Momentum conservation in electromagnetic systems

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Abstract

Newton's third law doesn't apply to electromagnetic systems. Nevertheless a relativistic dissertation, directly founded on Maxwell's equations and on relativistic dynamics, allows us to establish rigorously the law of total momentum for such systems. Some undervalued details about the role of internal forces in isolated systems are emphasized. The laws governing momentum in systems subject to electromagnetic forces are consistent in every situation. There are no reasons to postulate the existence of a hidden momentum to avoid non-existent paradoxes in the case of static fields.

1 Mechanical momentum

In special relativity (SR) the linear momentum of a particle of proper mass m_i moving, relatively to an inertial frame of reference, with velocity $\boldsymbol{u_i}$ is $\boldsymbol{p_i} = \gamma(u_i)m_i\boldsymbol{u_i}$. Therefore the *total mechanical momentum* of a many particles system is:

$$\boldsymbol{Q_m} = \sum_i \boldsymbol{p_i} = \sum_i \gamma(u_i)m_i\boldsymbol{u_i} \tag{1}$$

A *baricentric system of reference* always exists, in which the total momentum vanishes ($\boldsymbol{Q_m} = 0$). However the notion of center of gravity G, as a point (having its own motion) associated with the set of bodies, is not significant in SR [1].
The equation of motion $\boldsymbol{f_i} = d\boldsymbol{p_i}/dt$ is applicable to all particles of a many bodies system; therefore also in SR the fundamental equation of system-dynamics may be written as:

$$\sum_i \boldsymbol{f_i} = \boldsymbol{R} = \frac{d\boldsymbol{Q_m}}{dt} \tag{2}$$

But since in SR and in electromagnetism *Newton's third law doesn't apply*, we cannot state that the sum of internal forces acting in a system vanishes, so the resultant $\boldsymbol{R}$ must include all *internal and external* forces acting on the particles:

$$\boldsymbol{R} = \boldsymbol{R_{ext}} + \boldsymbol{R_{int}} \tag{3}$$

[1]A definition as $\sum(\gamma_i m_i)\,\boldsymbol{r_G} = \sum(\gamma_i m_i \boldsymbol{r_i})$ for the center of relativistic mass (or "*center of total energy*") of a system of material bodies wouldn't be Lorentz-invariant, because it implies the *simultaneous* knowledge of the positions $\boldsymbol{r_i}$ of all the individual points.

Of course this observation doesn't refer to the case of direct contact (i.e. in case of *local interactions*). The sum of internal forces is always zero in collisions and the sum of external forces is assumed to be irrelevant; being $\boldsymbol{R} = 0$, the principle of conservation of momentum is obviously true for collisions problems:

$$\boldsymbol{Q_m} = \sum \boldsymbol{p_i} = \text{constant in collisions} \tag{4}$$

2 Electromagnetic tensor

The well-known expressions of electromagnetic (em) potentials are:

$$\boldsymbol{B} = \operatorname{rot} \boldsymbol{A} \qquad\qquad \boldsymbol{E} = -\operatorname{grad} V - \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} \qquad\qquad A_\mu = (V\,, -\boldsymbol{A}) \tag{5}$$

The curl of the four-vector em-potential A_μ fully identifies the em-field and is the so called *electromagnetic tensor*:

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha \tag{6}$$

The em-tensor $F_{\alpha\beta}$ is antisymmetric and its components are:

$$F_{\alpha\beta} = -F_{\beta\alpha} \qquad F_{0i} = E^i \qquad F_{kj} = \epsilon_{ijk} B^i \tag{7}$$

$$F_{\alpha\beta} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} \tag{8}$$

The em-force acting on a particle is $\boldsymbol{f} = e(\boldsymbol{E} + \frac{1}{c}\boldsymbol{u}\times\boldsymbol{B})$ and the four-force becomes:

$$F^\mu = \gamma(u)\begin{pmatrix} \frac{1}{c}\boldsymbol{f}\cdot\boldsymbol{u} \\ \boldsymbol{f} \end{pmatrix} = e\,\gamma(u)\begin{pmatrix} \frac{1}{c}\boldsymbol{E}\cdot\boldsymbol{u} \\ \boldsymbol{E} + \frac{1}{c}\boldsymbol{u}\times\boldsymbol{B} \end{pmatrix} \tag{9}$$

The four-force F_μ acting over a charged particle moving with four-velocity U^ν in an em-field described by the tensor $F_{\mu\nu}$ consequentely is:

$$F_\mu = \frac{e}{c} F_{\mu\nu} U^\nu \tag{10}$$

A more detailed exposition of such well-known results (and also of those of the next section) can be found in every good book of relativity or in treatises of electromagnetism such as Jackson [7].

3 EM energy-momentum-stress tensor

The *density of electromagnetic four-force* may be expressed in the form:

$$K^\beta = -\partial_\alpha \mathcal{E}^{\alpha\beta} \tag{11}$$

where $\mathcal{E}^{\alpha\beta}$ is the *electromagnetic energy-momentum-stress tensor*:

$$\mathcal{E}^{\alpha\beta} = -\frac{1}{4\pi}\left[g^{\mu\beta}F_{\mu\nu}F^{\alpha\nu} - \frac{1}{4}g^{\alpha\beta}F_{\rho\nu}F^{\rho\nu}\right]$$
$$\mathcal{E}^{\alpha\beta} = \begin{pmatrix} \omega & c\pi_x & c\pi_y & c\pi_z \\ c\pi_x & p^{11} & p^{12} & p^{13} \\ c\pi_y & p^{21} & p^{22} & p^{23} \\ c\pi_z & p^{31} & p^{32} & p^{33} \end{pmatrix} = \left(\begin{array}{c|c} \omega & c\boldsymbol{\pi} \\ \hline c\boldsymbol{\pi} & p^{ij} \end{array}\right) \tag{12}$$

The tensor is simmetric ($\mathcal{E}^{\alpha\beta} = \mathcal{E}^{\beta\alpha}$) and its components are:

$$\omega = \frac{\boldsymbol{E}^2 + \boldsymbol{B}^2}{8\pi} \qquad \textit{energy density} \tag{13a}$$

$$\boldsymbol{\pi} = \frac{\boldsymbol{E}\times\boldsymbol{B}}{4\pi c} = \frac{\boldsymbol{\Pi}}{c^2} \qquad \boldsymbol{\Pi} = \frac{c}{4\pi}\boldsymbol{E}\times\boldsymbol{B} \qquad \textit{Poynting vector} \tag{13b}$$

$$p^{ij} = p^{ji} = -\frac{1}{4\pi}[E_iE_j + B_iB_j - \frac{1}{2}\delta_{ij}(\boldsymbol{E}^2 + \boldsymbol{B}^2)] \qquad \textit{stress 3-tensor} \tag{13c}$$

4 Electromagnetic energy and momentum

Splitting the time-component from the tensorial Eq.(11) we obtain the continuity equation for the electromagnetic energy:

$$\boldsymbol{E}\cdot\boldsymbol{j} = -\frac{\partial\omega}{\partial t} - \operatorname{div}\boldsymbol{\Pi} \tag{14}$$

Splitting the space-components from the tensorial Eq.(11) we obtain the following vectorial dynamical law:

$$\rho\boldsymbol{E} + \frac{1}{c}\boldsymbol{j}\times\boldsymbol{B} = -\frac{\partial\boldsymbol{\pi}}{\partial t} - \partial_j p^{ij} \tag{15}$$

Now let V be a region of space, bounded by a closed surface S, containing charged particles in arbitrary motion and subject to electromagnetic forces only.
By integrating Eq.(14) over the volume V we obtain the *first law of thermodynamics* for an electromagnetic system.
Similarly, by integrating Eq.(15) over V we obtain the *law of electromagnetic momentum*, as follows.
The resulting electromagnetic force acting on such a physical system is:

$$\boldsymbol{R} = \iiint_V \left(\rho\boldsymbol{E} + \frac{1}{c}\boldsymbol{j}\times\boldsymbol{B}\right) dV \tag{16}$$

By integrating the first term in the right hand side of Eq.(15) we obtain:

$$\iiint_V \frac{\partial\boldsymbol{\pi}}{\partial t}dV = \frac{d}{dt}\iiint \frac{\boldsymbol{\Pi}}{c^2}dV = \frac{d\boldsymbol{Q_{em}}}{dt} \tag{17}$$

where $\boldsymbol{Q_{em}}$ denotes the *electromagnetic momentum*:

$$\boldsymbol{Q_{em}} = \iiint_V \frac{\boldsymbol{\Pi}}{c^2} dV \tag{18}$$

The volume integral of the second term may be transformed in a surface integral by means of the divergence theorem; denoting the outward normal to the surface element dS by the unit vector $\boldsymbol{n} = (\alpha_1, \alpha_2, \alpha_3)$ we have:

$$\iiint_V \frac{\partial p^{ji}}{\partial x^j} dV = \iint_S \sum_j p^{ij} \alpha_j dS = -F_i \tag{19}$$

where $\boldsymbol{F} = (F_x, F_y, F_z)$ is the *electromagnetic stresses resultant* acting (from outside to inside) onto the boundary surface S.
So by integrating Eq.(15) we obtain the *electromagnetic momentum law*:

$$\boldsymbol{R} = \boldsymbol{F} - \frac{d\boldsymbol{Q_{em}}}{dt} \tag{20}$$

For a system subject to electromagnetic forces only, this relation (directly obtained from the electromagnetic theory) has to be combined with Eq.(2), that expresses the *mechanical momentum law* associated with the masses:

$$\boldsymbol{R} = \frac{d\boldsymbol{Q_m}}{dt} \tag{21}$$

and it's worth recalling that $\boldsymbol{R}$ is the *resultant of all the forces, both external and internal,* acting on the considered physical system.
Combining the laws Eq.(20) - (21) we get the *law of total momentum*:

$$\boldsymbol{F} = \frac{d}{dt}(\boldsymbol{Q_m} + \boldsymbol{Q_{em}}) \tag{22}$$

This result justifies the name of *electromagnetic momentum* attributed to $\boldsymbol{Q_{em}}$ and that of *total momentum* of a physical system attributed to the sum:

$$\boldsymbol{Q} = \boldsymbol{Q_m} + \boldsymbol{Q_{em}} \tag{23}$$

5 Polarization and magnetization of the medium

The discussion above refers to electromagnetic fields $(\boldsymbol{E}, \boldsymbol{B})$ *in vacuum* if ρ and $\boldsymbol{J}$ are defined as the macroscopic free sources ρ_{free} and $\boldsymbol{J}_{free}$ of the system. It's worth remembering that Maxwell's equations in a medium are obtained from equations in vacuum by adding the sources due to polarization:

$$\rho \;\rightarrow\; \rho_{free} + \rho_{pol} \qquad \boldsymbol{J} \;\rightarrow\; \boldsymbol{J}_{free} + \boldsymbol{J}_{pol} \tag{24}$$

Therefore the previous results remain true also in a medium, provided that with $\boldsymbol{Q_{em}}$ we mean the *total em-momentum*, including not only the field-momentum $\boldsymbol{Q}_{field}$ but also the momentum $\boldsymbol{Q}_{pol}$ generated by the polarized medium:

$$\boldsymbol{Q_{em}} = \boldsymbol{Q}_{field} + \boldsymbol{Q}_{pol} \tag{25}$$

The em-momentum of an electric dipole in a magnetic field and the em-momentum of a magnetic dipole in an electric field are discussed in [3] and in many other papers. Generally speaking such questions are of theoretical interest, but the decomposition Eq.(25) doesn't seem to affect our conclusions.
The quantities ρ_{pol} and $\boldsymbol{J}_{pol}$ are usually unknown and can be removed from the field equations by means of the relations inferred from the polarization theory:

$$\rho_{pol} = -\,\mathrm{div}\,\boldsymbol{P} \qquad \boldsymbol{J}_{pol} = c\;\mathrm{rot}\,\boldsymbol{M} \tag{26}$$

These suggest, as well known, the definitions of two new vectorial fields:

$$\boldsymbol{D} = \boldsymbol{E} + 4\pi\boldsymbol{P} \qquad \boldsymbol{H} = \boldsymbol{B} - 4\pi\boldsymbol{M} \tag{27}$$

In this way Maxwell's equations in a medium are usually written by means of the following four vectorial fields: $\boldsymbol{E}, \boldsymbol{B}, \boldsymbol{D}, \boldsymbol{H}$.
It's easy to see that in a non-dispersive medium the density of em-momentum is:

$$\boldsymbol{\pi} = \frac{\boldsymbol{\Pi}}{c^2} = \frac{\boldsymbol{E}\times\boldsymbol{H}}{4\pi c} \tag{28}$$

In articles [4]-[5] it is possible to find a in-depth discussion of how various expressions proposed for $\boldsymbol{\pi}$ are associated with different possible concepts of em-momentum and of hidden momentum. The different dissertations correspond to the different ways of splitting of the total em-momentum Eq.(25) into electromagnetic and material ones.
Using Maxwell's equations for the four fields $\boldsymbol{E}, \boldsymbol{B}, \boldsymbol{D}, \boldsymbol{H}$ would be useless and too heavy for our purposes. Fortunately in our discussion on the observable effects of internal forces we don't need to distinguish between the two terms of Eq.(25): with $\boldsymbol{Q}_{em}$ we mean always the total electromagnetic momentum, including polarization effects. In the present paper ρ and $\boldsymbol{J}$ have the meaning specified in Eq.(24), i.e. they include all sources, both free sources and polarization sources of the medium. That greatly simplifies the relativistic handling, without penalizing the logic and the conclusions of the work, as we are going to see.

6 Isolated systems

All relations previously written for a systems subject only to forces of electromagnetic nature are strictly true in classical (relativistic) physics. Unlike quantum mechanics and its theoretical developements, *classical physics* is characterized by a description of phenomena based on separeted concepts of *matter* and *radiation* (electromagnetic fields). Special relativity (SR) must be considered an integral part of classical physics, since only by this paradigm mechanics and electromagnetism form a coherent scheme.
Let's now discuss the formulation of the principle of momentum conservation for an isolated system subject to electromagnetic forces only. Such a system includes the sources (charges and currents) of the electromagnetic fields present in it. For

an isolated system it is possible to take $\boldsymbol{F} = 0$, provided that the border S is chosen far enough from the sources of the system, so that the em-stresses on it can be considered negligible. Indeed the coefficients p^{ij} of the stress 3-tensor are of order $1/r^4$ and the surface of integration in Eq.(19) is of order r^2, consequently $\boldsymbol{F}$ decreases as $1/r^2$. In this way from Eq.(22) we derive the *total momentum conservation principle* for an isolated physical system in which exclusively occur electromagnetic forces:

$$\frac{d}{dt}(\boldsymbol{Q_m} + \boldsymbol{Q_{em}}) = 0 \;\rightarrow\; \boldsymbol{Q} = \boldsymbol{Q_m} + \boldsymbol{Q_{em}} = \text{constant} \tag{29}$$

This isn't clearly an original result, but here we want to underline the rigorous logic sequence that led us to write it. In an isolated system the resultant of external forces vanishes $\boldsymbol{R_{ext}} = 0$, so $\boldsymbol{R} = \boldsymbol{R_{int}}$ reduces itself to be the resultant of the internal forces alone. *The fundamental laws of the mechanical momentum Eq.*(21) *and of the em-momentum Eq.*(20) *for an isolated system therefore take the form:*

$$\boldsymbol{R_{int}} = \frac{d\boldsymbol{Q_m}}{dt} \qquad\qquad \boldsymbol{R_{int}} = -\frac{d\boldsymbol{Q_{em}}}{dt} \tag{30}$$

These relations don't imply only the principle of conservation of total momentum Eq.(29). They show also that, accordingly to the classical physics, if for any reason it is observed a conversion of momentum $\Delta\boldsymbol{Q_m} \leftrightarrows \Delta\boldsymbol{Q_{em}}$ one must have:

$$\boldsymbol{R_{int}}\Delta t = \Delta\boldsymbol{Q_m} = -\Delta\boldsymbol{Q_{em}} \tag{31}$$

In an isolated system each conversion of electromagnetic momentum into mechanical momentum (or vice versa) is always associated with the action of an equivalent impulse operated by a non null resultant of the internal electromagnetic [2] *forces.*
An isolated physical system mechanically *at rest* , in which there are *electrical and magnetic stationary fields* generated by sources located inside the system (e.g. by a parallel-plate capacitor and a steady current flowing in a solenoid), will generally have a non null electromagnetic momentum ($\boldsymbol{Q_{em}} \neq 0$). Somebody considers "strange" such a circumstance (a "psychologism" would say Popper!), that instead should appear obvious when it is considered that the total momentum of a system at rest without any electromagnetic fields must be null: $\boldsymbol{Q} = \boldsymbol{Q_m} + \boldsymbol{Q_{em}} = 0$.
As emphasized, the creation of an electromagnetic momentum $\boldsymbol{Q_{em}}$ implies the action of an internal impulse and therefore also the generation of an equal and opposite mechanical momentum:

$$\int \boldsymbol{R_{int}}\, dt = -\boldsymbol{Q_{em}} = \boldsymbol{Q_m} \tag{32}$$

If the physical system has to be kept at rest, even after the *activation of its internal sources*, the internal forces have to be balanced with appropriate external forces,

[2]This conclusion remains true if inside the system a current generator is acting with non electrical forces on the charges. Indeed such local interactions satisfy Newton's third law and give therefore a null-contribution to the resultant $\boldsymbol{R_{int}}$ of the internal forces acting on the system.

such as to satisfy the condition:

$$\int \boldsymbol{R_{ext}}\, dt = \boldsymbol{Q_{em}} \tag{33}$$

In other words, the activation of *a non null electromagnetic momentum associated to the static fields necessarily requires the action of an equivalent impulse generated by external forces, designed to maintain the system at rest.*

7 Hidden momentum

Discussions about "hidden momentum" have had a revival following the publication on Am.J.Phys (2009) of an article by Babson, Reynolds, Bjornquist, Griffiths [1], in which it is stated that the total momentum of a physical system at rest has to be always null (even in presence of stationary electric and magnetic fields), thanks to an elusive relativistic "center of energy theorem". So it is needed to postulate the existence of a "hidden momentum", i.e. of some kind of additional momentum necessary to make ends meet.
You can't find traces of the center of energy theorem in any canonical relativity books. However many articles, accepting the hidden momentum theory, have recently appeared in literature. The concept of hidden momentum is illustrated by a lot of examples and formalized by sophisticated definitions: in [2] you can find a historical presentation, modern definitions and an extensive bibliography.
One of the few works against this fashion is an interesting article written by J.Franklin (2013) [3]; he concludes with the following textual words:
"Our conclusion is that the center of energy theorem does not apply to the EM momentum of a static charge-current distribution, and that hidden momentum is neither needed nor present in the charge-current distribution. The external force to keep matter at rest during the creation of charge-current distribution goes directly into EM momentum without moving any matter or hiding any momentum."
We have reached, in totally indipendent way, the same conclusions expressed by Franklin. But unlike [3] *we refuse Newton's third law and emphasize instead the role of internal forces.*
By means of the tensorial formalism we framed the principle of total momentum conservation Eq.(29) in a general rigorous concise theoretical dissertation, directly founded on Maxwell's equations and on relativistic dynamics. The fundamental law of electromagnetic momentum Eq.(15) is a consequence of Maxwell's equations and of the expression of the electromagnetic force: of course this isn't an original result, you can look it up e.g. on Jackson [7]. The four-dimensional tensorial form used guarantees the relativistic invariance of the result Eq.(20), which with the relativistic dynamic law Eq.(21) is the founding of our subsequent discussion of the dynamics of any mechanical-electromagnetic isolated system.
Since the general results obtained are valid for any isolated physical system in which only electromagnetic forces are acting, we haven't deliberately submitted any particular case: all examples available in literature, when properly discussed, must agree with the same general theoretical scheme.

8 Comments on an experimental challenge

The thesis of this paper is that the concept of "hidden momentum" seems to be useless (or sometimes the result of a hidden mistake) in the application of the laws of classical electromagnetism and relativistic dynamics.
An experimental discrimination between the theory developed in this article and the theory based on the hidden momentum should be possible, at least in principle. According to the theoretical dissertation carried out, in every physical system the activation or the deactivation of fields bearing an electromagnetic momentum is necessarely associated with impulses having observable mechanical motion effects, which should not occur if they were really balanced by a hidden momentum.
Given the smallness of the momentum quantities involved, one could try to perform such an experience by means of the low-thrust torsion balance used at NASA Johnson Space Center for the experiences described in the esoteric report [6]. It would however be necessary to change quite substantially the structure of the devices and the experimental procedure, designed formerly with the aim to observe a misterious *continuous* thrust, that would really violate the fundamental principles of classical physics. It might even be that the small results described in [6] have to be associated with the *impulsive internal forces* (we expect to observe) generated when turning on and off the EM-drive.

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