universe of discourse. Then, $\tilde{S}$ is a complex multi-fuzzy set in $X$of dimension three, as given below:

$$\tilde{S} = \{\langle\ x_1, (0.1).e^{i(\pi)}, (0.3)e^{i(0.4\pi)}, (0.2).e^{i(1.5\pi)}\rangle, \langle\ x_2, (0.0).e^{i(0.2\pi)}, (0.3).e^{i(0.7\pi)},$$

$$(0.2).e^{i(0.5\pi)}\rangle,\ \langle x_3,\ (0.3).e^{i(0.8\pi)}, (0.2).e^{i(0.1\pi)}, (0.5).e^{i(1.2\pi)}\rangle\}.$$

Now, we present the concept of the subset and equality operations on two complex multi-fuzzy sets in the following definition.

DEFINITION 3.4. Let k be a positive integer and let $\tilde{A}$ and $\tilde{B}$ be two complex multi-fuzzy sets in a universe of discourse X of dimension k, given as follows:

$$\tilde{A} = \{\langle x, r_{\tilde{A}_1}(x).e^{iarg_{\tilde{A}_1}(x)}, ..., r_{\tilde{A}_k}(x).e^{iarg_{\tilde{A}_k}(x)}\rangle : x \in X\},$$

$$\tilde{B} = \{\langle x, r_{\tilde{B}_1}(x).e^{iarg_{\tilde{B}_1}(x)}, ..., r_{\tilde{B}_k}(x).e^{iarg_{\tilde{B}_k}(x)}\rangle : x \in X\},$$

Then

1. $\tilde{A} \subset \tilde{B}$ if and only if $r_{\tilde{A}_j}(x) \leq r_{\tilde{B}_j}(x)$ and $arg_{\tilde{A}_j}(x) \leq arg_{\tilde{B}_j}(x)$, for all $x \in X$ and $j = 1, 2, ..., k$.
2. $\tilde{A} = \tilde{B}$ if and only if $r_{\tilde{A}_j}(x) = r_{\tilde{B}_j}(x)$ and $arg_{\tilde{A}_j}(x) = arg_{\tilde{B}_j}(x)$ for all $x \in X$ and $j = 1, 2, ..., k$.

4. Basic operations on complex multi-fuzzy sets

In this section, we introduce some basic theoretic operations on complex multi-fuzzy sets, which are complement, union and intersection, derive their properties and give some examples.

DEFINITION 4.1 Let k be a positive integer and let $\tilde{A}$ be a complex multi-fuzzy set in a universe of discourse X of dimension k, given by

$$\tilde{A} = \{ \langle x, r_{\tilde{A}_1}(x).e^{iarg_{\tilde{A}_1}(x)}, ..., r_{\tilde{A}_k}(x).e^{iarg_{\tilde{A}_k}(x)} \rangle : x \in X \},$$

The complement of $\tilde{A}$ is denoted by $\tilde{A}^c$, and is defined by

$$\tilde{A}^c = \{ \langle x, r_{\tilde{A}_1^c}(x).e^{iarg_{\tilde{A}_1^c}(x)}, \dots, r_{\tilde{A}_k^c}(x).e^{iarg_{\tilde{A}_k^c}(x)} \rangle : x \in X \}$$

$$= \{ \langle x, (1 - r_{\tilde{A}_1}(x)).e^{i(2\pi - arg_{\tilde{A}_1}(x))}, \dots, (1 - r_{\tilde{A}_k}(x)).e^{i(2\pi - arg_{\tilde{A}_k}(x))} \rangle : x \in X \}.$$

Example 4.2. Consider Example 3.3.By using the basic complex multi-fuzzy complement, we have

$$\tilde{A}^c = \{ \langle x_1, (1-0.1).e^{i(2\pi-\pi)}, (1-0.3).e^{i(2\pi-0.4\pi)}, (1-0.2).e^{i(2\pi-1.5\pi)} \rangle, \langle x_2, (1-0.0) .e^{i(2\pi-0.2\pi)}, (1-0.3).e^{i(2\pi-0.7\pi)}, (1-0.2).e^{i(2\pi-0.5\pi)} \rangle, \langle x_3, (1-0.3).e^{i(2\pi-0.8\pi)}, (1-0.2).e^{i(2\pi-0.1\pi)}, (1-0.5).e^{i(2\pi-1.2\pi)} \rangle \},$$

$$= \{ \langle x_1, (0.9).e^{i(\pi)}, (0.7).e^{i(1.6\pi)}, (0.8).e^{i(0.5\pi)} \rangle, \langle x_2, (1.0).e^{i(1.8\pi)}, (0.7).e^{i(1.3\pi)}, (0.8) .e^{i(1.5\pi)} \rangle, \langle x_3, (0.7).e^{i(1.2\pi)}, (0.8).e^{i(1.9\pi)}, (0.5).e^{i(0.8\pi)} \rangle \}.$$

Proposition 4.3. Let $\tilde{A}$ be a complex multi-fuzzy set in a universe of discourse X of dimension k. Then $\left(\tilde{A}^c\right)^c = \tilde{A}$.

Proof. By Definition 4.1, we have

$$\left(\tilde{A}^c\right)^c = \{ \langle x, r_{(\tilde{A_1}^c)^c}(x).e^{iarg_{(\tilde{A_1}^c)^c}(x)}, \dots, r_{(\tilde{A_k}^c)^c}(x).e^{iarg_{(\tilde{A_k}^c)^c}(x)} \rangle : x \in X \}$$

$$= \{ \langle x, (1 - r_{\tilde{A}_1^c}(x)).e^{i(2\pi - arg_{\tilde{A}_1^c}(x))}, \dots, (1 - r_{\tilde{A}_k^c}(x)).e^{i(2\pi - arg_{\tilde{A}_k^c}(x))} \rangle : x \in X \}.$$

$$= \{ \langle x, [1 - (1 - r_{\tilde{A}_1}(x))].e^{i[2\pi - (2\pi - arg_{\tilde{A}_1}(x))]}, \dots, [1 - (1 - r_{\tilde{A}_k}(x))]. e^{i[2\pi - (2\pi - arg_{\tilde{A}_k}(x))]} \rangle : x \in X \}$$

$$= \{ \langle x, r_{\tilde{A}_1}(x).e^{iarg_{\tilde{A}_1}(x)}, \dots, r_{\tilde{A}_k}(x).e^{iarg_{\tilde{A}_k}(x)} \rangle : x \in X \} = \tilde{A}.$$

Thus $\left(\tilde{A}^c\right)^c = \tilde{A}$.

In the following, we introduce the definition of the union of two complex multi-fuzzy sets with an illustrative example.

DEFINITION 4.4 Let k be a positive integer and let $\tilde{A}$ and $\tilde{B}$ be two complex multi-fuzzy sets in a universe of discourse X of dimension k, given as follows:

$$\tilde{A} = \{ \langle x, r_{\tilde{A}_1}(x).e^{iarg_{\tilde{A}_1}(x)}, ..., r_{\tilde{A}_k}(x).e^{iarg_{\tilde{A}_k}(x)} \rangle : x \in X \},$$

$$\tilde{B} = \{ \langle x, r_{\tilde{B}_1}(x).e^{iarg_{\tilde{B}_1}(x)}, ..., r_{\tilde{B}_k}(x).e^{iarg_{\tilde{B}_k}(x)} \rangle : x \in X \},$$

The union of $\tilde{A}$ and $\tilde{B}$, denoted by $\tilde{A} \cup \tilde{B}$, is defined as

$$\tilde{A} \cup \tilde{B} = \{ \langle x, r_{\tilde{A}_1 \cup \tilde{B}_2}(x).e^{iarg_{\tilde{A}_1 \cup \tilde{B}_1}(x)}, ..., r_{\tilde{A}_k \cup \tilde{B}_k}(x).e^{i\, arg_{\tilde{A}_k \cup \tilde{B}_k}(x)} \rangle : x \in X \}$$

$$= \{ \langle x, \vee (r_{\tilde{A}_1}(x), r_{\tilde{B}_1}(x)).e^{iarg_{\tilde{A}_1 \cup \tilde{B}_1}(x)}, \ldots, \vee (r_{\tilde{A}_k}(x), r_{\tilde{B}_k}(x)).e^{iarg_{\tilde{A}_k \cup \tilde{B}_k}(x)} \rangle : x \in X \}.$$

where $\vee$ denote the max operator.

The following definition of complex multi-fuzzy union is required to calculate the phase term $(e^{iarg_{\tilde{A}_j \cup \tilde{B}_j}(x)})$ (for $j = 1, ..., k$).

DEFINITION 4.5 Let k be a positive integer and let $\tilde{A}$ and $\tilde{B}$ be two complex multi-fuzzy sets in a universe of discourse X of dimension k, with the complex-valued multi membership functions $\mu_{\tilde{A}}(x)$ and $\mu_{\tilde{B}}(x)$. The complex multi-fuzzy union of $\tilde{A}$ and $\tilde{B}$, denoted by $\tilde{A} \cup \tilde{B}$, is specified by the function

$$\tilde{s} : \{ (a_1, ..., a_j) : a_1, ..., a_j \in C, |a_1|, \ldots, |a_j| \leq 1, \forall j = 1, \ldots, k \}$$

$$\times \{ (b_1, \ldots, b_j) : b_1, \ldots, b_j \in C, |b_1|, \ldots, |b_j| \leq 1, \forall j = 1, \ldots, k \}$$

$$\rightarrow \left\{ (d_1, \ldots, d_j) : d_1, \ldots, d_j \in C, |d_1|, \ldots, |d_j| \leq 1, \forall j = 1, \ldots, k \right\},$$

where $(a_1, \ldots, a_j), (b_1, \ldots, b_j)$ and $(d_1, \ldots, d_j)$ are the complex multi-membership functions of $\tilde{A}$, $\tilde{B}$ and $\tilde{A} \cup \tilde{B}$, respectively. $\tilde{s}$ assigns a complex value, for all $x \in X$ and $j = 1, 2, \ldots, k$.

$$\tilde{s}((a_1, \ldots, a_j), \ldots, (b_1, \ldots, b_j)) = (\tilde{s}_\mu(a_1, b_1), \ldots, \tilde{s}_\mu(a_j, b_j))$$

$$= \left(\mu_{\tilde{A}_1 \cup \tilde{B}_1}(x), \ldots, \mu_{\tilde{A}_j \cup \tilde{B}_j}(x) \right)$$
$$= (d_1, \ldots, d_j)$$

The complex multi-fuzzy union functions $\tilde{s}$ must satisfy at least the following axiomatic requirements, for any $(a_1, \ldots, a_j), (b_1, \ldots, b_j), (c_1, \ldots, c_j)$ and $(d_1, \ldots, d_j) \in \{x : x \in C, |x| \leq 1\}$.

1. Axiom 1: $|\tilde{s}_\mu(a_j, 0)| = |a_j|, \forall j = 1, 2, ..., k$ (boundary condition).
2. Axiom 2: $\tilde{s}_\mu(a_j, b_j) = \tilde{s}_\mu(b_j, a_j), \forall j = 1, 2, ..., k$ (commutative condition).
3. Axiom 3: if $|b_j| \leq |d_j|$ then $|\tilde{s}_\mu(a_j, b_j)| \leq |\tilde{s}_\mu(a_j, , d_j)|, \forall j = 1, 2, ..., k$ (monotonic condition).

4. Axiom 4: $\tilde{s}_\mu\left(a_j,\tilde{s}_\mu\left(b_j,c_j\right)\right)=\tilde{s}_\mu\left(\tilde{s}_\mu\left(a_j,b_j\right),c_j\right)$, $\forall j=1,2,...,k,$ (associative condition).

   In some cases, it may be desirable that $\tilde{s}$ also satisfy the following requirements:
5. Axiom 5: $\tilde{s}$ is a continuous function (continuity).
6. Axiom 6: $\left|\tilde{s}_\mu\left(a_j,a_j\right)\right|\geq\left|a_j\right|$, $\forall j=1,2,...,k$ (superidempotency).
7. Axiom 7: if : if $\left|a_j\right|\leq\left|c_j\right|$ and $\left|b_j\right|\leq\left|d_j\right|$ then $\left|\tilde{s}_\mu\left(a_j,b_j\right)\right|\leq\left|\tilde{s}_\mu\left(c_j,d_j\right)\right|$, $\forall j=1,2,...,k$ (strict monotonicity).

The phase term for complex multi-membership functions belongs to $(0,2\pi]$. To define the phase terms $arg_{\tilde{A}_j\cup\tilde{B}_j}(x)$ (for $j=1,...,k$.), we take those forms which Ramot et al.[8] presented to calculate $arg_{\tilde{A}_j\cup\tilde{B}_j}(x)$ as follows:

1. Sum: $arg_{\tilde{A}_j\cup\tilde{B}_j}(x)=arg_{\tilde{A}_j}(x)+arg_{\tilde{B}_j}(x)$, for all $j=1,2,...,k$.
2. Max: $arg_{\tilde{A}_j\cup\tilde{B}_j}(x)=max(arg_{\tilde{A}_j}(x)\ ,arg_{\tilde{B}_j}(x))$, for all $j=1,2,...,k$.
3. Min: $arg_{\tilde{A}_j\cup\tilde{B}_j}(x)=min(arg_{\tilde{A}_j}(x),arg_{\tilde{B}_j}(x))$, for all $j=1,2,...,k$.
4. "Winner Takes All":
   $$arg_{\tilde{A}_j\cup\tilde{B}_j}(x)=\begin{cases}arg_{\tilde{A}_j}(x) & r_{\tilde{A}_j}>r_{\tilde{B}_j}\\ arg_{\tilde{B}_j}(x) & r_{\tilde{A}_j}<r_{\tilde{B}_j}\end{cases}\quad \text{, for all } j=1,2,...,k.$$

Example 4.6. Let $\tilde{A}$ and $\tilde{B}$ be two complex multi-fuzz sets in $X=\{x_1,x_2\}$ of dimension four, given by

$$\tilde{A}=\{\langle x_1\ ,(0.4).e^{i(\pi)},(0.3).e^{i(0.4\pi)},(0.2).e^{i(1.5\pi)},(0.1).e^{i(0.1\pi)}\rangle,\langle x_2,\ (0.1).e^{i(\pi)},\ (0.2).e^{i(0.7\pi)},(0.2).e^{i(0.5\pi)},(0.5).e^{i(\pi)}\rangle\},$$

$$\tilde{B}=\{\langle x_1\ ,(0.3).e^{i(0.1\pi)},(0.2).e^{i(\pi)},(0.1).e^{i(1.1\pi)},(0.3).e^{i(0.2\pi)}\rangle,\langle x_2,\ (0.5).e^{i(\pi)},\ (0.3).e^{i(0.2\pi)},(0.0).e^{i(0.8\pi)},(0.1).e^{i(2\pi)}\rangle\}.$$

Using the max union function for calculating $(r_{\tilde{A}_k\cup\tilde{B}_k}(x))$ (for $k=1,...,4$.) and the "Winner Takes All" for determining $(arg_{\tilde{A}_k\cup\tilde{B}_k}(x))$ (for $k=1,...,4$.), the following results are obtained for $\tilde{A}\cup\tilde{B}$:

$$\tilde{A}\cup\tilde{B}=\{\langle x_1\ ,(0.4).e^{i(\pi)},(0.3).e^{i(0.4\pi)},(0.2).e^{i(1.5\pi)},(0.3).e^{i(0.2\pi)}\rangle,\langle x_2,\ (0.5).e^{i(\pi)},$$

$$(0.3).e^{i(0.2\pi)},(0.2).e^{i(0.5\pi)},(0.5).e^{i(\pi)}\rangle\}.$$

We introduce the definition of the intersection of two complex multi-fuzzy sets with an illustrative example as follows.

DEFINITION 4.7 Let k be a positive integer and let $\tilde{A}$ and $\tilde{B}$ be two complex multi-fuzzy sets in a universe of discourse X of dimension k, given as follows:

$$\tilde{A}=\{\langle x,r_{\tilde{A}_1}(x).e^{iarg_{\tilde{A}_1}(x)},...,r_{\tilde{A}_k}(x).e^{iarg_{\tilde{A}_k}(x)}\rangle:x\in X\},$$

$$\tilde{B}=\{\langle x,r_{\tilde{B}_1}(x).e^{iarg_{\tilde{B}_1}(x)},...,r_{\tilde{B}_k}(x).e^{iarg_{\tilde{B}_k}(x)}\rangle:x\in X\},$$

The intersection of $\tilde{A}$ and $\tilde{B}$, denoted by $\tilde{A}\cap\tilde{B}$, is defined as

$$\tilde{A}\cap\tilde{B} = \{\langle x, r_{\tilde{A}_1\cap\tilde{B}_2}(x).e^{iarg_{\tilde{A}_1\cap\tilde{B}_1}(x)},...,r_{\tilde{A}_k\cap\tilde{B}_k}(x).e^{i\ arg_{\tilde{A}_k\cap\tilde{B}_k}(x)}\rangle : x\in X\}$$

$$= \{\langle x, \wedge(r_{\tilde{A}_1}(x), r_{\tilde{B}_1}(x)).e^{iarg_{\tilde{A}_1\cap\tilde{B}_1}(x)},\ldots, \wedge(r_{\tilde{A}_k}(x), r_{\tilde{B}_k}(x)).e^{iarg_{\tilde{A}_k\cap\tilde{B}_k}(x)}\rangle : x\in X\}.$$

where $\wedge$ denote the max operator.

The following definition of complex multi-fuzzy intersection is required to calculate the phase term $(e^{iarg_{\tilde{A}_j\cap\tilde{B}_j}(x)})$ (for $j=1,...,k$).

DEFINITION 4.8 Let k be a positive integer and let $\tilde{A}$ and $\tilde{B}$ be two complex multi-fuzzy sets in a universe of discourse X of dimension k, with the complex-valued multi membership functions $\mu_{\tilde{A}}(x)$ and $\mu_{\tilde{B}}(x)$. The complex multi-fuzzy intersection of $\tilde{A}$ and $\tilde{B}$, denoted by $\tilde{A}\cap\tilde{B}$, is specified by the function

$$\tilde{t}: \{(a_1,...,a_j): a_1,...,a_j\in C, |a_1|,\ldots,|a_j|\le 1, \forall j=1,\ldots,k\}$$

$$\times\{(b_1,\ldots,b_j): b_1,\ldots,b_j\in C, |b_1|,\ldots,|b_j|\le 1, \forall j=1,\ldots,k\}$$

$$\rightarrow \{(d_1,\ldots,d_j): d_1,\ldots,d_j\in C, |d_1|,\ldots,|d_j|\le 1, \forall j=1,\ldots,k\},$$

where $(a_1,\ldots,a_j),(b_1,\ldots,b_j)$ and $(d_1,\ldots,d_j)$ are the complex multi-membership functions of $\tilde{A}$, $\tilde{B}$ and $\tilde{A}\cap\tilde{B}$, respectively. $\tilde{t}$ assigns a complex value, for all $x\in X$ and $j=1,2,\ldots,k$.

$$\tilde{t}((a_1,\ldots,a_j),\ldots,(b_1,\ldots,b_j)) = (\tilde{t}_\mu(a_1,b_1),\ldots,\tilde{t}_\mu(a_j,b_j))$$

$$= \left(\mu_{\tilde{A}_1\cap\tilde{B}_1}(x),\ldots,\mu_{\tilde{A}_j\cap\tilde{B}_j}(x)\right)$$
$$= (d_1,\ldots,d_j)$$

The complex multi-fuzzy intersection functions $\tilde{t}$ must satisfy at least the following axiomatic requirements, for any $(a_1,\ldots,a_j),(b_1,\ldots,b_j),(c_1,\ldots,c_j)$ and $(d_1,\ldots,d_j)\in\{x : x\in C, |x|\le 1\}$.

1. Axiom 1: if $|b_j|=1, |\tilde{t}_\mu(a_j,b_j)| = |a_j|, \forall j=1,2,...,k$ (boundary condition).
2. Axiom 2: $\tilde{t}_\mu(a_j,\ b_j) = \tilde{t}_\mu(b_j,\ a_j), \forall j=1,2,...,k$ (commutative condition).
3. Axiom 3: if $|b_j|\le|d_j|$ then $|\tilde{t}_\mu(a_j,b_j)|\le|\tilde{t}_\mu(a_j,d_j)|, \forall j=1,2,...,k$ (monotonic condition).
4. Axiom 4: $\tilde{t}_\mu(a_j,\tilde{t}_\mu(b_j,c_j)) = \tilde{t}_\mu(\tilde{t}_\mu(a_j,b_j),c_j)$, fo $\forall j=1,2,...,k$, (associative condition).

   In some cases, it may be desirable that $\tilde{t}$ also satisfy the following requirements:
5. Axiom 5: $\tilde{t}$ is a continuous function (continuity).
6. Axiom 6: $|\tilde{t}_\mu(a_j,a_j)|\ge|a_j|, \forall j=1,2,...,k$ (superidempotency).
7. Axiom 7: if $|a_j|\le|c_j|$ and $|b_j|\le|d_j|$ then $|\tilde{t}_\mu(a_j,b_j)|\le|\tilde{t}_\mu(c_j,d_j)|$, for all $j=1,2,...,k$ (strict monotonicity).

We consider some forms to calculate the phase term $arg_{\tilde{A}_j\cap\tilde{B}_j}(x)$(for $j=1,...,k.$), such that the same possible choices are given to calculate the $arg_{\tilde{A}_j\cup\tilde{B}_j}(x)$.

Example 4.9. Consider Example 4.6. By using the basic complex multi-fuzzy intersection, we have

$$\tilde{A}\cap\tilde{B} = \{\langle x_1\ ,(0.3).e^{i(0.1\pi)},(0.2).e^{i(0.4\pi)},(0.1)\,e^{i(1.1\pi)},(0.1).e^{i(0.1\pi)}\rangle,\langle x_2,\ (0.1).\ e^{i(\pi)},\ (0.3).e^{i(0.2\pi)},(0.0).e^{i(0.5\pi)},(0.1).e^{i(\pi)}\rangle\}.$$

We will now give some propositions on the union, intersection,complement of complex multi fuzzy sets. These propositions illustrate the relationship between the set theoretic operations that have been mentioned above.

Proposition 4.10 Let $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ be any three complex multi-fuzzy sets on X of dimension k. Then the following properties hold.

1. $\tilde{A}\cup\tilde{A}=\tilde{A},\ A\cap\tilde{A}=\tilde{A}$.
2. $\tilde{A}\cup\tilde{B}=\tilde{B}\cup\tilde{A},\ \ \tilde{A}\cap\tilde{B}=\tilde{B}\cap\tilde{A}$.
3. $\tilde{A}\cup(\tilde{B}\cup\tilde{C})=(\tilde{A}\cup\tilde{B})\cup\tilde{C},\ \ A\cap(\tilde{B}\cap\tilde{C})=(\tilde{A}\cap\tilde{B})\cap\tilde{C}$.
4. $\tilde{A}\cup\left(\tilde{B}\cap\tilde{C}\right)=\left(\tilde{A}\cup\tilde{B}\right)\cap\left(\tilde{A}\cup\tilde{C}\right),\ \ \tilde{A}\cap\left(\tilde{B}\cup\tilde{C}\right)=\left(\tilde{A}\cap\tilde{B}\right)\cup\left(\tilde{A}\cap\tilde{C}\right).$

Proof. The proof is straightforward by using Definitions 4.4 and 4.7.

Proposition 4.11 Let A and $\tilde{B}$ be two complex multi-fuzzy sets on X of dimension k. Then the following properties hold.

1. $\left(\tilde{A}\cup\tilde{B}\right)^c=\tilde{A}^c\cap\tilde{B}^c$.
2. $\left(\tilde{A}\cap\tilde{B}\right)^c=\tilde{A}^c\cup\tilde{B}^c$.

Proof. (1) Let the union of $\tilde{A}=\{\langle\ x,r_{\tilde{A}_1}(x).e^{iarg_{\tilde{A}_1}(x)},...,r_{\tilde{A}_k}(x)\ .e^{iarg_{\tilde{A}_k}(x)}\ \rangle:x\in X\}$ and $\tilde{B}=\{\langle\ x,r_{\tilde{B}_1}(x).e^{iarg_{\tilde{B}_1}(x)},...\ ,\ r_{\tilde{B}_k}(x).e^{iarg_{\tilde{B}_k}(x)}\ \rangle:x\in X\}$, be the basic max operator. Then

$$\tilde{A}\cup\tilde{B} = \{\langle\ x,max\,(r_{\tilde{A}_1}(x),r_{\tilde{B}_1}(x)).e^{i\ max(arg_{\tilde{A}_1}(x),arg_{\tilde{B}_1}(x))},\ldots,max\,(r_{\tilde{A}_k}(x),r_{\tilde{B}_k}(x))\ .e^{i\ max(arg_{\tilde{A}_k}(x),arg_{\tilde{B}_k}(x)}\ \rangle:x\in X\},$$

$$\tilde{A}^c = \{\langle x,\,(1-r_{\tilde{A}_1}(x)).e^{i\,(2\pi-arg_{\tilde{A}_1}(x))},\ldots\ ,(1-r_{\tilde{A}_k}(x)).e^{i\,(2\pi-arg_{\tilde{A}_k}(x))}\rangle:x\in X\},$$

$$\tilde{B}^c = \{\langle x,\,(1-r_{\tilde{B}_1}(x)).e^{i\,(2\pi-arg_{\tilde{B}_1}(x))},\ldots\ ,(1-r_{\tilde{B}_k}(x)).e^{i\,(2\pi-arg_{\tilde{B}_k}(x))}\rangle:x\in X\},$$

$$(\tilde{A}\cup\tilde{B})^c = \{\langle x,(1-max\Big(r_{\tilde{A}_1}(x),r_{\tilde{B}_1}(x)\Big)).e^{i(2\pi-max(arg_{\tilde{A}_1}(x),arg_{\tilde{B}_1}(x)))},\ldots,\ (1-max\Big(r_{\tilde{A}_k}(x),r_{\tilde{B}_k}(x)\Big)).e^{i(2\pi-max(arg_{\tilde{A}_k}(x),arg_{\tilde{B}_k}(x))}\rangle:x\in X\},$$

$$\tilde{A}^c\cap\tilde{B}^c = \{\langle x,\,(min\,(1-r_{\tilde{A}_1}(x),1-r_{\tilde{B}_1}(x))).e^{i\,(min\,(2\pi-arg_{\tilde{A}_1}(x),2\pi-arg_{\tilde{B}_1}(x)))},\ldots,\ (min\,(1-r_{\tilde{A}_k}(x),1-r_{\tilde{B}_k}(x))).e^{i\,(min\,(2\pi-arg_{\tilde{A}_k}(x),2\pi-arg_{\tilde{B}_k}(x)))}\rangle:x\in X\}.$$

To proof $(\tilde{A}\cup\tilde{B})^c=\tilde{A}^c\cap\tilde{B}^c$ we need to show that the amplitude term $1-max\,(r_{\tilde{A}_j}(x),r_{\tilde{B}_j}(x))=min\,(1-r_{\tilde{A}_j}(x),1-r_{\tilde{B}_j}(x))$ and that the phase term $(2\pi-max(arg_{\tilde{A}_j}(x),arg_{\tilde{B}_j}(x))=min(2\pi-arg_{\tilde{A}_j}(x),\ 2\pi-arg_{\tilde{B}_j}(x))$, for $j=1,...,k$.

To show the amplitude term, we consider the two possible cases:

Case 1: If $r_{\tilde{A}_j}(x) \geq r_{\tilde{B}_j}(x)$, then $1 - r_{\tilde{A}_j}(x) \leq 1 - r_{\tilde{B}_j}(x)$. Thus, $1 - max\,(r_{\tilde{A}_j}(x), r_{\tilde{B}_j}(x)) = 1 - r_{\tilde{A}_j}(x)$ and $min\,(1 - r_{\tilde{A}_j}(x),\ 1 - r_{\tilde{B}_j}(x)) = 1 - r_{\tilde{A}_j}(x)$.

Case 2: If $r_{\tilde{A}_j}(x) < r_{\tilde{B}_j}(x)$, then $1 - r_{\tilde{A}_j}(x) > 1 - r_{\tilde{B}_j}(x)$. Thus, $1 - max\,(r_{\tilde{A}_j}(x), r_{\tilde{B}_j}(x)) = 1 - r_{\tilde{B}_j}(x)$ and $min\,(1 - r_{\tilde{A}_j}(x), 1 - r_{\tilde{B}_j}(x)) = 1 - r_{\tilde{B}_j}(x)$.

Therefore, by Case 1 and Case 2:

$$1 - max\,(r_{\tilde{A}_j}(x), r_{\tilde{B}_j}(x)) = min\,(1 - r_{\tilde{A}_j}(x),\ 1 - r_{\tilde{B}_j}(x)), \forall j = 1, \ldots, k.$$

To show the phase term, we consider the two possible cases:
Case 1 : If $arg_{\tilde{A}_j}(x) \geq arg_{\tilde{B}_j}(x)$, then $2\pi - arg_{\tilde{A}_j}(x) \leq 2\pi - arg_{\tilde{B}_j}(x)$. Thus, $2\pi - max\,(arg_{A_j}(x), arg_{\tilde{B}_j}(x)) = 2\pi - arg_{A_j}(x)$ and $min\,(2\pi - arg_{\tilde{A}_j}(x),\ 2\pi - arg_{\tilde{B}_j}(x)) = 2\pi - arg_{\tilde{A}_j}(x)$.

Case 2 : If $arg_{\tilde{A}_j}(x) < arg_{\tilde{B}_j}(x)$, then $2\pi - arg_{\tilde{A}_j}(x) > 2\pi - arg_{\tilde{B}_j}(x)$. Thus, $2\pi - max\,(arg_{\tilde{A}_j}(x), arg_{\tilde{B}_j}(x)) = 2\pi - arg_{\tilde{B}_j}(x)$ and $min\,(2\pi - arg_{\tilde{A}_j}(x),\ 2\pi - arg_{\tilde{B}_j}(x)) = 2\pi - arg_{\tilde{B}_j}(x)$.

Therefore, by Case 1 and Case 2:
$2\pi - max\,(arg_{\tilde{A}_j}(x), arg_{\tilde{B}_j}(x)) = min\,(2\pi - arg_{\tilde{A}_j}(x), 2\pi - arg_{\tilde{B}_j}(x)), \forall j = 1, \ldots, k.$

(2) The proof is similar to that in part (1) and therefore is omitted.

In the following, we introduce the concept of the product of two multi-fuzzy sets and the Cartesian product of multi-fuzzy sets.

DEFINITION 4.12 Let k be a positive integer and let $\tilde{A}$ and $\tilde{B}$ be two multi-fuzzy sets in a universe of discourse X of dimension k, given as follows:
$\tilde{A} = \{\langle x, \mu_{\tilde{A}_1}(x), ..., \mu_{\tilde{A}_k}(x)\rangle : x \in X\}$ and $\tilde{B} = \{\langle x, \mu_{\tilde{B}_1}(x), ..., \mu_{\tilde{B}_k}(x)\rangle : x \in X\}$. The multi-fuzzy product of $\tilde{A}$ and $\tilde{B}$, denoted by $\tilde{A} \circ \tilde{B}$, is defined as

$$\begin{aligned}\tilde{A} \circ \tilde{B} &= \{\langle x, r_{\tilde{A}_1 \circ \tilde{B}_1}(x), ..., r_{\tilde{A}_k \circ \tilde{B}_k}(x)\rangle : x \in X\} \\ &= \{\langle x, (r_{\tilde{A}_1}(x) \cdot r_{\tilde{B}_1}(x)), \ldots,\ (r_{\tilde{A}_k}(x) \cdot r_{\tilde{B}_k}(x))\rangle : x \in X\}.\end{aligned}$$

DEFINITION 4.13. Let k be a positive integer and let $\tilde{A}_1, \ldots, \tilde{A}_n$ be multi-fuzzy sets in $X_1, \ldots, X_n$ of dimension k. The Cartesian product $\tilde{A}_1 \times \ldots \times \tilde{A}_n$ is a multi-fuzzy set defined by

$$\tilde{A}_1 \times \ldots \times \tilde{A}_n = \{\langle (x_1, \ldots, x_n), \mu^1_{\tilde{A}_1 \times \ldots \times \tilde{A}_n}(x_1, \ldots, x_n), \ldots, \mu^k_{\tilde{A}_1 \times \ldots \times \tilde{A}_n}(x_1, \ldots, x_n)\rangle : (x_1, \ldots, x_n) \in X_1 \times \ldots \times X_n\}$$

Where $\mu^j_{\tilde{A}_1 \times \ldots \times \tilde{A}_n}(x_1, \ldots, x_n) = min\,[\mu^j_{\tilde{A}_1}(x_1), \ldots, \mu^j_{\tilde{A}_n}(x_n)], \forall j = 1, 2, \ldots, k$

In the following, we introduce the concept of the product of two complex multi-fuzzy sets and the Cartesian product of complex multi-fuzzy sets with an illustrative example of the product operation.

DEFINITION 4.14 Let k be a positive integer and let $\tilde{A}$ and $\tilde{B}$ be two complex multi-fuzzy sets in a universe of discourse X of dimension k, given as follows:

$$\tilde{A} = \{ \langle x, r_{\tilde{A}_1}(x) .e^{iarg_{\tilde{A}_1}(x)}, ..., r_{\tilde{A}_k}(x) .e^{iarg_{\tilde{A}_k}(x)} \rangle : x \in X \},$$

$$\tilde{B} = \{ \langle x, r_{\tilde{B}_1}(x) .e^{iarg_{\tilde{B}_1}(x)}, ..., r_{\tilde{B}_k}(x) .e^{iarg_{\tilde{B}_k}(x)} \rangle : x \in X \},$$

The complex multi-fuzzy product of $\tilde{A}$ and $\tilde{B}$, denoted by $\tilde{A} \circ \tilde{B}$, is defined as

$$\tilde{A} \circ \tilde{B} = \{ \langle x, r_{\tilde{A}_1 \circ \tilde{B}_1}(x) .e^{iarg_{\tilde{A}_1 \circ \tilde{B}_1}(x)}, ..., r_{\tilde{A}_k \circ \tilde{B}_k}(x) .e^{iarg_{\tilde{A}_k \circ \tilde{B}_k}(x)} \rangle : x \in X \},$$

$$= \{ \langle x, (r_{\tilde{A}_1}(x) \cdot r_{\tilde{B}_1}(x)).e^{i2\pi(\frac{arg_{\tilde{A}_1}(x)}{2\pi} \cdot \frac{arg_{\tilde{B}_1}(x)}{2\pi})}, \ldots, (r_{\tilde{A}_k}(x) \cdot r_{\tilde{B}_k}(x)). \\ .\, e^{i2\pi(\frac{arg_{\tilde{A}_k}(x)}{2\pi} \cdot \frac{arg_{\tilde{B}_k}(x)}{2\pi})} \rangle : x \in X \}.$$

Example 4.15. Consider Example 4.6. By using the basic complex multi-fuzzy product,

$$\tilde{A} \circ \tilde{B} = \{ \langle x_1, (0.12) .e^{i(0.05\pi)}, (0.06) .e^{i(0.2\pi)}, (0.02) .e^{i(0.83\pi)}, (0.03) .e^{i(0.01\pi)} \rangle, \langle x_2,$$

$$(0.05) .e^{i(0.1\pi)}, (0.09) .e^{i(0.35\pi)}, (0.0)\, e^{i(0.2\pi)}, (0.05)\, e^{i(\pi)} \rangle \}.$$

DEFINITION 4.16. Let k be a positive integer and let $\tilde{A}_1, \ldots, \tilde{A}_n$ be complex multi-fuzzy sets in $X_1, \ldots, X_n$ of dimension k , respectively. The Cartesian product $\tilde{A}_1 \times \ldots \times \tilde{A}_n$ is a complex multi-fuzzy set defined by

$$\tilde{A}_1 \times \ldots \times \tilde{A}_n = \{ \langle (x_1, \ldots, x_n), \mu^1_{\tilde{A}_1 \times \ldots \times \tilde{A}_n}(x_1, \ldots, x_n), \ldots, \mu^k_{\tilde{A}_1 \times \ldots \times \tilde{A}_n}(x_1, \ldots, x_n) \rangle \\ : (x_1, \ldots, x_n) \in X_1 \times \ldots \times X_n \}$$

where

$$\mu^j_{\tilde{A}_1 \times \ldots \times \tilde{A}_n}(x_1, \ldots, x_n) = min\, [\mu^j_{\tilde{A}_1}(x_1), \ldots, \mu^j_{\tilde{A}_n}(x_n) \cdot e^{imin\,[arg^j_{\tilde{A}_1}(x_1), \ldots, arg^j_{\tilde{A}_n}(x_n)]}], \\ \forall j = 1, 2, \ldots, k.$$

5. Distance measure and δ-equalities of complex multi-fuzz sets

In this section we give the structure of distance measure on complex multi-fuzzy sets by extending the structure of distance measure of complex fuzzy sets [16].

We will first introduce the axiom definition of distance measure between complex multi-fuzzy sets and give an illustrative example.

DEFINITION 5.1. The distance between complex multi-fuzzy sets is a function $\hat{d} : \mathrm{CM}^k\mathrm{FS(X)} \times \mathrm{CM}^k\mathrm{FS(X)} \to [0,1]$ for any $\tilde{A}, \tilde{B}, \tilde{C} \in CM^kFS(X)$, satisfying the following properties:

1. $0 \leq \hat{d}\left(\tilde{A},\tilde{B}\right) \leq 1$,
2. $\hat{d}\left(\tilde{A},\tilde{B}\right) = 0$ if and only if $\tilde{A} = \tilde{B}$,
3. $\hat{d}\left(\tilde{A},\tilde{B}\right) = \hat{d}\left(\tilde{B},\tilde{A}\right)$,
4. $\hat{d}\left(\tilde{A},\tilde{B}\right) \leq \hat{d}\left(\tilde{A},\tilde{C}\right) + \hat{d}\left(\tilde{C},\tilde{B}\right)$.

We define

$$\hat{d}\left(\tilde{A},\tilde{B}\right) = max\begin{pmatrix} max\ (sup_{x\in X}\ |r_{\tilde{A}_1}(x) - r_{\tilde{B}_1}(x)\ |,\dots,sup_{x\in X}\ |r_{\tilde{A}_k}(x) - r_{\tilde{B}_k}(x)\ |), \\ max\begin{pmatrix} \frac{1}{2\pi} sup_{x\in X}\ |arg_{\tilde{A}_1}(x) - arg_{\tilde{B}_1}(x)\ |,\dots, \\ \frac{1}{2\pi} sup_{x\in X}\ |arg_{\tilde{A}_k}(x) - arg_{\tilde{B}_k}(x)\ | \end{pmatrix} \end{pmatrix}$$

Example 5.2. Let $\tilde{A}$ and $\tilde{B}$ be two complex multi-fuzzy sets in $X = \{x_1, x_2\}$ of dimension four, given by

$$\tilde{A} = \{\langle x_1\ , (0.2)\,.e^{i(\pi)}, (0.3)\,.e^{i(0.4\pi)}, (0.0)\,.e^{i(0.1\pi)}, (0.3)\,.e^{i(2\pi)}\rangle, \langle x_2,\ (0.2)\,.e^{i(2\pi)}, (0.1)\,.e^{i(0.5\pi)}, (0.3)\,.e^{i(0.2\pi)}, (0.4)\,.e^{i(0.2\pi)}\rangle\},$$

$$\tilde{B} = \{\langle x_1\ , (0.6)\,.e^{i(0.1\pi)}, (0.0)\,.e^{i(\pi)}, (0.3)\,.e^{i(1.1\pi)}, (0.1)\,.e^{i(0.7\pi)}\rangle, \langle x_2,\ (0.5)\,.e^{i(\pi)}, (0.3)\,.e^{i(0.8\pi)}, (0.1)\,.e^{i(0.6\pi)}, (0.2)\,.e^{i(2\pi)}\rangle\}.$$

we see

$sup_{x\in X}\ |r_{\tilde{A}_1}(x) - r_{\tilde{B}_1}(x)\ | = (|-0.4|,|-0.3|) = 0.4$,
$sup_{x\in X}\ |r_{\tilde{A}_2}(x) - r_{\tilde{B}_2}(x)\ | = 0.3$,
$sup_{x\in X}\ |r_{\tilde{A}_3}(x) - r_{\tilde{B}_3}(x)\ | = 0.3$,
$sup_{x\in X}\ |r_{\tilde{A}_4}(x) - r_{\tilde{B}_4}(x)\ | = 0.2$

and

$\frac{1}{2\pi} sup_{x\in X}\ |arg_{\tilde{A}_1}(x) - arg_{\tilde{B}_1}(x)\ | = \frac{1}{2\pi}(|0.9\pi|,|\pi|) = \frac{1}{2\pi}(\pi) = 0.5$,
$\frac{1}{2\pi} sup_{x\in X}\ |arg_{\tilde{A}_1}(x) - arg_{\tilde{B}_1}(x)\ | = 0.5$,
$\frac{1}{2\pi} sup_{x\in X}\ |arg_{\tilde{A}_1}(x) - arg_{\tilde{B}_1}(x)\ | = 1.8$,
$\frac{1}{2\pi} sup_{x\in X}\ |arg_{\tilde{A}_1}(x) - arg_{\tilde{B}_1}(x)\ | = 0.9$

Therefore

$\hat{d}(\tilde{A},\tilde{B}) = max[max(0.4, 0.3, 0.3, 0.2), max(0.5, 0.3, 0.5, 0.9)] = max[0.4.0.9] = 0.9.$

Now, we propose the definition of δ-Equalities of complex multi-fuzzy sets.

DEFINITION 5.3. Let k be a positive integer and let $\tilde{A}$ and $\tilde{B}$ be two complex multi-fuzzy sets in a universe of discourse X of dimension k, given as follows:

$$\tilde{A} = \{\langle x, r_{\tilde{A}_1}(x)\,.e^{iarg_{\tilde{A}_1}(x)},\dots, r_{\tilde{A}_k}(x)\,.e^{iarg_{\tilde{A}_k}(x)}\rangle : x \in X\},$$

$$\tilde{B} = \{\langle x, r_{\tilde{B}_1}(x)\,.e^{iarg_{\tilde{B}_1}(x)},\dots, r_{\tilde{B}_k}(x)\,.e^{iarg_{\tilde{B}_k}(x)}\rangle : x \in X\},$$

Then $\tilde{A}$ and $\tilde{B}$ are said to be δ-equal, denoted by $\tilde{A} = (\delta)\,\tilde{B}$, if and only if $\hat{d}\left(\tilde{A},\tilde{B}\right) \leq 1-\delta$, where $0 \leq \delta \leq 1$.

Given the definition above, we can easily obtain the following proposition.

Proposition 5.4 Let $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ be any three complex multi-fuzzy sets in X of dimension k. Then

1. $\tilde{A} = (0)\tilde{B}$,
2. $\tilde{A} = (1)\tilde{B}$ if and only if $\tilde{A} = \tilde{B}$,
3. if $\tilde{A} = (\delta)\tilde{B}$ if and only if $\tilde{B} = (\delta)\tilde{A}$,
4. $\tilde{A} = (\delta_1)\tilde{B}$ and $\delta_2 \leq \delta_1$, then $\tilde{A} = (\delta_2)\tilde{B}$,
5. if for all $\alpha \in I$, $\tilde{A} = (\delta_\alpha)\tilde{B}$, where I is an index set, then $\tilde{A} = (sup_{\alpha\in I}\delta_\alpha)\tilde{B}$,
6. for all $\tilde{A}$ and $\tilde{B}$, there exists a unique δ such that $\tilde{A} = (\delta)\tilde{B}$ and if $\tilde{A} = (\delta')\tilde{B}$ then $\delta' \leq \delta$,
7. if $\tilde{A} = (\delta_1)\tilde{B}$ and $\tilde{A} = (\delta_2)\tilde{B}$, then $\tilde{A} = (\delta)\tilde{C}$, where $\delta = \delta_1 * \delta_2$.

Proof. Properties 1- 4 can be easily proven using Definitions 5.1 and 5.3. Here we only prove properties 5, 6 and 7.
5. Since for all $\alpha \in I$, $\tilde{A} = (\delta_\alpha)$, we have

$$\hat{d}\left(\tilde{A},\tilde{B}\right) = max \begin{pmatrix} max\ (sup_{x\in X}\ |r_{\tilde{A}_1}(x) - r_{\tilde{B}_1}(x)\ |, \ldots, sup_{x\in X}\ |r_{\tilde{A}_k}(x) - r_{\tilde{B}_k}(x)\ |), \\ max \begin{pmatrix} \frac{1}{2\pi} sup_{x\in X}\ |arg_{\tilde{A}_1}(x) - arg_{\tilde{B}_1}(x)\ |, \ldots, \\ \frac{1}{2\pi} sup_{x\in X}\ |arg_{\tilde{A}_k}(x) - arg_{\tilde{B}_k}(x)\ | \end{pmatrix} \end{pmatrix}$$

$$\leq 1 - \delta_\alpha.$$

Therefore $sup_{x\in X}\ |r_{\tilde{A}_j}(x) - r_{\tilde{B}_j}(x)\ | \leq 1 - sup_{\alpha\in I}\delta_\alpha,\ \forall j = 1,2,\ldots,k.$
and $\frac{1}{2\pi} sup_{x\in X}\ |arg_{\tilde{A}_j}(x) - arg_{\tilde{B}_j}(x)\ | \leq 1 - sup_{\alpha\in I}\delta_\alpha,\ \forall j = 1,2,\ldots,k.$
Thus

$$\hat{d}\left(\tilde{A},\tilde{B}\right) = max \begin{pmatrix} max\ (sup_{x\in X}\ |r_{\tilde{A}_1}(x) - r_{\tilde{B}_1}(x)\ |, \ldots, sup_{x\in X}\ |r_{\tilde{A}_k}(x) - r_{\tilde{B}_k}(x)\ |), \\ max \begin{pmatrix} \frac{1}{2\pi} sup_{x\in X}\ |arg_{\tilde{A}_1}(x) - arg_{\tilde{B}_1}(x)\ |, \ldots, \\ \frac{1}{2\pi} sup_{x\in X}\ |arg_{\tilde{A}_k}(x) - arg_{\tilde{B}_k}(x)\ | \end{pmatrix} \end{pmatrix}$$

$$\leq 1 - sup_{\alpha\in I}\delta_\alpha.$$

Hence $\tilde{A} = (sup_{\alpha\in I}\delta_\alpha)\tilde{B}$.

6. Let $\delta = 1 - \hat{d}\left(\tilde{A},\tilde{B}\right)$. Then $\tilde{A} = (\delta)\tilde{B}$. If $\tilde{A} = (\delta')\tilde{B}$, we have $1 - \delta = \hat{d}\left(\tilde{A},\tilde{B}\right) \leq 1 - \delta'$. Consequently $\delta' \leq \delta$. Now assume there exist two constants δ_1 and δ_2 which simultaneously satisfy the required properties, then $\delta_1 \leq \delta_2$ and $\delta_2 \leq \delta_1$. This implies $\delta_1 = \delta_2$. Hence the desired δ is unique.

7. Since $\tilde{A} = (\delta_1)\tilde{B}$, we have

$$\hat{d}\left(\tilde{A},\tilde{B}\right) = max \begin{pmatrix} max\ (sup_{x\in X}\ |r_{\tilde{A}_1}(x) - r_{\tilde{B}_1}(x)\ |, \ldots, sup_{x\in X}\ |r_{\tilde{A}_k}(x) - r_{\tilde{B}_k}(x)\ |), \\ max \begin{pmatrix} \frac{1}{2\pi} sup_{x\in X}\ |arg_{\tilde{A}_1}(x) - arg_{\tilde{B}_1}(x)\ |, \ldots, \\ \frac{1}{2\pi} sup_{x\in X}\ |arg_{\tilde{A}_k}(x) - arg_{\tilde{B}_k}(x)\ | \end{pmatrix} \end{pmatrix}$$

$$\leq 1 - \delta_1,$$

which implies
$sup_{x\in X}\ |r_{\tilde{A}_j}(x)-r_{\tilde{B}_j}(x)\ | \leq 1-\delta_1$ and $\frac{1}{2\pi}sup_{x\in X}\ |arg_{\tilde{A}_j}(x)-arg_{\tilde{B}_j}(x)\ | \leq 1-\delta_1, \forall j = 1,2,\ldots,k$.

Also we have $\tilde{B}=(\delta_1)\tilde{C}$, thus

$$\hat{d}\left(\tilde{B},\tilde{C}\right)=max\begin{pmatrix} max\ (sup_{x\in X}\ |r_{\tilde{B}_1}(x)-r_{\tilde{C}_1}(x)\ |,\ldots,sup_{x\in X}\ |r_{\tilde{B}_k}(x)-r_{\tilde{C}_k}(x)\ |), \\ max\begin{pmatrix} \frac{1}{2\pi}sup_{x\in X}\ |arg_{\tilde{B}_1}(x)-arg_{\tilde{C}_1}(x)\ |,\ldots, \\ \frac{1}{2\pi}sup_{x\in X}\ |arg_{\tilde{B}_k}(x)-arg_{\tilde{C}_k}(x)\ | \end{pmatrix}\end{pmatrix}$$

$$\leq 1-\delta_2,$$

which implies
$sup_{x\in X}\ |r_{\tilde{B}_j}(x)-r_{\tilde{C}_j}(x)\ | \leq 1-\delta_2$ and $\frac{1}{2\pi}sup_{x\in X}\ |arg_{\tilde{B}_j}(x)-arg_{\tilde{C}_j}(x)\ | \leq 1-\delta_2, \forall j = 1,2,\ldots,k$.
Now, we have

$$\hat{d}\left(\tilde{A},\tilde{C}\right)=max\begin{pmatrix} max\ (sup_{x\in X}\ |r_{\tilde{A}_1}(x)-r_{\tilde{C}_1}(x)\ |,\ldots,sup_{x\in X}\ |r_{\tilde{A}_k}(x)-r_{\tilde{C}_k}(x)\ |), \\ max\begin{pmatrix} \frac{1}{2\pi}sup_{x\in X}\ |arg_{\tilde{A}_1}(x)-arg_{\tilde{C}_1}(x)\ |,\ldots, \\ \frac{1}{2\pi}sup_{x\in X}\ |arg_{\tilde{A}_k}(x)-arg_{\tilde{C}_k}(x)\ | \end{pmatrix}\end{pmatrix}$$

$$\leq max\begin{pmatrix} max\begin{pmatrix} (sup_{x\in X}\ |r_{\tilde{A}_1}(x)-r_{\tilde{B}_1}(x)\ |+sup_{x\in X}\ |r_{\tilde{B}_1}(x)-r_{\tilde{C}_1}(x)\ |),\ldots, \\ (sup_{x\in X}\ |r_{\tilde{A}_k}(x)-r_{\tilde{B}_k}(x)\ |+sup_{x\in X}\ |r_{\tilde{B}_k}(x)-r_{\tilde{C}_k}(x)\ |) \end{pmatrix}, \\ max\begin{pmatrix} \begin{pmatrix} \frac{1}{2\pi}sup_{x\in X}\ |arg_{\tilde{A}_1}(x)-arg_{\tilde{B}_1}(x)\ |+ \\ \frac{1}{2\pi}sup_{x\in X}\ |arg_{\tilde{B}_1}(x)-arg_{C_1}(x)\ | \end{pmatrix}, \\ \ldots,\begin{pmatrix} \frac{1}{2\pi}sup_{x\in X}\ |arg_{\tilde{A}_k}(x)-arg_{\tilde{B}_k}(x)\ |+ \\ \frac{1}{2\pi}sup_{x\in X}\ |arg_{\tilde{B}_k}(x)-arg_{\tilde{C}_k}(x)\ | \end{pmatrix}\end{pmatrix}\end{pmatrix}$$

$$\leq max\begin{pmatrix} max\ (((1-\delta_1)+(1-\delta_2)),\ldots,(\ (1-\delta_1)+(1-\delta_2))), \\ max(((1-\delta_1)+(1-\delta_2)),\ldots,(\ (1-\delta_1)+(1-\delta_2))) \end{pmatrix}$$

$$\leq max((1-\delta_1)+(1-\delta_2),(1-\delta_1)+(1-\delta_2))=(1-\delta_1)+(1-\delta_2) \\ =1-(\delta_1+\delta_2-1)$$

From definition 5.1 , since $\hat{d}\left(\tilde{A},\tilde{C}\right)\leq 1$. Therefore $d\left(\tilde{A},\tilde{C}\right)\leq 1-\delta_1 * \delta_2 = 1-\delta$, which implies $\tilde{A}=(\delta)\tilde{C}$.

Now, we give a theorem of the complement of δ-Equalities of complex multi-fuzzy sets .

THEOREM 5.6. If $\tilde{A}=(\delta)\tilde{B}$, then $\tilde{A}^c=(\delta)\tilde{B}^c$, where $\tilde{A}^c$ and $\tilde{B}^c$ are the complement of the complex multi-fuzzy sets $\tilde{A}$ and $\tilde{B}$.

Proof. Since

$$\hat{d}\left(\tilde{A}^c,\tilde{B}^c\right)=max\begin{pmatrix} max\left(sup_{x\in X}\ |r_{\tilde{A}_1^c}(x)-r_{\tilde{B}_1^c}(x)\ |,\ldots,sup_{x\in X}\ |r_{\tilde{A}_k^c}(x)-r_{\tilde{B}_k^c}(x)\ |\right), \\ max\begin{pmatrix} \frac{1}{2\pi}sup_{x\in X}\ |arg_{\tilde{A}_1^c}(x)-arg_{\tilde{B}_1^c}(x)\ |,\ldots, \\ \frac{1}{2\pi}sup_{x\in X}\ |arg_{\tilde{A}_k^c}(x)-arg_{\tilde{B}_k^c}(x)\ | \end{pmatrix}\end{pmatrix}$$

$$= max \begin{pmatrix} max \begin{pmatrix} sup_{x\in X} \mid (1-r_{\tilde{A}_1}(x)) - (1-r_{\tilde{B}_1}(x)) \mid, \ldots, \\ sup_{x\in X} \mid (1-r_{\tilde{A}_k}(x)) - (1-r_{\tilde{B}_k}(x)) \mid \end{pmatrix}, \\ max \begin{pmatrix} \frac{1}{2\pi} sup_{x\in X} \mid (2\pi - arg_{\tilde{A}_1}(x)) - (2\pi - arg_{\tilde{B}_1}(x)) \mid, \ldots, \\ \frac{1}{2\pi} sup_{x\in X} \mid (2\pi - arg_{\tilde{A}_k}(x)) - (2\pi - arg)_{\tilde{B}_k}(x) \mid \end{pmatrix} \end{pmatrix}$$

$$= max \begin{pmatrix} max \left(sup_{x\in X} \, |r_{\tilde{A}_1}(x) - r_{\tilde{B}_1}(x) \,|, \ldots, \, sup_{x\in X} \, |r_{\tilde{A}_k}(x) - r_{\tilde{B}_k}(x) \,| \right), \\ max \begin{pmatrix} \frac{1}{2\pi} sup_{x\in X} \, |arg_{\tilde{A}_1}(x) - arg_{\tilde{B}_1}(x) \,|, \ldots, \\ \frac{1}{2\pi} sup_{x\in X} \, |arg_{\tilde{A}_k}(x) - arg_{\tilde{B}_k}(x) \,| \end{pmatrix} \end{pmatrix}$$

$$= \hat{d}\left(\tilde{A}, \tilde{B}\right) \le 1 - \delta$$

Hence $\hat{d}\left(\tilde{A}^c, \tilde{B}^c\right) \le 1 - \delta$, thus $\tilde{A}^c = (\delta)\,\tilde{B}^c$.

6. Conclusion

Complex multi-fuzzy set theory is an extension of complex fuzzy set and complex Atanassov intuitionistic fuzzy set theory. In this paper, we have introduced the novel concept of complex multi-fuzzy set and studied the basic theoretic operations of this new concept which are complement, union and intersection on complex multi-fuzzy sets. We also presented some properties of these basic theoretic operations and other relevant laws pertaining to the concept of complex multi-fuzzy sets. Finally, we present the distance measure between two complex multi-fuzzy sets. This distance measure is used to define δ-equalities of complex multi-fuzzy sets.

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