Why are odd Bernoulli numbers equal to zero?*
(Extended Version)

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Abstract

In this note we will use Faulhaber's Formula to explain why the odd Bernoulli numbers are equal to zero.

1 Introduction

For odd numbers greater than or equal to seven, why are the Bernoulli numbers equal to zero? Because Faulhaber's Formula tells us that $\sum_{k=1}^{n} k^{2m+1}$ is a polynomial in $\left(\sum_{k=1}^{n} k\right)^2$, and $\left(\sum_{k=1}^{n} k\right)^2 = \frac{n^2+2n^3+n^4}{4}$.

2 Faulhaber's Formula

We might know already

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2 = \left(\sum_{k=1}^{n} k\right)^2.$$

†Updated on 11 August 2018 to correct typos and make a minor change.

Through inductive reasoning like that in [6] we might discover further

$$
\begin{aligned}
2^1 \cdot \left(\frac{n(n+1)}{2}\right)^2 &= \binom{2}{1} \cdot \sum k^3 \\
2^2 \cdot \left(\frac{n(n+1)}{2}\right)^3 &= \binom{3}{0} \cdot \sum k^3 + \binom{3}{2} \cdot \sum k^5 \\
2^3 \cdot \left(\frac{n(n+1)}{2}\right)^4 &= \binom{4}{1} \cdot \sum k^5 + \binom{4}{3} \cdot \sum k^7 \\
2^4 \cdot \left(\frac{n(n+1)}{2}\right)^5 &= \binom{5}{0} \cdot \sum k^5 + \binom{5}{2} \cdot \sum k^7 + \binom{5}{4} \cdot \sum k^9 \\
2^5 \cdot \left(\frac{n(n+1)}{2}\right)^6 &= \binom{6}{1} \cdot \sum k^7 + \binom{6}{3} \cdot \sum k^9 + \binom{6}{5} \cdot \sum k^{11}.
\end{aligned} \tag{1}
$$

(We abbreviate $\sum_{k=1}^{n} k^m$ by $\sum k^m$.) The general case is

$$
\begin{aligned}
2^{m-1} \cdot \left(\frac{n(n+1)}{2}\right)^m \quad = \quad & \binom{m}{0} \cdot \sum k^m + \binom{m}{2} \cdot \sum k^{m+2} \qquad (2) \\
& + \cdots + \binom{m}{m-3} \cdot \sum k^{2m-3} + \binom{m}{m-1} \cdot \sum k^{2m-1}
\end{aligned}
$$

or

$$
\begin{aligned}
2^{m-1} \cdot \left(\frac{n(n+1)}{2}\right)^m \quad = \quad & \binom{m}{1} \cdot \sum k^{m+1} + \binom{m}{3} \cdot \sum k^{m+3} \qquad (3) \\
& + \cdots + \binom{m}{m-3} \cdot \sum k^{2m-3} + \binom{m}{m-1} \cdot \sum k^{2m-1}.
\end{aligned}
$$

If we wish to prove such expressions, by [1, 2, 3] we may proceed using Pascal's observation of telescoping sums. Consider the special case of

$$
\begin{aligned}
\left(\frac{3 \cdot 4}{2}\right)^m \quad = \quad & \left(\frac{1 \cdot 2}{2}\right)^m - 0 + \left(\frac{2 \cdot 3}{2}\right)^m - \left(\frac{1 \cdot 2}{2}\right)^m + \left(\frac{3 \cdot 4}{2}\right)^m - \left(\frac{2 \cdot 3}{2}\right)^m \\
= & \sum_{k=1}^{3} \left[\left(\frac{k(k+1)}{2}\right)^m - \left(\frac{(k-1)k}{2}\right)^m \right].
\end{aligned}
$$

A general expression like

$$
\left(\frac{n(n+1)}{2}\right)^m = \sum_{k=1}^{n} \left[\left(\frac{k(k+1)}{2}\right)^m - \left(\frac{(k-1)k}{2}\right)^m \right]
$$

we may rewrite as (2) or (3), and then in a proof by mathematical induction we may pass from n to $n+1$.

What if we want to rewrite such expressions in terms of a particular $\sum k^{2m+1}$? For example, by looking at (1) we may write

$$\begin{aligned}\sum k^5 &= \frac{1}{\binom{3}{2}} \cdot \left[2^2 \cdot \left(\frac{n(n+1)}{2}\right)^3 - \binom{3}{0} \cdot \sum k^3\right] \\ &= \frac{1}{3} \cdot \left[4 \cdot \frac{n(n+1)}{2} - 1\right] \cdot \left(\sum k\right)^2,\end{aligned}$$

which then implies

$$\begin{aligned}\sum k^7 &= \frac{1}{\binom{4}{3}} \cdot \left[2^3 \cdot \left(\frac{n(n+1)}{2}\right)^4 - \binom{4}{1} \cdot \sum k^5\right] \\ &= \frac{1}{4} \cdot \left[2^3 \cdot \left(\frac{n(n+1)}{2}\right)^2 - 4 \cdot \frac{1}{3} \cdot \left(4 \cdot \frac{n(n+1)}{2} - 1\right)\right] \cdot \left(\sum k\right)^2.\end{aligned}$$

Well, by another proof by mathematical induction, this time on the m of (2) or (3), we may arrive at the general case of

$$\begin{aligned}\sum k^{2m+1} &= \frac{1}{m+1} \cdot \left[2^m \cdot \left(\frac{n(n+1)}{2}\right)^{m-1} - a_2 \cdot \left(\frac{n(n+1)}{2}\right)^{m-2} + a_3 \cdot \left(\frac{n(n+1)}{2}\right)^{m-3}\right. \\ &\left.\mp \cdots \mp a_{m-2} \cdot \left(\frac{n(n+1)}{2}\right)^2 \mp a_{m-1} \cdot \frac{1}{3} \cdot \left(4 \cdot \frac{n(n+1)}{2} - 1\right)\right] \cdot \left(\sum k\right)^2, \quad (4)\end{aligned}$$

where the a_i are rational numbers and $m \geqq 3$. This relationship we will call Faulhaber's Formula. (For some of the history of the problem, see [1, 2, 3, 4].)

3 Bernoulli Numbers

By [1, 2, 4, 5] we may define the Bernoulli numbers B_n by

$$B_0 = 1, \quad \sum_{k=0}^{n} \binom{n+1}{k} \cdot B_k = 0,$$

where $n \geqq 1$. For example, to find B_1 we write

$$\sum_{k=0}^{1} \binom{1+1}{k} \cdot B_k = \binom{2}{0} \cdot B_0 + \binom{2}{1} \cdot B_1 = 0,$$

which implies $B_1 = -\frac{1}{2}$. It turns out $B_3 = B_5 = 0$. With a bit of work we can establish

$$\begin{aligned}\sum k^{2m+1} &= \frac{1}{2m+2} \cdot \left[\binom{2m+2}{0} \cdot B_0 \cdot n^{2m+2} - \binom{2m+2}{1} \cdot B_1 \cdot n^{2m+1}\right. \\ &\left.\pm \cdots + \binom{2m+2}{2m} \cdot B_{2m} \cdot n^2 - \binom{2m+2}{2m+1} \cdot B_{2m+1} \cdot n\right], \quad (5)\end{aligned}$$

for which we will assume $m \geqq 3$.

4 Conclusion

With regard to the claim at the start, suppose we set (4) and (5) equal to one another. If we multiply out (4), we see it does not contain the term n. That means the last term of (5),

$$-\frac{1}{2m+2} \cdot \binom{2m+2}{2m+1} \cdot B_{2m+1} \cdot n,$$

must be equal to zero. In other words, for all $m \geqq 3$, $B_{2m+1} = 0$.

References

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