An exact GR-solution for the relativistic rotator

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Abstract
A relativistic rotator is a pair of black-holes moving around their center-of mass (com) on GR-orbits in their own gravitational field. First we present a GR-solution in the com-frame for non-rotating (Schwarzschild) black-holes in the usual spherical coordinates *(t,r,θ,φ)* using a complex transformation of the radius *r* . with the condition that for *r*→∞ the resulting orbit equations must be the Newtonian equations. We analyze the solution and show examples of orbits. In a second step we generalize it to the case of rotating (Kerr) black-holes

1. The basics: Schwarzschild and Kerr spacetime and the Newtonian and GR energy equation

We start with exact solutions of Einstein equations in spherical coordinates for the non-rotating (Schwarzschild) and rotating (Kerr) black-hole.
The Kerr line element reads [3]

$$ds^2 = \left(1-\frac{rr_s}{r^2+\alpha^2\cos^2\theta}\right)(dt)^2 + \left(\frac{2rr_s\alpha\sin^2\theta}{r^2+\alpha^2\cos^2\theta}\right)dt\,d\varphi$$
$$-\left(\frac{r^2+\alpha^2\cos^2\theta}{r^2-rr_s+\alpha^2}\right)dr^2 -$$
$$\left(r^2+\alpha^2+\frac{rr_s\alpha^2\sin^2\theta}{r^2+\alpha^2\cos^2\theta}\right)\sin^2\theta\,d\varphi^2 - \left(r^2+\alpha^2\cos^2\theta\right)\left(d\theta^2\right) \tag{1}$$

where $r_s = \frac{2GM}{c^2}$ is the Schwarzschild radius, and $\alpha = \frac{J}{Mc}$ is the angular momentum radius (amr) , α has the dimension of a distance: $[\alpha]=[r]$, and *J* is the angular momentum.
In the limit α→0 the Kerr line element becomes the standard Schwarzschild line element

$$ds^2 = \left(1-\frac{r_s}{r}\right)c^2dt^2 - \frac{dr^2}{\left(1-\frac{r_s}{r}\right)} - r^2\left(d\theta^2+\sin^2\theta\,d\varphi\right) \tag{2}$$

The total energy for a mass *m* in Newtonian gravitation field of a mass *M* is:

$$\frac{m\dot{r}^2}{2}+\frac{m\dot{\varphi}^2r^2}{2}-\frac{GmM}{r} = E_t = \varepsilon_t mc^2 \tag{3}$$

where E_t is the *total energy* and ε_t the *relative total energy*. We use in the following the terminology of [2] for the GR energy and radial orbit equation: $\varepsilon_t = \frac{F^2-1}{2}$, where $F^2 = 2\varepsilon_t+1$ is the (dimensionless) *relativistic velocity factor*.

Because of conservation of angular momentum *L* is $l = \frac{L}{m} = \dot{\varphi}\,r^2 = const$, *l* = reduced angular momentum is a constant. Using this relation, (3) becomes the Newtonian orbit differential equation for the orbit radius *r* , with the parameters *l* and ε_t to be determined from the initial condition.
From the first (time t) Schwarzschild orbit equation (see below) we get
$\dot{t}\left(1-\frac{1}{r}\right) = const = F$ [2] , where *F* is the above *relativistic velocity factor* .
In the general relativistic Schwarzschild case the Newtonian approximation (3) becomes the exact relativistic energy equation [2] :

$$\frac{m\dot{r}^2}{2}+\frac{ml^2}{2r^2}\left(1-\frac{2GM}{c^2r}\right)-\frac{GmM}{r} = \varepsilon_t = \frac{F^2-1}{2} \tag{4}$$

We consider now the case of the Newtonian gravitational rotator (Ngr): two point masses m_1 and m_2 with

$m_{1} \geq m_2$ rotating around their center-of mass (com), in the com reference-frame with orbit radii r_1 and r_2 resp. Because of the com-condition $m_2 r_2 = m_1 r_1$ r_1 and r_2 can be calculated from the distance r_0 between m_1 and m_2

$r_1 = \frac{m_2}{m} r_0 = \frac{m_r}{m_1} r_0$ and $r_2 = \frac{m_1}{m} r_0 = \frac{m_r}{m_2} r_0$ where $m_r = \frac{m_1 m_2}{m_1 + m_2}$ is the *reduced mass* , and $m = m_1 + m_2$ is the total mass .

The Newtonian energy equation for the both orbit radii r_1 and r_2 read :

$$\frac{m_1 \dot{r}_1^{\,2}}{2} + \frac{m_1 \dot{\varphi}^2 r_1^{\,2}}{2} - \frac{G m_1 m_2}{r_0} = \varepsilon_{t1} m c^2 \tag{5a}$$

$$\frac{m_2 \dot{r}_2^{\,2}}{2} + \frac{m_2 \dot{\varphi}^2 r_2^{\,2}}{2} - \frac{G m_1 m_2}{r_0} = \varepsilon_{t2} m c^2 \tag{5b}$$

From the energy balance $E_{kin1} + E_{kin2} - \frac{G m_1 m_2}{r_0} = \varepsilon_t m_r c^2$

we reduce (5ab) to the well-known rotator equation with mass m_r

$$m_r \frac{\dot{r}_0^{\,2}}{2} + m_r \frac{\dot{\varphi}^2 r_0^{\,2}}{2} - \frac{r_s}{2 r_0} = \varepsilon_t m_r c^2 \quad \text{with } r_s = \frac{2Gm}{c^2} \tag{6}$$

The basic orbit angular frequency for a circular orbit results from the force equilibrium condition

$m_r \omega_0^{\,2} r_0 = \frac{G m_1 m_2}{r_0^{\,2}}$, so $\omega_0^{\,2} = \frac{Gm}{r_0^{\,3}}$

We introduce the dimensionless distance $\bar{r}_0 = \frac{r_0}{r_s}$, and dimensionless $\bar{l} = \frac{\dot{\varphi} r^2}{c r_s}$, and get from (6)

$$\frac{\dot{\bar{r}}_0^{\,2}}{2} + \frac{\bar{l}^2}{2\bar{r}_0^{\,2}} - \frac{1}{2\bar{r}_0} = \varepsilon_t \tag{7}$$

We generalize this to the GR Schwarzschild energy equation

$$\frac{\dot{\bar{r}}_0^{\,2}}{2} + \frac{\bar{l}^2}{2\bar{r}_0^{\,2}}\left(1 - \frac{1}{\bar{r}_0}\right) - \frac{1}{2\bar{r}_0} = \varepsilon_t \tag{8}$$

(7) is the Newtonian approximation of (8), valid for the dimensionless distance $\bar{r}_0 = \frac{r_0}{r_s}$ and with the parameters $\bar{l}$ and $\varepsilon_t \leq 0$.

Accordingly result the dimensionless $\bar{\omega}_0 = \sqrt{\frac{1}{2\bar{r}_0^{\,3}}}$ and $\bar{l} = \bar{\omega}_0 \bar{r}_0^{\,2} = \sqrt{\frac{\bar{r}_0}{2}}$

We calculate the minimal and maximal radius $\{r_{p1}, r_{p2}\}$ of the (in general) elliptical Newtonian orbit by setting $\dot{\bar{r}} = 0$

$r_{p1} = \frac{2\bar{l}^2}{1 + \sqrt{1 + 8\bar{l}^2 \varepsilon_t}}$, $r_{p2} = \frac{2\bar{l}^2}{1 - \sqrt{1 + 8\bar{l}^2 \varepsilon_t}}$, r_0 is the harmonic mean of r_{p1} and r_{p2} : $\frac{1}{r_0} = \frac{1}{r_{p1}} + \frac{1}{r_{p2}}$

For the circle orbit: $r_{p1} = r_{p2} = \bar{r}_0 = 2\bar{l}^2$

In the following, we will drop the bar in $\bar{r}_0$ and work exclusively with dimensionless coordinates r in units r_s , also we set c=1.

2. The orbit equations in Kerr-spacetime

The Einstein field equations are [2,4,5]:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R_0 - \Lambda g_{\mu\nu} = -\kappa T_{\mu\nu} \tag{9}$$

where $R_{\mu\nu}$ is the Ricci tensor, R_0 the Ricci curvature, $\kappa = \frac{8\pi G}{c^4}$, $T_{\mu\nu}$ is the energy-momentum tensor, Λ is the cosmological constant (in the following neglected, i.e. set 0), with the Christoffel symbols (second kind)

$$\Gamma^{\lambda}{}_{\mu\nu} = \frac{1}{2} g^{\lambda\kappa} \left(\frac{\partial g_{\kappa\mu}}{\partial x^{\nu}} + \frac{\partial g_{\kappa\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\kappa}} \right) \tag{10}$$

and the Ricci tensor

$$R_{\mu\nu} = \frac{\partial \Gamma^{\rho}_{\mu\rho}}{\partial x^{\nu}} - \frac{\partial \Gamma^{\rho}_{\mu\nu}}{\partial x^{\rho}} + \Gamma^{\sigma}_{\mu\rho} \Gamma^{\rho}_{\sigma\nu} - \Gamma^{\sigma}_{\mu\nu} \Gamma^{\rho}_{\sigma\rho} \tag{11}$$

The orbit equations O1…O4 in vacuum ($T_{\mu\nu} = 0$) are:

$$\frac{d^2 x^{\kappa}}{d\lambda^2} + \Gamma^{\kappa}_{\mu\nu} \frac{dx^{\mu}}{d\lambda} \frac{dx^{\nu}}{d\lambda} = 0 \qquad \kappa=0\ldots3 \tag{12}$$

with the usual setting $\lambda=\tau$ = proper time

For $\lambda=\tau$ we get for the line-element $ds=c\,d\lambda= d\lambda$ and therefore trivially:

$$g_{\mu\nu} \frac{dx^{\mu}}{d\lambda} \frac{dx^{\nu}}{d\lambda} - 1 = 0 \tag{13}$$

This relation yields for the Kerr- and Schwarzschild-spacetimes the GR energy relation, we choose the denomination E1 for it.

The explicit form of E1 and O1…O4 for Kerr-spacetime as a series in α is given in the appendix.

In the following, we use the expression for the τ-derivative with dot or with prime: $t' = \dot{t} = \frac{dt}{d\tau}$

3. The ansatz for the GR rotator as complex Kerr spacetime

We introduce now the ansatz for the GR rotator: it should contain both radii *(r1, r2)* and the mass ratio as a parameter and it should of course satisfy the Einstein equations. Furthermore, it is clear that it should have axial, and not spherical, symmetry, as the rotator has its rotation axis as the symmetry axis.
Consequently, we make an ansatz with a Kerr spacetime with complex orbit radius: it has axial symmetry, one more parameter because of its imaginary part, and it satisfies the Einstein equations. We have to verify, that in the limit $r\to\infty$ in order *O(1/r)* the correct Newtonian orbit equations emerge.
The Kerr metric is transformed for binary BH (*m1, m2*) at distance r_0, rotating around the center-of-mass (com) at distance *(r1, r2)* from com as follows:

$r_1 m_1 = r_2 m_2 = m_r r_0 \quad m_r = \dfrac{m_1 m_2}{m_1+m_2}$ is the reduced mass , $\mu = \dfrac{m_2}{m_1} \le 1$ is the mass ratio.

We generate now a **new complex Kerr spacetime** from the Kerr spacetime of r_2 Kerr(*r2*) by the transformation $r_2 \to \dfrac{\tilde r}{1+i\mu}$

The transformation $\tilde r = r_2(1+i\mu) = r_1 \dfrac{(1+i\mu)}{\mu} = r_2 + i r_1$ maps r_2 into the complex orbit of the binary rotator, the resulting Kerr metric satisfies the Einstein equations in $\tilde r$ and
$\mathrm{Re}[\tilde r] = r_2$ and $\mathrm{Im}[\tilde r] = r_1$, i.e. *orbit(*$\tilde r$*)= orbit(r2) + i*orbit(r1)* , the complex orbit $\tilde r$ yields the orbits of the two masses.
We get immediately the following relations:

$r_0 = r_2(1+\mu) = \dfrac{\tilde r(1+\mu)}{(1+i\mu)}$, $l = \dfrac{\tilde l}{(1+i\mu)^2}$

The transformed $\tilde r = r_2(1+i\mu) = r_1 \dfrac{(1+i\mu)}{\mu} = r_2 + i r_1$ Kerr energy equation (left side) as series in α, with Christoffel symbols $\Gamma^{\kappa}{}_{\mu\nu}$ and coordinates x^{μ} =*(t , r , θ , φ)* is then with $\tilde r = r$ (we drop the tilde for convenience)
E1d=

$$-1+\frac{\left(\alpha^2(-i+\mu)^2\,\mathrm{Cos}[\theta[\tau]]^2-r[\tau]^2\right)r'[\tau]^2}{(-i+\mu)^2\left(\alpha^2(-i+\mu)^2+r[\tau]+i\,\mu\,r[\tau]-r[\tau]^2\right)}+$$
$$\left(1-\frac{i\,(-i+\mu)\,r[\tau]}{-\alpha^2(-i+\mu)^2\,\mathrm{Cos}[\theta[\tau]]^2+r[\tau]^2}\right)t'[\tau]^2+$$
$$\left(-\alpha^2\,\mathrm{Cos}[\theta[\tau]]^2+\frac{r[\tau]^2}{(-i+\mu)^2}\right)\theta'[\tau]^2+\frac{2\,\alpha\,(1+i\,\mu)\,r[\tau]\,\mathrm{Sin}[\theta[\tau]]^2\,t'[\tau]\,\phi'[\tau]}{-\alpha^2(-i+\mu)^2\,\mathrm{Cos}[\theta[\tau]]^2+r[\tau]^2}-$$
$$\mathrm{Sin}[\theta[\tau]]^2\left(\alpha^2+\frac{r[\tau]^2}{(1+i\,\mu)^2}+\frac{\alpha^2(1+i\,\mu)\,r[\tau]\,\mathrm{Sin}[\theta[\tau]]^2}{-\alpha^2(-i+\mu)^2\,\mathrm{Cos}[\theta[\tau]]^2+r[\tau]^2}\right)\phi'[\tau]^2$$

and the orbit equations (left side) O1d…O4d are given in the appendix.
How should we interpret the transformed Kerr spacetime in complex $r = \tilde r$?
For the gravitational rotator itself the arising orbit equations O1d…O4d describe through the complex solution $\tilde r[\tau]$ the orbits of the two masses (*m1, m2*) via
$\mathrm{Re}[\tilde r[\tau]] = r_2[\tau]$ and $\mathrm{Im}[\tilde r[\tau]] = r_1[\tau]$. We will show in 5 that we recover the Schwarzschild energy equation (which is the radial orbit equation) for $r_2[\tau]$ and $r_1[\tau]$ for $\alpha=0$,
For a remote observer in the transformed Kerr spacetime we take as his orbit $\mathrm{Re}[\tilde r[\tau]]$, since the orbit must be real. We will show in section 6 that in lowest order in 1/r-powers the orbit equations of the transformed Kerr spacetime are identical with the Newtonian acceleration equations calculated directly.

4. The relativistic time-derivative dt/dτ

Of special importance is the solution of O1 , which gives the derivative $t'=\frac{dt}{d\tau}$.

In the Newtonian approximation, is of course $t=\tau$ and $t'=1$.

In the Schwarzschild spacetime , O1 can be solved analytically, and the well-known solution is [2, 5]

$$t'=\frac{F}{1-\frac{1}{r}}\text{ , where } F^2=2\varepsilon_t+1$$

In the Kerr spacetime, the solution cannot be given in analytical form, but it can be expressed as a series in r and α , it seems that it is derived here for the first time.

First, we bring O1 into a new form using $\theta=\pi/2$, $\theta'=0$ and $\varphi' r^2=l$ (see 5.), thus eliminating φ' and θ' :

$$-\frac{l\,\alpha\left(1+\frac{2\,r[\tau]^2}{\alpha^2+r[\tau]^2}\right)r'[\tau]}{r[\tau]^4\left(1-\frac{r[\tau]}{\alpha^2+r[\tau]^2}\right)}+\frac{r'[\tau]\,t'[\tau]}{r[\tau]^2\left(1-\frac{r[\tau]}{\alpha^2+r[\tau]^2}\right)}+t''[\tau]=0$$

This has the general form

$t''+t'r'f_2(r)-r'f_3(r)=0$ and after multiplication with a function $f1(r)$ it can be made a total differential

$t''f_1(r)+t'r'f_2(r)f_1(r)-r'f_3(r)f_1(r)=0$ with $f_1'(r)=r'f_2(r)f_1(r)$

And with this condition the formal solution can be derived immediately:

$$f_1(r)=\exp(\int f_1(r)) \quad\text{and}\quad t'=\frac{\int f_3(r)f_1(r)+F}{f_1(r)}$$ with an integration constant F .

In the Schwarzschild case with $\alpha=0$ and $f_3(r)=0$ this results immediately in $f_1(r)=1-\frac{1}{r}$ and $t'=\frac{F}{1-\frac{1}{r}}$

In the Kerr case, $f_2(r)=\frac{1}{r^2\left(1-\frac{r}{r^2+\alpha^2}\right)}$, $f_3(r)=\frac{\alpha\left(1+\frac{2}{1+\frac{\alpha^2}{r^2}}\right)}{r^4\left(1-\frac{1}{r\left(1+\frac{\alpha^2}{r^2}\right)}\right)}$

and after turning the integral in the numerator of t' into a series in α and 1/r , t' becomes

$$t'=ts(r,\alpha)=\frac{e^{-\frac{\alpha^2}{4r^4}}F\,r}{-1+r}+\frac{e^{-\frac{\alpha^2}{4r^4}}\,l\,\alpha\left(-4+\frac{1}{5r^2}-\frac{2\alpha^2}{7r^4}+\frac{\alpha^2}{3r^3}\right)}{(-1+r)\,r^2},$$

which for $\alpha=0$ results again in $t'=\frac{F}{1-\frac{1}{r}}$

So the Kerr-correction to t' is of the order $F\frac{\alpha^2}{r^4}$ from the F-term (total energy) and of the order $l\,\frac{\alpha}{r^3}$ from the l-term (rotational energy) .

5. The equations in the “equatorial” form and α as radiation energy parameter

Without loss of generality we can set $\theta=\pi/2$, $\theta'=0$: the orbit plane is the equatorial plane, we introduce the conserved l = reduced angular momentum from conservation of angular momentum $\varphi' r^2=l$, eliminating φ'

Further on, we use the solution of *O1* to eliminate $t'=\frac{dt}{d\tau}$:

In Schwarzschild spacetime $t'=\frac{F}{1-\frac{1+i\mu}{r}}$ (Schwarzschild replacement)

in Kerr spacetime $t'=ts(\frac{r}{1+i\mu},\alpha)$ (see above, Kerr replacement)

E1 with Schwarzschild-replacement (approximate Kerr)
Left side E1dS=

$$-1-\frac{2\,F\,l\,\alpha\,(-i+\mu)}{(-i+\mu+i\,r[\tau])\,r[\tau]^2}+\frac{i\,F^2\,r[\tau]}{-i+\mu+i\,r[\tau]}-\frac{l^2\left(\alpha^2+\frac{\alpha^2(1+i\mu)}{r[\tau]}+\frac{r[\tau]^2}{(1+i\mu)^2}\right)}{r[\tau]^4}-\frac{r[\tau]^2\,r'[\tau]^2}{(-i+\mu)^2\left(\alpha^2(-i+\mu)^2+r[\tau]+i\,\mu\,r[\tau]-r[\tau]^2\right)}$$

E1 with Kerr-replacement (fullKerr)
Left side E1dA=

$$-1-\frac{l^2\left(\alpha^2+\frac{\alpha^2(1+i\mu)}{r[\tau]}+\frac{r[\tau]^2}{(1+i\mu)^2}\right)}{r[\tau]^4}+$$

$$\left(2\,e^{-\frac{\alpha^2(-i+\mu)^4}{4\,r[\tau]^4}}\,l\,\alpha\,(-i+\mu)\left(-30\,i\,l\,\alpha^3(-i+\mu)^7+35\,l\,\alpha^3(-i+\mu)^6\,r[\tau]-21\,i\,l\,\alpha\,(-i+\mu)^5\,r[\tau]^2-\right.\right.$$
$$\left.\left.420\,i\,l\,\alpha\,(-i+\mu)^3\,r[\tau]^4-105\,F\,r[\tau]^7\right)\right)\Big/\left(105\,(-i+\mu+i\,r[\tau])\,r[\tau]^9\right)+$$
$$\left(i\,e^{-\frac{\alpha^2(-i+\mu)^4}{2\,r[\tau]^4}}\left(30\,i\,l\,\alpha^3(-i+\mu)^7-35\,l\,\alpha^3(-i+\mu)^6\,r[\tau]+21\,l\,\alpha\,(1+i\mu)^5\,r[\tau]^2+\right.\right.$$
$$\left.\left.420\,l\,\alpha\,(-1-i\mu)^3\,r[\tau]^4+105\,F\,r[\tau]^7\right)^2\right)\Big/$$
$$\left(11025\,(-i+\mu+i\,r[\tau])\,r[\tau]^{13}\right)-\frac{r[\tau]^2\,r'[\tau]^2}{(-i+\mu)^2\left(\alpha^2(-i+\mu)^2+r[\tau]+i\,\mu\,r[\tau]-r[\tau]^2\right)}$$

E1dA series in α is as follows:

$$\left(-1+\frac{l^2}{(-i+\mu)^2\,r[\tau]^2}+\frac{i\,F^2\,r[\tau]}{-i+\mu+i\,r[\tau]}+\frac{i\,r[\tau]\,r'[\tau]^2}{(-i+\mu)^2\,(-i+\mu+i\,r[\tau])}\right)+$$
$$\frac{2\,F\,(-i+\mu)^3\left(-i\,l+4\,l\,\mu+6\,i\,l\,\mu^2-4\,l\,\mu^3-i\,l\,\mu^4+15\,i\,l\,r[\tau]^2-40\,l\,\mu\,r[\tau]^2-20\,i\,l\,\mu^2\,r[\tau]^2\right)\alpha}{5\,(1+i\mu)^2\,r[\tau]^4\,(-1-i\mu+r[\tau])}+$$
$$\left(\frac{-l^2-i\,l^2\mu-l^2\,r[\tau]}{r[\tau]^5}-\frac{2\,i\,l^2\,(-i+\mu)^4\left(-1-2\,i\,\mu+\mu^2+20\,r[\tau]^2\right)}{5\,(-i+\mu+i\,r[\tau])\,r[\tau]^7}+\right.$$
$$\frac{i\left(-\frac{11025}{2}F^2(-i+\mu)^4\,r[\tau]^{10}+\left(21\,l\,(1+i\mu)^5\,r[\tau]^2+420\,l\,(-1-i\mu)^3\,r[\tau]^4\right)^2\right)}{11025\,(-i+\mu+i\,r[\tau])\,r[\tau]^{13}}-$$
$$\left.\frac{r'[\tau]^2}{(-i+\mu+i\,r[\tau])^2}\right)\alpha^2+$$
$$\left((-1-i\mu)^7\left(42\,F\,l+168\,i\,F\,l\,\mu-252\,F\,l\,\mu^2-168\,i\,F\,l\,\mu^3+42\,F\,l\,\mu^4-615\,F\,l\,r[\tau]^2-\right.\right.$$
$$\left.\left.1440\,i\,F\,l\,\mu\,r[\tau]^2+720\,F\,l\,\mu^2\,r[\tau]^2-140\,F\,l\,r[\tau]^3-140\,i\,F\,l\,\mu\,r[\tau]^3\right)\alpha^3\right)\Big/$$
$$\left(210\,(1+i\mu)^2\,r[\tau]^8\,(-1-i\mu+r[\tau])\right)+O[\alpha]^4 \qquad = 0$$

For $\alpha=0$ we get after division by $(1+i\,\mu)^2$

$$-1+F^2-\frac{l^2\left(1-\frac{1+i\mu}{r[\tau]}\right)}{(1+i\mu)^2\,r[\tau]^2}+\frac{1+i\mu}{r[\tau]}-\frac{r'[\tau]^2}{(1+i\mu)^2}=0$$

And in the original r_2-coordinates $r_2=\dfrac{r[\tau]}{1+i\mu}$ $\quad l_2=\dfrac{l}{(1+i\mu)^2}$

$$\frac{\dot{\bar{r}}_2^{\,2}}{2}+\frac{l_2^{\,2}}{2\bar{r}_2^{\,2}}\left(1-\frac{1}{\bar{r}_2}\right)-\frac{1}{2\bar{r}_2}=\varepsilon_t \qquad (14)$$

which is the Schwarzschild energy equation for $r_2[\tau]$, and $r_1[\tau]=\mu\, r_2[\tau]$, so we have verified the Scharzschild limit (*α=0*) of the transformed Kerr spacetime.

α-term in E1dA is $2\Delta E_\alpha = \frac{2l\,(1+i\mu)\alpha F}{r[\tau]^3}\left(3+8i\mu-4\mu^2\right)$ with rotation period

$T = \frac{2\pi}{\omega} = \frac{2\pi r[\tau]^2}{l}$, BH–distance $r_0 = r[\tau]\frac{(1+\mu)}{(1+i\mu)}$

The Einstein-formula gravitational radiation power [2] is

$$P_{gr} = \frac{32}{5} m_1{}^2 m_1{}^2 m \frac{G^4}{r_0{}^5 c^5} \tag{15}$$

or dimensionless and in transformed Kerr coordinates $\tilde{r}[\tau] = \frac{r_2[\tau]}{1+i\mu}$ and $m=1$, $m_1 = \frac{1}{1+\mu}$, $m_2 = \frac{\mu}{1+\mu}$

$$\overline{P}_{gr} = \frac{2}{5}\frac{m_r}{m}\frac{1}{\overline{r}_0{}^5} = \frac{2}{5}\frac{\mu}{(1+\mu)^2}\frac{(1+i\,\mu)^5 1}{r[\tau]^5(1+\mu)^5}$$

and E1 power-correction from the series E1dA in *α*

$$P_\alpha = \frac{\Delta E_\alpha}{T} = \frac{(1+i\mu)\alpha}{r[\tau]^3}\frac{l^2 F}{2\pi r[\tau]^2}\left(3+8i\mu-4\mu^2\right) \tag{16}$$

It is remarkable that the E1 power-correction and the gravitational radiation power formula have the same r-dependence $1/r^5$,
so we **interpret the parameter *α* of the rotator spacetime as the gravitational radiation loss**.

$P_{gr} = P_\alpha$ if $\alpha = \left(\frac{4\pi}{5 l_{cp}{}^2 F}\frac{\mu}{(1+\mu)^7\left(3+8i\mu-4\mu^2\right)}\right)$ where $l = \tilde{l} = l_{cp}(1+i\mu)^2$ and l_{cp} is the original l_2 in the Newton radial equation

or with $l_{cp} = \omega\, r_0{}^2 = \frac{1}{\sqrt{2r_0{}^3}} r_0{}^2 = \sqrt{\frac{r_0}{2}}$:

$$\alpha_{gr} = \frac{8\pi}{5 r_0 F}\frac{\mu}{(1+\mu)^7\left(3+8i\mu-4\mu^2\right)} \tag{17}$$

The actual radiation energy is real of course, $E_{rad} = \overline{P}_{gr}$.

α_{gr} contains no factor $(1+i\,\mu)$, the complex factor in the denominator comes from the Kerr t-derivative from section 4, for the Schwarzschild t-derivative α_{gr} is real.

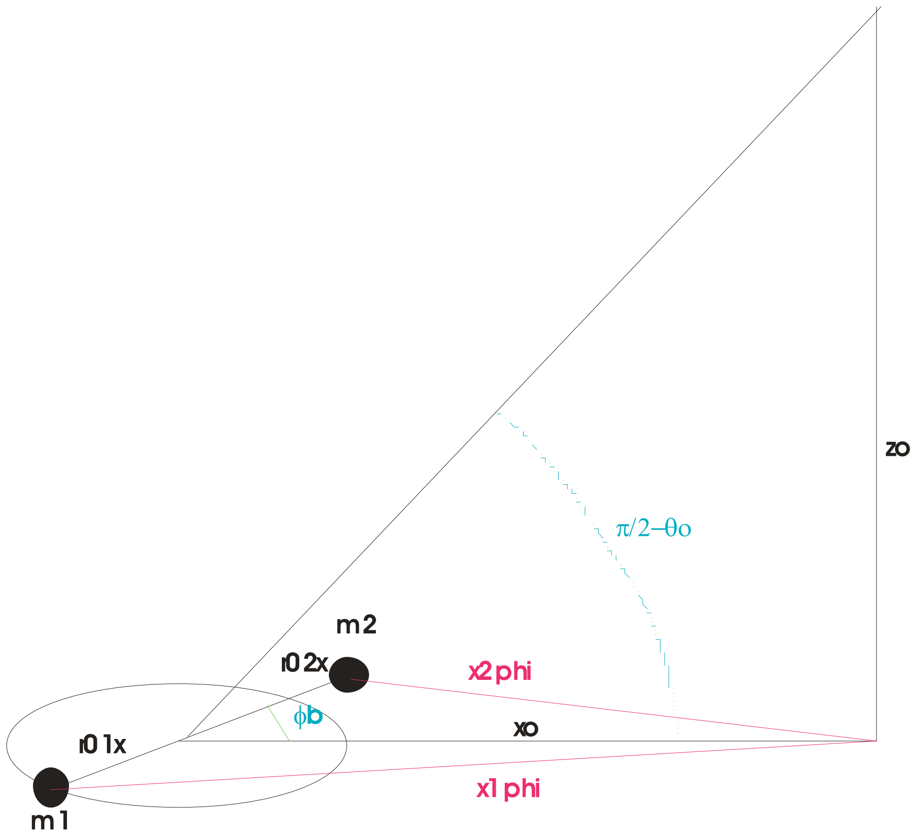
For $\mu \to 0$, that is for a planet orbiting a star, α_{gr} becomes real , also $\alpha_{gr}(\mu = \frac{\sqrt{3}}{2}) = \frac{\pi}{i\,5 F r_0 (1+\frac{\sqrt{3}}{2})^7}$

becomes purely imaginary.
With $F_c=1$, $r_0=2$, $\mu=1$: $\alpha_{gr} = -0.00030-0.0024\, i$, so the gravitational correction is very small even for close orbits. Note that that $\alpha_{gr} \to 0$ for $\mu \to 0$: the transformed Kerr spacetime becomes the Schwarzschild spacetime of a single point mass, and a single star emits no gravitational radiation.

6. The orbit equations for the remote observer in transformed Kerr and in the Newtonian limit

First, we calculate the Newtonian gravitational acceleration of the remote free falling observer towards the rotator *(m1,m2)* in the com-frame of the rotator.
Second, we calculate the orbit equations of the observer from the transformed Kerr spacetime of the rotator, and take the real part as the valid observer orbit.
Third, we compare both in the lowest order in *1/r* .
The result is that they are identical, which proves that the transformed Kerr spacetime is indeed the correct physical description of the gravitational rotator.

We define the variables of the remote observer and calculate the acceleration from the Newtonian gravitation law.



$\vec{r}o$ =vector(observer,com rotator)
distance d(observer,m1), d(observer,m2):
r0x=d(m1,m2), {x1phi,x2phi}=projection(ro,θ=π/2)
{ro1x,ro2x}={d(observer,m1),d(observer,m2)}
$\vec{r}o$ ={xo,yo,zo}=observer coordinates={ro,θo,φo}
BH-masses={m1,m2,m=(m1+m2),mr=m1 m2/m}
BH-distance com={r0x,r1x=r0x*mr/m1=m2/m,r2x=r0x*mr/m2}
μ=m2/m1 mass ratio
φb=φ(binBH,observer x-axis)!= φo
xo=ro Sin[θo];
zo=ro Cos[θo];
m1=m-m2;
x1phi=Sqrt[xo^2+(m2/m)^2r0x^2-2 xo(m2/m)r0x Sin[φb]]
x2phi=Sqrt[xo^2+(m1/m)^2r0x^2+2 xo(m1/m)r0x Sin[φb]]
ro1x=ro Sqrt[1+(m2/m)^2(r0x/ro)^2-2 (m2/m)(r0x /ro)Sin[φb]Sin[θo]]
ro2x=ro Sqrt[1+(m1/m)^2(r0x/ro)^2+2 (m1/m)(r0x /ro)Sin[φb]Sin[θo]]

The dimensionless Newtonian acceleration vector in {x,y,z} of the observer towards the rotator is the vector sum of the accelerations towards $m1$ and $m2$, dimensionless gravitational potential is $E_{gr} = -\frac{m_x}{2r}$, so the dimensionless acceleration=force is $F_{gr} = -\frac{m_x}{2r^3}\vec{r}$, where $\vec{n}_r = \frac{1}{r}\vec{r}$ is the unit vector from the mass attractor to the observer, and m_x is the dimensionless mass ($m=m_1+m_2=1$ is the mass of the rotator) .

```
(*  Newtonian vector  acceleration aro=nr1(1/2ro1x^2)+nr2(1/2ro2x^2)
   in {xo,yo,zo} *)
x2o = Sqrt[x2phi^2 - (m1 / m)^2 r0x^2 Sin[ϕb]^2]
x1o = Sqrt[x1phi^2 - (m2 / m)^2 r0x^2 Sin[ϕb]^2]
y2o = (m1 / m) r0x Sin[ϕb]
y1o = -(m2 / m) r0x Sin[ϕb]
aro = ((m1 / m) / (2 ro1x^3)) {x1o, y1o, zo} + ((m2 / m) / (2 ro2x^3)) {x2o, y2o, zo}
```

We calculate the acceleration vector in spherical coordinates (r,θ,ϕ) of the observer, we use the initial conditions $r[0]=r_o$, $\theta[0]=\theta_o$, $\varphi[0]=\varphi_o=0$

aror=

$$\Big\{-\frac{1}{128\,ro^4} r0x^2 (-1+\mu)\,\mu\,(6+24\,Cos[2\,\theta o]-30\,Cos[4\,\theta o]+15\,Cos[4\,\theta o-2\,\phi b]-36\,Cos[2\,(\theta o-\phi b)]+10\,Cos[2\,\phi b]-36\,Cos[2\,(\theta o+\phi b)]+15\,Cos[2\,(2\,\theta o+\phi b)])\,Csc[\theta o]-\frac{Sin[\theta o]}{2\,ro^2},$$
$$-\frac{3\,r0x^2\,(-1+\mu)\,\mu\,Sin[\theta o]\,Sin[\phi b]^2}{2\,ro^4},\ -\frac{Cos[\theta o]}{2\,ro^2}-\frac{1}{32\,ro^4}3\,r0x^2\,(-1+\mu)\,\mu\,Cos[\theta o]\,(-2+10\,Cos[2\,\theta o]-5\,Cos[2\,(\theta o-\phi b)]+10\,Cos[2\,\phi b]-5\,Cos[2\,(\theta o+\phi b)])\Big\}$$

Setting $(xo'', yo'', zo'')=aror$ we get the Newtonian equations of motion in spherical coordinates (the respective left side, the right side is 0)

deqgN1

$$\frac{1}{128\,ro^4} r0x^2 (-1+\mu)\,\mu\,(6+24\,Cos[2\,\theta 0]-30\,Cos[4\,\theta 0]+15\,Cos[4\,\theta 0-2\,\phi b]-36\,Cos[2\,(\theta 0-\phi b)]+10\,Cos[2\,\phi b]-36\,Cos[2\,(\theta 0+\phi b)]+15\,Cos[2\,(2\,\theta 0+\phi b)])\,Csc[\theta 0]+\frac{Sin[\theta 0]}{2\,ro^2}+2\,Cos[\theta 0]\,r'[\tau]\,\theta'[\tau]-ro\,Sin[\theta 0]\,\theta'[\tau]^2+Sin[\theta 0]\,r''[\tau]+ro\,Cos[\theta 0]\,\theta''[\tau]$$

deqgN2: φ'' equation

$$\frac{3\,r0x^2\,(-1+\mu)\,\mu\,Sin[\theta 0]\,Sin[\phi b]^2}{2\,r[\tau]^4}+2\,Sin[\theta 0]\,r'[\tau]\,\phi'[\tau]+r[\tau]\,(2\,Cos[\theta 0]\,\theta'[\tau]\,\phi'[\tau]+Sin[\theta 0]\,\phi''[\tau])$$

deqgN3

$$\frac{Cos[\theta 0]}{2\,ro^2}+\frac{1}{32\,ro^4}3\,r0x^2\,(-1+\mu)\,\mu\,Cos[\theta 0]\,(-2+10\,Cos[2\,\theta 0]-5\,Cos[2\,(\theta 0-\phi b)]+10\,Cos[2\,\phi b]-5\,Cos[2\,(\theta 0+\phi b)])-2\,Sin[\theta 0]\,r'[\tau]\,\theta'[\tau]-ro\,Cos[\theta 0]\,\theta'[\tau]^2+Cos[\theta 0]\,r''[\tau]-ro\,Sin[\theta 0]\,\theta''[\tau]$$

We form linear combinations to get pure θ'' equation and r'' equation:

deqgN13= Cos[θ0]deqgN1−Sin[θ0]deqgN3: θ'' equation

$$2\,r0x^2\,(-1+\mu)\,\mu\,Cos[\theta 0]\,(6-6\,Cos[2\,\theta 0]+3\,Cos[2\,(\theta 0-\phi b)]-10\,Cos[2\,\phi b]+3\,Cos[2\,(\theta 0+\phi b)])+64\,ro^4\,Sin[\theta 0]\,r'[\tau]\,\theta'[\tau]+32\,ro^5\,Sin[\theta 0]\,\theta''[\tau]$$

deqgN31 Sin[θ0]deqgN1+Cos[θ0]deqgN3 : r'' equation

$$16\,ro^2-6\,r0x^2\,\mu+6\,r0x^2\,\mu^2-18\,r0x^2\,\mu\,Cos[2\,\theta 0]+18\,r0x^2\,\mu^2\,Cos[2\,\theta 0]+9\,r0x^2\,\mu\,Cos[2\,(\theta 0-\phi b)]-9\,r0x^2\,\mu^2\,Cos[2\,(\theta 0-\phi b)]-10\,r0x^2\,\mu\,Cos[2\,\phi b]+10\,r0x^2\,\mu^2\,Cos[2\,\phi b]+9\,r0x^2\,\mu\,Cos[2\,(\theta 0+\phi b)]-9\,r0x^2\,\mu^2\,Cos[2\,(\theta 0+\phi b)]-32\,ro^5\,\theta'[\tau]^2+32\,ro^4\,r''[\tau]$$

The corresponding transformed-Kerr orbit equations O2dn, O3dn, O4dn are:

O2dnA

$$\begin{aligned}&\frac{1}{2}\Bigg(\frac{\mathrm{Fc}^2}{(-1+\mathrm{ro})\,\mathrm{ro}}+\frac{r'[\tau]^2}{\mathrm{ro}-\mathrm{ro}^2}-2\,(-1+\mathrm{ro})\,\theta'[\tau]^2-\\&\quad\frac{2\,\mathrm{Fc}\,\alpha\,\mathrm{Sin}[\theta 0]^2\,\phi'[\tau]}{\mathrm{ro}^2}-2\,(-1+\mathrm{ro})\,\mathrm{Sin}[\theta 0]^2\,\phi'[\tau]^2+\frac{1}{2\,\mathrm{ro}^3}\\&\quad\alpha^2\Bigg(-\frac{2\,\mathrm{Fc}^2\,(-2+\mathrm{ro}+2\,(-1+\mathrm{ro})\,\mathrm{Cos}[2\,\theta 0])}{(-1+\mathrm{ro})^2}+\frac{2\,\left(-1+\mathrm{ro}+\mathrm{ro}^2-(-1+\mathrm{ro})^2\,\mathrm{Cos}[2\,\theta 0]\right)\,r'[\tau]^2}{(-1+\mathrm{ro})^2}-\\&\qquad 8\,\mathrm{ro}\,\mathrm{Cos}[\theta 0]\,\mathrm{Sin}[\theta 0]\,r'[\tau]\,\theta'[\tau]+4\,\mathrm{ro}\,\left(-\mathrm{ro}+(-1+\mathrm{ro})\,\mathrm{Cos}[\theta 0]^2\right)\,\theta'[\tau]^2+\\&\qquad\left(-1-\mathrm{ro}-2\,\mathrm{ro}^2+\left(1-3\,\mathrm{ro}+2\,\mathrm{ro}^2\right)\,\mathrm{Cos}[2\,\theta 0]\right)\,\mathrm{Sin}[\theta 0]^2\,\phi'[\tau]^2\Bigg)+2\,r''[\tau]\Bigg)\end{aligned}$$

O3dnA

$$\begin{aligned}&\frac{2\,r'[\tau]\,\theta'[\tau]}{\mathrm{ro}}+\frac{\mathrm{Fc}\,\alpha\,\mathrm{Sin}[2\,\theta 0]\,\phi'[\tau]}{(-1+\mathrm{ro})\,\mathrm{ro}^2}-\mathrm{Cos}[\theta 0]\,\mathrm{Sin}[\theta 0]\,\phi'[\tau]^2-\\&\frac{1}{2\,\mathrm{ro}^3}\alpha^2\Bigg(\frac{\mathrm{Fc}^2\,\mathrm{Sin}[2\,\theta 0]}{(-1+\mathrm{ro})^2}-\frac{\mathrm{Sin}[2\,\theta 0]\,r'[\tau]^2}{-1+\mathrm{ro}}+4\,\mathrm{Cos}[\theta 0]^2\,r'[\tau]\,\theta'[\tau]+\\&\quad\mathrm{ro}\,\mathrm{Sin}[2\,\theta 0]\,\theta'[\tau]^2+2\,(2+\mathrm{ro})\,\mathrm{Cos}[\theta 0]\,\mathrm{Sin}[\theta 0]^3\,\phi'[\tau]^2\Bigg)+\theta''[\tau]\end{aligned}$$

O4dnA

$$\begin{aligned}&\frac{\mathrm{Fc}\,\alpha\,(r'[\tau]-2\,(-1+\mathrm{ro})\,\mathrm{Cot}[\theta 0]\,\theta'[\tau])}{(-1+\mathrm{ro})^2\,\mathrm{ro}^2}+\frac{2\,r'[\tau]\,\phi'[\tau]}{\mathrm{ro}}+2\,\mathrm{Cot}[\theta 0]\,\theta'[\tau]\,\phi'[\tau]+\\&\frac{\alpha^2\,((1-4\,\mathrm{ro}+3\,\mathrm{Cos}[2\,\theta 0])\,r'[\tau]+2\,(-1+\mathrm{ro})\,\mathrm{Sin}[2\,\theta 0]\,\theta'[\tau])\,\phi'[\tau]}{2\,(-1+\mathrm{ro})\,\mathrm{ro}^3}+\phi''[\tau]\end{aligned}$$

Comparing deqgN2 and O4dnA yields in lowest order in $1/r_o$ after cancellation of common factors the φ'' equation *eqphi*=

$$2Cot[\theta_o]\theta'[\tau]\phi'[\tau]+\phi''[\tau]=0 \qquad (18)$$

The comparison of deqgN13 and O3dnA gives the θ'' equation, where the last term appears only in O3dnA, *eqth*=

$$\frac{2r'[\tau]\theta'[\tau]}{r_o}+\theta''[\tau]-Cos[\theta_o]Sin[\theta_o]\phi'[\tau]^2=0 \qquad (19)$$

The comparison of deqgN31 and O2dnA gives the r'' equation, where the last term appears only in O2dnA, *eqr*=

$$\frac{1}{2r_o^{\,2}}+r''[\tau]-\theta'[\tau]^2 r_o-r_o Sin[\theta_o]^2\phi'[\tau]^2=0\ , \qquad (20)$$

From the conservation of angular momentum for the observer $\phi'[\tau]=\frac{l_o}{r_o^{\,2}}$, and for the free falling observer we can assume that initially there is a negligible rotation around the remote rotator, so $l_o=0$, and we can neglect the terms $\phi'[\tau]^2$ in *eqth* and *eqr* .

The three equations *eqphi*, *eqth* and *eqr* follow directly from the Schwarzschild spacetime in lowest order in *1/r* , and it is interesting to solve them explicitly, replacing $r_o\to r[\tau]$.

From *eqphi* we get by integration

$Log[\phi'[\tau]]=-2Cot[\theta_o]\theta[\tau]+cphi$, and $\phi'[\tau]=Exp[-2Cot[\theta_o]\theta[\tau]]cphi1$ with $cphi1=Exp[cphi]$

But from *eqr* we see that φ' must be of order $\phi'[\tau]^2\le O(\frac{1}{r_o^{\,3}})$, which is possible only if $\varphi'=0$.

After having inserted this into *eqth* , we can integrate it and get $\theta'[\tau]=\frac{cth}{r[\tau]}$.

We insert this into *eqr* , integrate and get a solution for r :

$r'[\tau]=\sqrt{\frac{1}{2r[\tau]}+\frac{cth^2}{3r[\tau]^2}}$, the *cth*-term is a GR-modification of the Newtonian force law.

7. Taking into account self-rotation of the participating masses m_1 and m_2

The Kerr ansatz is valid as long as there is a θ-symmetry , i.e. the system is independent of ϕ . If there is self-rotation (around z-axis) for the masses m_1 and m_2 , the θ-symmetry is not disturbed and self-rotation (spin-) angular momentums L_1 and L_2 add up and contribute to the Kerr parameter α according to the formula

$\alpha=\alpha_1+\alpha_2$. For a rotating blackhole $\alpha_x=\frac{L_x}{m_x c}=\frac{\kappa\omega_x r_s(m_x)^2 m_x}{m_x c}$, where $r_s(m_x)=\frac{2Gm_x}{c^2}$ is the Schwarzschild radius , κ the inertia-factor (κ=2/3 for a spherical shell) and ω_x the angular frequency. For the masses (m_1, m_2) (dimensionless, i.e. $m=1$) rotating with angular frequencies (ω_1, ω_2) we get dimensionless

$$\alpha=\alpha_1+\alpha_2=\frac{\kappa(\omega_1+\omega_2\mu^2)}{(1+\mu^2)} \qquad (21)$$

For $\mu\to 0$ the gravitational rotator becomes simply a single rotating Kerr- blackhole with $\alpha=\alpha_1$, as it should be. Also, the contribution α_{gr} from gravitational rotation becomes 0 for $\mu\to 0$, and the spacetime becomes the normal real Kerr spacetime of a rotating blackhole, which emits no gravitational radiation.
So the total α of the gravitational rotator (m_1, m_2) with spin-contributions (α_1 , α_2) becomes

$$\alpha=\alpha_1+\alpha_2+\alpha_{gr}=\frac{\kappa(\omega_1+\omega_2\mu^2)}{(1+\mu^2)}+\alpha_{gr} \quad ,$$

where α_{gr} is complex and α_1 and α_2 are real and $|\alpha_{gr}|<<1$.

8. Numerical examples

In the following, we present gravitational rotator orbits calculated numerically from the transformed Kerr spacetime.

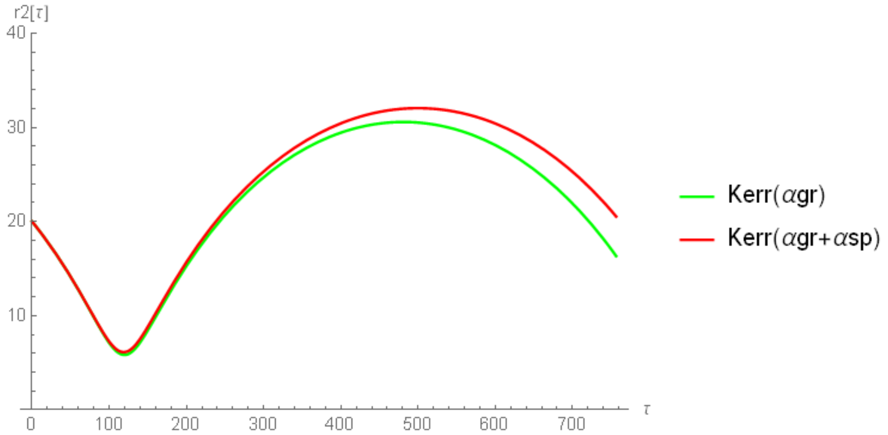
8.1. Binary blackhole with mass ratio μ=1/2

We consider an example of a binary blackhole with mass ratio $\mu=\frac{m_2}{m_1}$ with mean distance $r_0=16.80$, basic angular frequency $\omega_0=\frac{1}{\sqrt{2r_0{}^3}}$ *=0.01026* , high spinning frequencies (ω_1,ω_2) *=(0.8,0.8)* (within the limit $\omega\le 1$) , the velocity factor *F=0.987* (ε_t*= −0.0132)* and the reduced angular momentum *l=1.77+2.36i* are chosen for the initial condition to ensure a bounded orbit.
The resulting Kerr parameters are α=0.297−0.00099 *i*, α_{gr}=0.00050−0.00099 *i* .
We calculate the orbits for the Schwarzschild case (α=0 , no radiation), without spins: Kerr(α_{gr}) , and the full Kerr case with spins and radiation Kerr($\alpha_{gr}+\alpha_{sp}$) .
The Newtonian maximal and minimal basic (circular) r_2-radii from section 1 are: r_{p1}=30.91, r_{p2}=6,84, the basic Newtonian period $T_0=\frac{2\pi}{\omega_0}$ *=612.1*

The results are as follows.

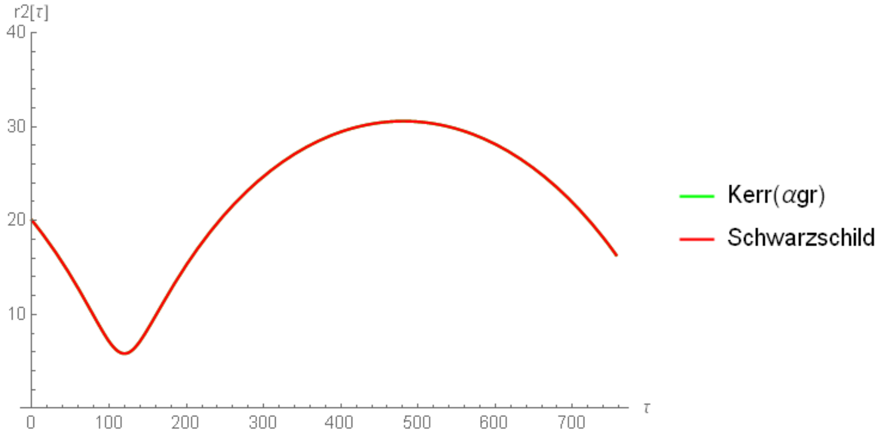
The r_2-orbit for the Kerr($\alpha_{gr} + \alpha_{sp}$) case and the Kerr(α_{gr}) case

Out[6762]=



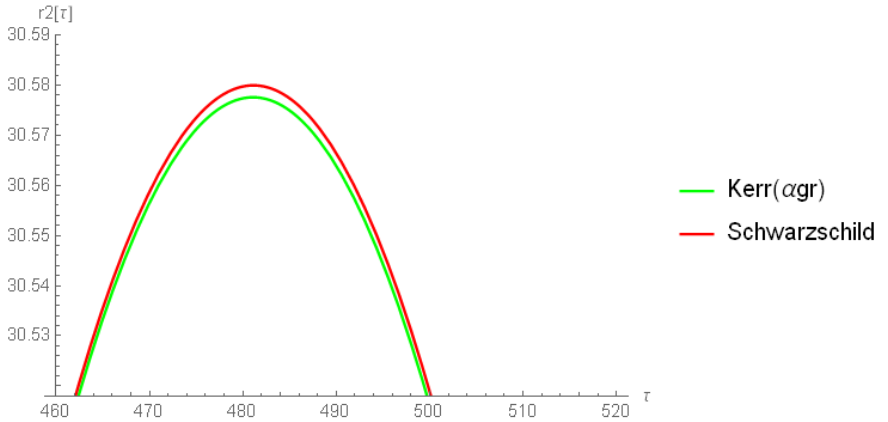
The r_2-orbit for the Schwarzschild case and the Kerr(α_{gr}) case

Out[4674]=



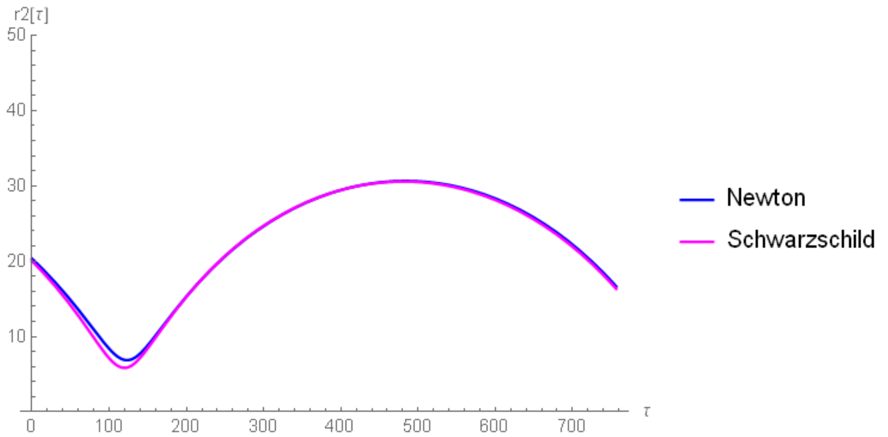
and a magnified part of it to show the difference

Out[4675]=



The r_2-orbit for the Schwarzschild and for Newton compared

Out[6663]=



In the Kerr($\alpha_{gr} + \alpha_{sp}$) case, period T=761.80, r_2-radii=(32.0485,6.127) ,
in the Kerr(α_{gr}) case, period T=722.631, r_2-radii=(30.577,5.834) ,
in the Schwarzschild case, period T=722.707, r_2-radii=(30.580,5.833) ,
in the Newtonian case, period T=721.4, r_2-radii=(3.66,6.85),
It is interesting to study the behavior of the orbit period: comparing the Schwarzschild and the Kerr(α_{gr}) case the (very small) radiation loss through α_{gr} decreases the period by 0.076, i.e. 0.010% . On the other hand, the strong self-rotation in the Kerr($\alpha_{gr} + \alpha_{sp}$) case accelerates the orbit through the "dragging" (Thirring-Lense effect) , therefore the period becomes longer by 39.17, i.e. by 5.42% .

Appendix A1

Schwarzschild spacetime in matrix form

$$\begin{pmatrix} 1-\frac{1}{r} & 0 & 0 & 0 \\ 0 & -\frac{1}{1-\frac{1}{r}} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\,\text{Sin}[\theta]^2 \end{pmatrix}$$

Kerr spacetime in matrix form

$$\begin{pmatrix} 1-\frac{r}{r^2+\alpha^2\,\text{Cos}[\theta]^2} & 0 & 0 & \frac{r\,\alpha\,\text{Sin}[\theta]^2}{r^2+\alpha^2\,\text{Cos}[\theta]^2} \\ 0 & \frac{-r^2-\alpha^2\,\text{Cos}[\theta]^2}{-r+r^2+\alpha^2} & 0 & 0 \\ 0 & 0 & -r^2-\alpha^2\,\text{Cos}[\theta]^2 & 0 \\ \frac{r\,\alpha\,\text{Sin}[\theta]^2}{r^2+\alpha^2\,\text{Cos}[\theta]^2} & 0 & 0 & -\text{Sin}[\theta]^2\left(r^2+\alpha^2+\frac{r\,\alpha^2\,\text{Sin}[\theta]^2}{r^2+\alpha^2\,\text{Cos}[\theta]^2}\right) \end{pmatrix}$$

Christoffel symbols $\Gamma^{\kappa}{}_{\mu\nu}$ (Schwarzschild) have the values

$\Gamma^{0}{}_{\mu\nu} =$

$$\left\{\left\{0,\ \frac{1}{2\left(1-\frac{1}{r}\right)r^2},\ 0,\ 0\right\},\ \left\{\frac{1}{2\left(1-\frac{1}{r}\right)r^2},\ 0,\ 0,\ 0\right\},\ \{0,\ 0,\ 0,\ 0\},\ \{0,\ 0,\ 0,\ 0\}\right\}$$

$\Gamma^{1}{}_{\mu\nu} =$

$$\left\{\left\{\frac{\text{Csc}[\theta]^2\left(-r^3\,\text{Sin}[\theta]^2+r^4\,\text{Sin}[\theta]^2\right)}{2\,r^6},\ 0,\ 0,\ 0\right\},\right.$$

$$\left\{0,\ -\frac{\text{Csc}[\theta]^2\left(-r^3\,\text{Sin}[\theta]^2+r^4\,\text{Sin}[\theta]^2\right)}{2\left(1-\frac{1}{r}\right)^2 r^6},\ 0,\ 0\right\},$$

$$\left\{0,\ 0,\ -\frac{\text{Csc}[\theta]^2\left(-r^3\,\text{Sin}[\theta]^2+r^4\,\text{Sin}[\theta]^2\right)}{r^3},\ 0\right\},$$

$$\left.\left\{0,\ 0,\ 0,\ -\frac{-r^3\,\text{Sin}[\theta]^2+r^4\,\text{Sin}[\theta]^2}{r^3}\right\}\right\}$$

$\Gamma^{2}{}_{\mu\nu} =$

$$\left\{\{0,\ 0,\ 0,\ 0\},\ \left\{0,\ 0,\ \frac{1}{r},\ 0\right\},\ \left\{0,\ \frac{1}{r},\ 0,\ 0\right\},\ \{0,\ 0,\ 0,\ -\text{Cos}[\theta]\,\text{Sin}[\theta]\}\right\}$$

$\Gamma^{3}{}_{\mu\nu} =$

$$\left\{\{0,\ 0,\ 0,\ 0\},\ \left\{0,\ 0,\ 0,\ \frac{1}{r}\right\},\ \{0,\ 0,\ 0,\ \text{Cot}[\theta]\},\ \left\{0,\ \frac{1}{r},\ \text{Cot}[\theta],\ 0\right\}\right\}$$

Christoffel symbols $\Gamma^{\kappa}{}_{\mu\nu}$ (Kerr) have the values

$\Gamma^{0}{}_{\mu\nu} =$

$$\Big\{\Big\{0,\ \frac{(r^2+\alpha^2)\,(2r^2-\alpha^2-\alpha^2\cos[2\theta])}{(-r+r^2+\alpha^2)\,(2r^2+\alpha^2+\alpha^2\cos[2\theta])^2},\ -\frac{2r\alpha^2\sin[2\theta]}{(2r^2+\alpha^2+\alpha^2\cos[2\theta])^2},\ 0\Big\},$$

$$\Big\{\frac{(r^2+\alpha^2)\,(2r^2-\alpha^2-\alpha^2\cos[2\theta])}{(-r+r^2+\alpha^2)\,(2r^2+\alpha^2+\alpha^2\cos[2\theta])^2},\ 0,\ 0,$$
$$\frac{\alpha\,(-6r^4-3r^2\alpha^2+\alpha^4+(-r^2\alpha^2+\alpha^4)\cos[2\theta])\,\sin[\theta]^2}{(-r+r^2+\alpha^2)\,(2r^2+\alpha^2+\alpha^2\cos[2\theta])^2}\Big\},$$

$$\Big\{-\frac{2r\alpha^2\sin[2\theta]}{(2r^2+\alpha^2+\alpha^2\cos[2\theta])^2},\ 0,\ 0,\ \frac{4r\alpha^3\cos[\theta]\sin[\theta]^3}{(2r^2+\alpha^2+\alpha^2\cos[2\theta])^2}\Big\},\ \Big\{0,$$
$$\frac{\alpha\,(-6r^4-3r^2\alpha^2+\alpha^4+(-r^2\alpha^2+\alpha^4)\cos[2\theta])\,\sin[\theta]^2}{(-r+r^2+\alpha^2)\,(2r^2+\alpha^2+\alpha^2\cos[2\theta])^2},\ \frac{4r\alpha^3\cos[\theta]\sin[\theta]^3}{(2r^2+\alpha^2+\alpha^2\cos[2\theta])^2},\ 0\Big\}\Big\}$$

$\Gamma^{1}{}_{\mu\nu} =$

$$\Big\{\Big\{\frac{(-r+r^2+\alpha^2)\,(r^2-\alpha^2\cos[\theta]^2)}{2\,(r^2+\alpha^2\cos[\theta]^2)^3},\ 0,\ 0,\ \frac{\alpha\,(-r+r^2+\alpha^2)\,(-r^2+\alpha^2\cos[\theta]^2)\,\sin[\theta]^2}{2\,(r^2+\alpha^2\cos[\theta]^2)^3}\Big\},$$

$$\Big\{0,\ -\frac{r\,(r-2\alpha^2)+(-1+2r)\,\alpha^2\cos[\theta]^2}{2\,(-r+r^2+\alpha^2)\,(r^2+\alpha^2\cos[\theta]^2)},\ -\frac{\alpha^2\cos[\theta]\sin[\theta]}{r^2+\alpha^2\cos[\theta]^2},\ 0\Big\},$$

$$\Big\{0,\ -\frac{\alpha^2\cos[\theta]\sin[\theta]}{r^2+\alpha^2\cos[\theta]^2},\ -\frac{r\,((-1+r)\,r+\alpha^2)}{r^2+\alpha^2\cos[\theta]^2},\ 0\Big\},$$

$$\Big\{\frac{\alpha\,(-r+r^2+\alpha^2)\,(-r^2+\alpha^2\cos[\theta]^2)\,\sin[\theta]^2}{2\,(r^2+\alpha^2\cos[\theta]^2)^3},\ 0,\ 0,$$
$$-\Big(\Big(\Big((-r+r^2+\alpha^2)\,\sin[\theta]^2\,\big(2r^5+2r\alpha^4\cos[\theta]^4-r^2\alpha^2\sin[\theta]^2+$$
$$\cos[\theta]^2\,(4r^3\alpha^2+\alpha^4\sin[\theta]^2)\big)\Big)\Big)\Big/\Big(2\,(r^2+\alpha^2\cos[\theta]^2)^3\Big)\Big)\Big\}\Big\}$$

$\Gamma^{2}{}_{\mu\nu} =$

$$\Big\{\Big\{-\frac{r\alpha^2\cos[\theta]\sin[\theta]}{(r^2+\alpha^2\cos[\theta]^2)^3},\ 0,\ 0,\ \frac{r\alpha\,(r^2+\alpha^2)\cos[\theta]\sin[\theta]}{(r^2+\alpha^2\cos[\theta]^2)^3}\Big\},$$

$$\Big\{0,\ \frac{\alpha^2\cos[\theta]\sin[\theta]}{(-r+r^2+\alpha^2)\,(r^2+\alpha^2\cos[\theta]^2)},\ \frac{r}{r^2+\alpha^2\cos[\theta]^2},\ 0\Big\},$$

$$\Big\{0,\ \frac{r}{r^2+\alpha^2\cos[\theta]^2},\ -\frac{\alpha^2\cos[\theta]\sin[\theta]}{r^2+\alpha^2\cos[\theta]^2},\ 0\Big\},\ \Big\{\frac{r\alpha\,(r^2+\alpha^2)\cos[\theta]\sin[\theta]}{(r^2+\alpha^2\cos[\theta]^2)^3},$$
$$0,\ 0,\ -\frac{1}{(r^2+\alpha^2\cos[\theta]^2)^3}\cos[\theta]\sin[\theta]\,\big(r^6+r^4\alpha^2+\alpha^4\,(r^2+\alpha^2)\cos[\theta]^4+$$
$$2r^3\alpha^2\sin[\theta]^2+r\alpha^4\sin[\theta]^4+2r\alpha^2\cos[\theta]^2\,(r^3+r\alpha^2+\alpha^2\sin[\theta]^2)\big)\Big\}\Big\}$$

$\Gamma^{3}{}_{\mu\nu} =$

$$\left\{\left\{0, \frac{2\alpha\left(r^2-\alpha^2\,\mathrm{Cos}[\theta]^2\right)}{\left(-r+r^2+\alpha^2\right)\left(2r^2+\alpha^2+\alpha^2\,\mathrm{Cos}[2\theta]\right)^2}, -\frac{4r\alpha\,\mathrm{Cot}[\theta]}{\left(2r^2+\alpha^2+\alpha^2\,\mathrm{Cos}[2\theta]\right)^2}, 0\right\},\right.$$

$$\left\{\frac{2\alpha\left(r^2-\alpha^2\,\mathrm{Cos}[\theta]^2\right)}{\left(-r+r^2+\alpha^2\right)\left(2r^2+\alpha^2+\alpha^2\,\mathrm{Cos}[2\theta]\right)^2}, 0,\right.$$

$$0, \left(2\left(2(-1+r)r^4+2r\alpha^4\,\mathrm{Cos}[\theta]^4-r^2\alpha^2\,\mathrm{Sin}[\theta]^2+\alpha^2\,\mathrm{Cos}[\theta]^2\left(2r^2(-1+2r)+\alpha^2\,\mathrm{Sin}[\theta]^2\right)\right)\right)/$$

$$\left.\left(\left(-r+r^2+\alpha^2\right)\left(2r^2+\alpha^2+\alpha^2\,\mathrm{Cos}[2\theta]\right)^2\right)\right\}, \left\{-\frac{4r\alpha\,\mathrm{Cot}[\theta]}{\left(2r^2+\alpha^2+\alpha^2\,\mathrm{Cos}[2\theta]\right)^2}, 0,\right.$$

$$\left.0, \frac{\left(8r^4+4r\alpha^2+8r^2\alpha^2+3\alpha^4+4\alpha^2\left(-r+2r^2+\alpha^2\right)\mathrm{Cos}[2\theta]+\alpha^4\,\mathrm{Cos}[4\theta]\right)\mathrm{Cot}[\theta]}{2\left(2r^2+\alpha^2+\alpha^2\,\mathrm{Cos}[2\theta]\right)^2}\right\},$$

$$\left\{0, \left(2\left(2(-1+r)r^4+2r\alpha^4\,\mathrm{Cos}[\theta]^4-r^2\alpha^2\,\mathrm{Sin}[\theta]^2+\alpha^2\,\mathrm{Cos}[\theta]^2\right.\right.\right.$$

$$\left.\left.\left(2r^2(-1+2r)+\alpha^2\,\mathrm{Sin}[\theta]^2\right)\right)\right)/\left(\left(-r+r^2+\alpha^2\right)\left(2r^2+\alpha^2+\alpha^2\,\mathrm{Cos}[2\theta]\right)^2\right),$$

$$\left.\left.\frac{\left(8r^4+4r\alpha^2+8r^2\alpha^2+3\alpha^4+4\alpha^2\left(-r+2r^2+\alpha^2\right)\mathrm{Cos}[2\theta]+\alpha^4\,\mathrm{Cos}[4\theta]\right)\mathrm{Cot}[\theta]}{2\left(2r^2+\alpha^2+\alpha^2\,\mathrm{Cos}[2\theta]\right)^2}, 0\right\}\right\}$$

General Kerr energy and orbit equations, series in α

E1: $\sum_{\mu=0}^{3}\sum_{\nu=0}^{3} g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}-1=0$ total differential of line-element

$$-1+\left(-\frac{r[\tau]}{-1+r[\tau]}+\alpha^2\left(\frac{1}{(-1+r[\tau])^2}-\frac{\mathrm{Cos}[\theta[\tau]]^2}{-r[\tau]+r[\tau]^2}\right)\right)r'[\tau]^2+$$

$$\left(1+\frac{\alpha^2\,\mathrm{Cos}[\theta[\tau]]^2}{r[\tau]^3}-\frac{1}{r[\tau]}\right)t'[\tau]^2+$$

$$\left(-\alpha^2\,\mathrm{Cos}[\theta[\tau]]^2-r[\tau]^2\right)\theta'[\tau]^2+\frac{2\alpha\,\mathrm{Sin}[\theta[\tau]]^2\,t'[\tau]\,\phi'[\tau]}{r[\tau]}+$$

$$\left(-r[\tau]^2\,\mathrm{Sin}[\theta[\tau]]^2+\alpha^2\left(-\mathrm{Sin}[\theta[\tau]]^2-\frac{\mathrm{Sin}[\theta[\tau]]^4}{r[\tau]}\right)\right)\phi'[\tau]^2==0$$

O1: $\frac{d^2x^0}{d\tau^2}=-\sum_{\mu=0}^{3}\sum_{\nu=0}^{3}\Gamma^0{}_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}$

$$2\left(\frac{1}{2(-1+r[\tau])\,r[\tau]}+\frac{\alpha^2\left(1+3\,\mathrm{Cos}[2\theta[\tau]]-3r[\tau]-3\,\mathrm{Cos}[2\theta[\tau]]\,r[\tau]\right)}{4(-1+r[\tau])^2\,r[\tau]^3}\right)r'[\tau]\,t'[\tau]-$$

$$\frac{\alpha^2\,\mathrm{Sin}[2\theta[\tau]]\,t'[\tau]\,\theta'[\tau]}{r[\tau]^3}-\frac{3\alpha\,\mathrm{Sin}[\theta[\tau]]^2\,r'[\tau]\,\phi'[\tau]}{(-1+r[\tau])\,r[\tau]}+t''[\tau]==0$$

O2: $\frac{d^2x^1}{d\tau^2}=-\sum_{\mu=0}^{3}\sum_{\nu=0}^{3}\Gamma^1{}_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}$

$$\left(-\frac{1}{2\,(-1+r[\tau])\,r[\tau]}+\frac{\alpha^2\left(-2\,\mathrm{Cos}[\theta[\tau]]^2-r[\tau]+4\,\mathrm{Cos}[\theta[\tau]]^2\,r[\tau]+2\,r[\tau]^2-2\,\mathrm{Cos}[\theta[\tau]]^2\,r[\tau]^2\right)}{2\,(-1+r[\tau])^2\,r[\tau]^3}\right)$$

$$r'[\tau]^2+\left(\frac{-1+r[\tau]}{2\,r[\tau]^3}+\frac{\alpha^2\left(4\,\mathrm{Cos}[\theta[\tau]]^2+r[\tau]-4\,\mathrm{Cos}[\theta[\tau]]^2\,r[\tau]\right)}{2\,r[\tau]^5}\right)t'[\tau]^2-$$

$$\frac{2\,\alpha^2\,\mathrm{Cos}[\theta[\tau]]\,\mathrm{Sin}[\theta[\tau]]\,r'[\tau]\,\theta'[\tau]}{r[\tau]^2}+$$

$$\left(1-r[\tau]+\frac{\alpha^2\left(-\mathrm{Cos}[\theta[\tau]]^2-r[\tau]+\mathrm{Cos}[\theta[\tau]]^2\,r[\tau]\right)}{r[\tau]^2}\right)\theta'[\tau]^2-$$

$$\frac{\alpha\,(-1+r[\tau])\,\mathrm{Sin}[\theta[\tau]]^2\,t'[\tau]\,\phi'[\tau]}{r[\tau]^3}+\left(\mathrm{Sin}[\theta[\tau]]^2-r[\tau]\,\mathrm{Sin}[\theta[\tau]]^2+\frac{1}{2\,r[\tau]^3}\right.$$

$$\alpha^2\left(-2\,\mathrm{Cos}[\theta[\tau]]^2\,r[\tau]\,\mathrm{Sin}[\theta[\tau]]^2-2\,r[\tau]^2\,\mathrm{Sin}[\theta[\tau]]^2+2\,\mathrm{Cos}[\theta[\tau]]^2\right.$$

$$\left.\left.r[\tau]^2\,\mathrm{Sin}[\theta[\tau]]^2-\mathrm{Sin}[\theta[\tau]]^4+r[\tau]\,\mathrm{Sin}[\theta[\tau]]^4\right)\right)\phi'[\tau]^2+r''[\tau]==0$$

O3: $\frac{d^2x^2}{d\tau^2}=-\sum_{\mu=0}^{3}\sum_{\nu=0}^{3}\Gamma^2{}_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}$

$$\frac{\alpha^2\,\mathrm{Cos}[\theta[\tau]]\,\mathrm{Sin}[\theta[\tau]]\,r'[\tau]^2}{(-1+r[\tau])\,r[\tau]^3}-$$

$$\frac{\alpha^2\,\mathrm{Cos}[\theta[\tau]]\,\mathrm{Sin}[\theta[\tau]]\,t'[\tau]^2}{r[\tau]^5}+2\left(-\frac{\alpha^2\,\mathrm{Cos}[\theta[\tau]]^2}{r[\tau]^3}+\frac{1}{r[\tau]}\right)r'[\tau]\,\theta'[\tau]-$$

$$\frac{\alpha^2\,\mathrm{Cos}[\theta[\tau]]\,\mathrm{Sin}[\theta[\tau]]\,\theta'[\tau]^2}{r[\tau]^2}+\frac{2\,\alpha\,\mathrm{Cos}[\theta[\tau]]\,\mathrm{Sin}[\theta[\tau]]\,t'[\tau]\,\phi'[\tau]}{r[\tau]^3}+$$

$$\left(-\mathrm{Cos}[\theta[\tau]]\,\mathrm{Sin}[\theta[\tau]]+\frac{1}{r[\tau]^3}\alpha^2\left(-\mathrm{Cos}[\theta[\tau]]\,r[\tau]\,\mathrm{Sin}[\theta[\tau]]+\right.\right.$$

$$\left.\left.\mathrm{Cos}[\theta[\tau]]^3\,r[\tau]\,\mathrm{Sin}[\theta[\tau]]-2\,\mathrm{Cos}[\theta[\tau]]\,\mathrm{Sin}[\theta[\tau]]^3\right)\right)\phi'[\tau]^2+\theta''[\tau]==0$$

O4: $\frac{d^2x^3}{d\tau^2}=-\sum_{\mu=0}^{3}\sum_{\nu=0}^{3}\Gamma^3{}_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}$

$$\frac{\alpha\,r'[\tau]\,t'[\tau]}{(-1+r[\tau])\,r[\tau]^3}-\frac{2\,\alpha\,\mathrm{Cot}[\theta[\tau]]\,t'[\tau]\,\theta'[\tau]}{r[\tau]^3}+$$

$$2\left(\frac{1}{r[\tau]}+\frac{1}{2\,(-1+r[\tau])\,r[\tau]^3}\alpha^2\left(2-2\,\mathrm{Cos}[\theta[\tau]]^2+2\,\mathrm{Cos}[2\,\theta[\tau]]-4\,r[\tau]+\right.\right.$$

$$\left.\left.4\,\mathrm{Cos}[\theta[\tau]]^2\,r[\tau]-2\,\mathrm{Cos}[2\,\theta[\tau]]\,r[\tau]-\mathrm{Sin}[\theta[\tau]]^2\right)\right)r'[\tau]\,\phi'[\tau]+$$

$$2\left(\mathrm{Cot}[\theta[\tau]]+\frac{\alpha^2\left(\mathrm{Cot}[\theta[\tau]]-\mathrm{Cos}[2\,\theta[\tau]]\,\mathrm{Cot}[\theta[\tau]]\right)}{2\,r[\tau]^3}\right)\theta'[\tau]\,\phi'[\tau]+\phi''[\tau]==0$$

The transformed-Kerr energy equation (left side) as series in α, with Christoffel symbols $\Gamma^{\kappa}{}_{\mu\nu}$ and coordinates $x^{\mu}=(t, r, \theta, \phi)$ is E1d=

$$
\begin{aligned}
&-1+\frac{\left(\alpha^2 (-i+\mu)^2 \mathrm{Cos}[\theta[\tau]]^2-r[\tau]^2\right) r'[\tau]^2}{(-i+\mu)^2 \left(\alpha^2 (-i+\mu)^2+r[\tau]+i \mu\, r[\tau]-r[\tau]^2\right)}+\\
&\left(1-\frac{i (-i+\mu)\, r[\tau]}{-\alpha^2 (-i+\mu)^2 \mathrm{Cos}[\theta[\tau]]^2+r[\tau]^2}\right) t'[\tau]^2+\\
&\left(-\alpha^2 \mathrm{Cos}[\theta[\tau]]^2+\frac{r[\tau]^2}{(-i+\mu)^2}\right) \theta'[\tau]^2+\frac{2 \alpha (1+i \mu)\, r[\tau] \mathrm{Sin}[\theta[\tau]]^2 t'[\tau] \phi'[\tau]}{-\alpha^2 (-i+\mu)^2 \mathrm{Cos}[\theta[\tau]]^2+r[\tau]^2}-\\
&\mathrm{Sin}[\theta[\tau]]^2 \left(\alpha^2+\frac{r[\tau]^2}{(1+i \mu)^2}+\frac{\alpha^2 (1+i \mu)\, r[\tau] \mathrm{Sin}[\theta[\tau]]^2}{-\alpha^2 (-i+\mu)^2 \mathrm{Cos}[\theta[\tau]]^2+r[\tau]^2}\right) \phi'[\tau]^2
\end{aligned}
$$

An the orbit equations (left side) O1d…O4d

O1d=

$$
\begin{aligned}
&-\Big(\Big(4 i (-i+\mu) \left(\alpha^2 (-i+\mu)^2-r[\tau]^2\right) \left(\alpha^2 (-i+\mu)^2 \mathrm{Cos}[\theta[\tau]]^2+r[\tau]^2\right) r'[\tau]\, t'[\tau]\Big)\Big/\\
&\quad\Big(\left(\alpha^2 (-i+\mu)^2+\alpha^2 (-i+\mu)^2 \mathrm{Cos}[2 \theta[\tau]]-2 r[\tau]^2\right)^2\\
&\qquad\left(-\alpha^2 (-i+\mu)^2+(-1-i \mu)\, r[\tau]+r[\tau]^2\right)\Big)\Big)+\\
&\frac{4 i \alpha^2 (-i+\mu)^3 r[\tau] \mathrm{Sin}[2 \theta[\tau]]\, t'[\tau]\, \theta'[\tau]}{\left(\alpha^2 (-i+\mu)^2+\alpha^2 (-i+\mu)^2 \mathrm{Cos}[2 \theta[\tau]]-2 r[\tau]^2\right)^2}-\\
&\Big(4 i \alpha (-i+\mu)\\
&\quad\left(-\alpha^2 (-i+\mu)^2 \mathrm{Cos}[\theta[\tau]]^2 \left(\alpha^2 (-i+\mu)^2+r[\tau]^2\right)+r[\tau]^2 \left(-\alpha^2 (-i+\mu)^2+3 r[\tau]^2\right)\right)\\
&\quad\mathrm{Sin}[\theta[\tau]]^2 r'[\tau]\, \phi'[\tau]\Big)\Big/\Big(\left(\alpha^2 (-i+\mu)^2+\alpha^2 (-i+\mu)^2 \mathrm{Cos}[2 \theta[\tau]]-2 r[\tau]^2\right)^2\\
&\quad\left(-\alpha^2 (-i+\mu)^2+(-1-i \mu)\, r[\tau]+r[\tau]^2\right)\Big)-\\
&\frac{8 i \alpha^3 (-i+\mu)^3 \mathrm{Cos}[\theta[\tau]]\, r[\tau] \mathrm{Sin}[\theta[\tau]]^3 \theta'[\tau]\, \phi'[\tau]}{\left(\alpha^2 (-i+\mu)^2+\alpha^2 (-i+\mu)^2 \mathrm{Cos}[2 \theta[\tau]]-2 r[\tau]^2\right)^2}+\\
&t''[\tau]
\end{aligned}
$$

O2d=

$$
\begin{aligned}
&-\Big(\Big(i (-i+\mu) \left(\alpha^2 (-i+\mu) \mathrm{Cos}[\theta[\tau]]^2 (-i+\mu+2 i\, r[\tau])+r[\tau] \left(-2 i \alpha^2 (-i+\mu)+r[\tau]\right)\right)\\
&\qquad r'[\tau]^2\Big)\Big/\\
&\quad\Big(2 \left(-\alpha^2 (-i+\mu)^2 \mathrm{Cos}[\theta[\tau]]^2+r[\tau]^2\right) \left(-\alpha^2 (-i+\mu)^2+(-1-i \mu)\, r[\tau]+r[\tau]^2\right)\Big)\Big)-\\
&\Big(i (-i+\mu)^3 \left(\alpha^2 (-i+\mu)^2 \mathrm{Cos}[\theta[\tau]]^2+r[\tau]^2\right) \left(-\alpha^2 (-i+\mu)^2+(-1-i \mu)\, r[\tau]+r[\tau]^2\right)\\
&\quad t'[\tau]^2\Big)\Big/\Big(2 \left(-\alpha^2 (-i+\mu)^2 \mathrm{Cos}[\theta[\tau]]^2+r[\tau]^2\right)^3\Big)+\\
&\frac{2 \alpha^2 \mathrm{Cos}[\theta[\tau]] \mathrm{Sin}[\theta[\tau]]\, r'[\tau]\, \theta'[\tau]}{-\alpha^2 \mathrm{Cos}[\theta[\tau]]^2+\frac{r[\tau]^2}{(-i+\mu)^2}}-\\
&\frac{r[\tau] \left(-\alpha^2 (-i+\mu)^2+(-1-i \mu)\, r[\tau]+r[\tau]^2\right) \theta'[\tau]^2}{-\alpha^2 (-i+\mu)^2 \mathrm{Cos}[\theta[\tau]]^2+r[\tau]^2}+\\
&\Big(i \alpha (-i+\mu)^3 \left(\alpha^2 (-i+\mu)^2 \mathrm{Cos}[\theta[\tau]]^2+r[\tau]^2\right) \left(-\alpha^2 (-i+\mu)^2+(-1-i \mu)\, r[\tau]+r[\tau]^2\right)\\
&\quad\mathrm{Sin}[\theta[\tau]]^2 t'[\tau]\, \phi'[\tau]\Big)\Big/\left(-\alpha^2 (-i+\mu)^2 \mathrm{Cos}[\theta[\tau]]^2+r[\tau]^2\right)^3+\\
&\Bigg(\left(\alpha^2 (-i+\mu)^2+r[\tau]+i \mu\, r[\tau]-r[\tau]^2\right) \mathrm{Sin}[\theta[\tau]]^2 \Big(2 \alpha^4 (-i+\mu)^4 \mathrm{Cos}[\theta[\tau]]^4 r[\tau]+\\
&\qquad\alpha^2 \mathrm{Cos}[\theta[\tau]]^2 \left(-4 (-i+\mu)^2 r[\tau]^3+\alpha^2 \left(1+5 i \mu+5 \mu^4+i \mu^5\right) \mathrm{Sin}[\theta[\tau]]^2\right)+\\
&\qquad\frac{1}{2} \left(4 r[\tau]^5+2 \alpha^2 (-1-i \mu)^3 r[\tau]^2 \mathrm{Sin}[\theta[\tau]]^2-5 i \alpha^4 \mu^2 (-i+\mu) \mathrm{Sin}[2 \theta[\tau]]^2\right)\Big)\\
&\quad\phi'[\tau]^2\Bigg)\Big/\left(2 \left(-\alpha^2 (-i+\mu)^2 \mathrm{Cos}[\theta[\tau]]^2+r[\tau]^2\right)^3\right)+r''[\tau]
\end{aligned}
$$

O3d=

$$-\frac{\alpha^2\,(-i+\mu)^2\,\mathrm{Sin}[2\,\theta[\tau]]\,r'[\tau]^2}{2\left(-\alpha^2\,(-i+\mu)^2\,\mathrm{Cos}[\theta[\tau]]^2+r[\tau]^2\right)\left(-\alpha^2\,(-i+\mu)^2+(-1-i\,\mu)\,r[\tau]+r[\tau]^2\right)}-$$

$$\frac{i\,\alpha^2\,(-i+\mu)^5\,\mathrm{Cos}[\theta[\tau]]\,r[\tau]\,\mathrm{Sin}[\theta[\tau]]\,t'[\tau]^2}{\left(-\alpha^2\,(-i+\mu)^2\,\mathrm{Cos}[\theta[\tau]]^2+r[\tau]^2\right)^3}+$$

$$\frac{2\,r[\tau]\,r'[\tau]\,\theta'[\tau]}{-\alpha^2\,(-i+\mu)^2\,\mathrm{Cos}[\theta[\tau]]^2+r[\tau]^2}+\frac{\alpha^2\,\mathrm{Cos}[\theta[\tau]]\,\mathrm{Sin}[\theta[\tau]]\,\theta'[\tau]^2}{-\alpha^2\,\mathrm{Cos}[\theta[\tau]]^2+\frac{r[\tau]^2}{(-i+\mu)^2}}-$$

$$\frac{2\,i\,\alpha\,(-i+\mu)^3\,\mathrm{Cos}[\theta[\tau]]\,r[\tau]\left(-\alpha^2\,(-i+\mu)^2+r[\tau]^2\right)\mathrm{Sin}[\theta[\tau]]\,t'[\tau]\,\phi'[\tau]}{\left(-\alpha^2\,(-i+\mu)^2\,\mathrm{Cos}[\theta[\tau]]^2+r[\tau]^2\right)^3}+$$

$$\left(\mathrm{Cos}[\theta[\tau]]\,\mathrm{Sin}[\theta[\tau]]\left(\alpha^2+\frac{r[\tau]^2}{(1+i\,\mu)^2}+\right.\right.$$

$$\left.\frac{2\,\alpha^2\,(1+i\,\mu)\,r[\tau]\,\mathrm{Sin}[\theta[\tau]]^2}{-\alpha^2\,(-i+\mu)^2\,\mathrm{Cos}[\theta[\tau]]^2+r[\tau]^2}+\frac{\alpha^4\,r[\tau]\,\mathrm{Sin}[\theta[\tau]]^4}{(1+i\,\mu)\left(\alpha^2\,\mathrm{Cos}[\theta[\tau]]^2+\frac{r[\tau]^2}{(1+i\,\mu)^2}\right)^2}\right)$$

$$\left.\phi'[\tau]^2\right)\Big/\left(-\alpha^2\,\mathrm{Cos}[\theta[\tau]]^2+\frac{r[\tau]^2}{(-i+\mu)^2}\right)+\theta''[\tau]$$

O4d=

$$-\Big(\Big(4\,i\,\alpha\,(-i+\mu)^3\left(\alpha^2\,(-i+\mu)^2\,\mathrm{Cos}[\theta[\tau]]^2+r[\tau]^2\right)r'[\tau]\,t'[\tau]\Big)\Big/$$
$$\Big(\left(\alpha^2\,(-i+\mu)^2+\alpha^2\,(-i+\mu)^2\,\mathrm{Cos}[2\,\theta[\tau]]-2\,r[\tau]^2\right)^2$$
$$\left(-\alpha^2\,(-i+\mu)^2+(-1-i\,\mu)\,r[\tau]+r[\tau]^2\right)\Big)\Big)+$$

$$\frac{8\,\alpha\,(-1-i\,\mu)^3\,\mathrm{Cot}[\theta[\tau]]\,r[\tau]\,t'[\tau]\,\theta'[\tau]}{\left(\alpha^2\,(-i+\mu)^2+\alpha^2\,(-i+\mu)^2\,\mathrm{Cos}[2\,\theta[\tau]]-2\,r[\tau]^2\right)^2}+$$

$$\Big(4\,\Big(2\,\alpha^4\,(-i+\mu)^4\,\mathrm{Cos}[\theta[\tau]]^4\,r[\tau]+\alpha^2\,\mathrm{Cos}[\theta[\tau]]^2$$
$$\left(-2\,(-i+\mu)^2\,r[\tau]^2\,(-1-i\,\mu+2\,r[\tau])+\alpha^2\left(1+5\,i\,\mu+5\,\mu^4+i\,\mu^5\right)\mathrm{Sin}[\theta[\tau]]^2\right)+$$
$$\frac{1}{2}\,\Big(4\,r[\tau]^4\,(-1-i\,\mu+r[\tau])+2\,\alpha^2\,(-1-i\,\mu)^3\,r[\tau]^2\,\mathrm{Sin}[\theta[\tau]]^2-$$
$$5\,i\,\alpha^4\,\mu^2\,(-i+\mu)\,\mathrm{Sin}[2\,\theta[\tau]]^2\Big)\Big)\,r'[\tau]\,\phi'[\tau]\Big)\Big/$$
$$\Big(\left(\alpha^2\,(-i+\mu)^2+\alpha^2\,(-i+\mu)^2\,\mathrm{Cos}[2\,\theta[\tau]]-2\,r[\tau]^2\right)^2$$
$$\left(-\alpha^2\,(-i+\mu)^2+(-1-i\,\mu)\,r[\tau]+r[\tau]^2\right)\Big)+$$

$$\Big(8\,\mathrm{Cot}[\theta[\tau]]\,\Big(\alpha^4\,(-i+\mu)^4\,\mathrm{Cos}[\theta[\tau]]^4\left(-\alpha^2\,(-i+\mu)^2+r[\tau]^2\right)+$$
$$\alpha^2\,(-i+\mu)\,\mathrm{Cos}[\theta[\tau]]^2\,r[\tau]\,\Big(-5\,\alpha^2\,\mu\left(-1-i\,\mu+\mu^2\right)+\alpha^2\left(-i-\mu+i\,\mu^2+\mu^3-i\,\mu^4\right)$$
$$\mathrm{Cos}[2\,\theta[\tau]]+2\,\alpha^2\,(-i+\mu)^3\,r[\tau]+i\,(-i+\mu)^2\,r[\tau]^2-2\,(-i+\mu)\,r[\tau]^3\Big)+$$
$$\frac{1}{2}\,r[\tau]\,\Big(2\,r[\tau]^2\,(-1-i\,\mu+r[\tau])\left(-\alpha^2\,(-i+\mu)^2+r[\tau]^2\right)-$$
$$2\,i\,\alpha^2\,(-i+\mu)^3\,r[\tau]\,(-1-i\,\mu+2\,r[\tau])\,\mathrm{Sin}[\theta[\tau]]^2+2\,\alpha^4\,(1+i\,\mu)^5\,\mathrm{Sin}[\theta[\tau]]^4+$$
$$5\,\alpha^4\,\mu\left(i-2\,\mu-2\,i\,\mu^2+\mu^3\right)\mathrm{Sin}[2\,\theta[\tau]]^2\Big)\Big)\,\theta'[\tau]\,\phi'[\tau]\Big)\Big/$$
$$\Big(\left(\alpha^2\,(-i+\mu)^2+\alpha^2\,(-i+\mu)^2\,\mathrm{Cos}[2\,\theta[\tau]]-2\,r[\tau]^2\right)^2$$
$$\left(-\alpha^2\,(-i+\mu)^2+(-1-i\,\mu)\,r[\tau]+r[\tau]^2\right)\Big)+\phi''[\tau]$$

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