ON THE UNIVERSALITY OF SOME SMARANDACHE LOOPS OF BOL-MOUFANG TYPE *†

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Abstract

A Smarandache quasigroup(loop) is shown to be universal if all its f, g-principal isotopes are Smarandache f, g-principal isotopes. Also, weak Smarandache loops of Bol-Moufang type such as Smarandache: left(right) Bol, Moufang and extra loops are shown to be universal if all their f, g-principal isotopes are Smarandache f, g-principal isotopes. Conversely, it is shown that if these weak Smarandache loops of Bol-Moufang type are universal, then some autotopisms are true in the weak Smarandache sub-loops of the weak Smarandache loops of Bol-Moufang type relative to some Smarandache elements. Futhermore, a Smarandache left(right) inverse property loop in which all its f, g-principal isotopes are Smarandache f, g-principal isotopes is shown to be universal if and only if it is a Smarandache left(right) Bol loop in which all its f, g-principal isotopes are Smarandache f, g-principal isotopes. Also, it is established that a Smarandache inverse property loop in which all its f, g-principal isotopes are Smarandache f, g-principal isotopes is universal if and only if it is a Smarandache Moufang loop in which all its f, g-principal isotopes are Smarandache f, g-principal isotopes. Hence, some of the autotopisms earlier mentioned are found to be true in the Smarandache sub-loops of universal Smarandache: left(right) inverse property loops and inverse property loops.

1 Introduction

W. B. Vasantha Kandasamy initiated the study of Smarandache loops (S-loop) in 2002. In her book [27], she defined a Smarandache loop (S-loop) as a loop with at least a subloop

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§All correspondence to be addressed to this author which forms a subgroup under the binary operation of the loop called a Smarandache subloop (S-subloop). In [11], the present author defined a Smarandache quasigroup (S-quasigroup) to be a quasigroup with at least a non-trivial associative subquasigroup called a Smarandache subquasigroup (S-subquasigroup). Examples of Smarandache quasigroups are given in Muktibodh [21]. For more on quasigroups, loops and their properties, readers should check [24], [2],[4], [5], [8] and [27]. In her (W.B. Vasantha Kandasamy) first paper [28], she introduced Smarandache : left(right) alternative loops, Bol loops, Moufang loops, and Bruck loops. But in [10], the present author introduced Smarandache : inverse property loops (IPL), weak inverse property loops (WIPL), G-loops, conjugacy closed loops (CC-loop), central loops, extra loops, A-loops, K-loops, Bruck loops, Kikkawa loops, Burn loops and homogeneous loops. The isotopic invariance of types and varieties of quasigroups and loops described by one or more equivalent identities, especially those that fall in the class of Bol-Moufang type loops as first named by Fenyves [7] and [6] in the 1960s and later on in this 21^{st} century by Phillips and Vojtěchovský [25], [26] and [18] have been of interest to researchers in loop theory in the recent past. For example, loops such as Bol loops, Moufang loops, central loops and extra loops are the most popular loops of Bol-Moufang type whose isotopic invariance have been considered. Their identities relative to quasigroups and loops have also been investigated by Kunen [20] and [19]. A loop is said to be universal relative to a property $\mathcal{P}$ if it is isotopic invariant relative to $\mathcal{P}$, hence such a loop is called a universal $\mathcal{P}$ loop. This language is well used in [22]. The universality of most loops of Bol-Moufang types have been studied as summarised in [24]. Left(Right) Bol loops, Moufang loops, and extra loops have all been found to be isotopic invariant. But some types of central loops were shown to be universal in Jaíyéọlá [13] and [12] under some conditions. Some other types of loops such as A-loops, weak inverse property loops and cross inverse property loops (CIPL) have been found be universal under some neccessary and sufficient conditions in [3], [23] and [1] respectively. Recently, Michael Kinyon et. al. [16], [14], [15] solved the Belousov problem concerning the universality of F-quasigroups which has been open since 1967 by showing that all the isotopes of F-quasigroups are Moufang loops.

In this work, the universality of the Smarandache concept in loops is investigated. That is, will all isotopes of an S-loop be an S-loop? The answer to this could be 'yes' since every isotope of a group is a group (groups are G-loops). Also, the universality of weak Smarandache loops, such as Smarandache Bol loops (SBL), Smarandache Moufang loops (SML) and Smarandache extra loops (SEL) will also be investigated despite the fact that it could be expected to be true since Bol loops, Moufang loops and extra loops are universal. The universality of a Smarandache inverse property loop (SIPL) will also be considered.

2 Preliminaries

Definition 2.1 *A loop is called a Smarandache left inverse property loop (SLIPL) if it has at least a non-trivial subloop with the LIP.*

A loop is called a Smarandache right inverse property loop (SRIPL) if it has at least a non-trivial subloop with the RIP.

A loop is called a Smarandache inverse property loop (SIPL) if it has at least a non-trivial subloop with the IP.

A loop is called a Smarandache right Bol-loop (SRBL) if it has at least a non-trivial subloop that is a right Bol(RB)-loop.

A loop is called a Smarandache left Bol-loop (SLBL) if it has at least a non-trivial subloop that is a left Bol(LB)-loop.

A loop is called a Smarandache central-loop (SCL) if it has at least a non-trivial subloop that is a central-loop.

A loop is called a Smarandache extra-loop (SEL) if it has at least a non-trivial subloop that is a extra-loop.

A loop is called a Smarandache A-loop (SAL) if it has at least a non-trivial subloop that is a A-loop.

A loop is called a Smarandache Moufang-loop (SML) if it has at least a non-trivial subloop that is a Moufang-loop.

Definition 2.2 *Let $(G,\oplus)$ and $(H,\otimes)$ be two distinct quasigroups. The triple (A,B,C) such that $A,B,C\ :\ (G,\oplus)\rightarrow(H,\otimes)$ are bijections is said to be an isotopism if and only if*

$$xA\otimes yB=(x\oplus y)C\ \forall\ x,y\in G.$$

Thus, H is called an isotope of G and they are said to be isotopic. If $C=I$, then the triple is called a principal isotopism and $(H,\otimes)=(G,\otimes)$ is called a principal isotope of $(G,\oplus)$. If in addition, $A=R_g$, $B=L_f$, then the triple is called an f,g-principal isotopism, thus $(G,\otimes)$ is reffered to as the f,g-principal isotope of $(G,\oplus)$.

A subloop(subquasigroup) $(S,\otimes)$ of a loop(quasigroup) $(G,\otimes)$ is called a Smarandache f,g-principal isotope of the subloop(subquasigroup) $(S,\oplus)$ of a loop(quasigroup) $(G,\oplus)$ if for some $f,g\in S$,

$$xR_g\otimes yL_f=(x\oplus y)\ \forall\ x,y\in S.$$

On the other hand $(G,\otimes)$ is called a Smarandache f,g-principal isotope of $(G,\oplus)$ if for some $f,g\in S$,

$$xR_g\otimes yL_f=(x\oplus y)\ \forall\ x,y\in G$$

where $(S,\oplus)$ is a S-subquasigroup(S-subloop) of $(G,\oplus)$. In these cases, f and g are called Smarandache elements(S-elements).

Theorem 2.1 *([2]) Let $(G,\oplus)$ and $(H,\otimes)$ be two distinct isotopic loops(quasigroups). There exists an f,g-principal isotope $(G,\circ)$ of $(G,\oplus)$ such that $(H,\otimes)\cong(G,\circ)$.*

Corollary 2.1 *Let $\mathcal{P}$ be an isotopic invariant property in loops(quasigroups). If $(G,\oplus)$ is a loop(quasigroup) with the property $\mathcal{P}$, then $(G,\oplus)$ is a universal loop(quasigroup) relative to the property $\mathcal{P}$ if and only if every f,g-principal isotope $(G,\circ)$ of $(G,\oplus)$ has the property $\mathcal{P}$.*

Proof
If $(G,\oplus)$ is a universal loop relative to the property $\mathcal{P}$ then every distinct loop isotope $(H,\otimes)$ of $(G,\oplus)$ has the property $\mathcal{P}$. By Theorem 2.1, there exists an f,g-principal isotope $(G,\circ)$ of $(G,\oplus)$ such that $(H,\otimes)\cong(G,\circ)$. Hence, since $\mathcal{P}$ is an isomorphic invariant property, every $(G,\circ)$ has it.

Conversely, if every f,g-principal isotope $(G,\circ)$ of $(G,\oplus)$ has the property $\mathcal{P}$ and since by Theorem 2.1 for each distinct isotope $(H,\otimes)$ there exists an f,g-principal isotope $(G,\circ)$ of $(G,\oplus)$ such that $(H,\otimes)\cong(G,\circ)$, then all $(H,\otimes)$ has the property, Thus, $(G,\oplus)$ is a universal loop relative to the property $\mathcal{P}$.

Lemma 2.1 *Let $(G,\oplus)$ be a loop(quasigroup) with a subloop(subquasigroup) $(S,\oplus)$. If $(G,\circ)$ is an arbitrary f,g-principal isotope of $(G,\oplus)$, then $(S,\circ)$ is a subloop(subquasigroup) of $(G,\circ)$ if $(S,\circ)$ is a Smarandache f,g-principal isotope of $(S,\oplus)$.*

Proof
If $(S,\circ)$ is a Smarandache f,g-principal isotope of $(S,\oplus)$, then for some $f,g\in S$,

$$xR_g\circ yL_f=(x\oplus y)\ \forall\ x,y\in S\Rightarrow x\circ y=xR_g^{-1}\oplus yL_f^{-1}\in S\ \forall\ x,y\in S$$

since $f,g\in S$. So, $(S,\circ)$ is a subgroupoid of $(G,\circ)$. $(S,\circ)$ is a subquasigroup follows from the fact that $(S,\oplus)$ is a subquasigroup. $f\oplus g$ is a two sided identity element in $(S,\circ)$. Thus, $(S,\circ)$ is a subloop of $(G,\circ)$.

3 Main Results

Universality of Smarandache Loops

Theorem 3.1 *A Smarandache quasigroup is universal if all its f,g-principal isotopes are Smarandache f,g-principal isotopes.*

Proof
Let $(G,\oplus)$ be a Smarandache quasigroup with a S-subquasigroup $(S,\oplus)$. If $(G,\circ)$ is an arbitrary f,g-principal isotope of $(G,\oplus)$, then by Lemma 2.1, $(S,\circ)$ is a subquasigroup of $(G,\circ)$ if $(S,\circ)$ is a Smarandache f,g-principal isotope of $(S,\oplus)$. Let us choose all $(S,\circ)$ in this manner. So,

$$x\circ y=xR_g^{-1}\oplus yL_f^{-1}\ \forall\ x,y\in S.$$

It shall now be shown that

$$(x\circ y)\circ z=x\circ(y\circ z)\ \forall\ x,y,z\in S.$$

But in the quasigroup $(G,\oplus)$, xy will have preference over $x\oplus y\ \forall\ x,y\in G$.

$$(x\circ y)\circ z=(xR_g^{-1}\oplus yL_f^{-1})\circ z=(xg^{-1}\oplus f^{-1}y)\circ z=(xg^{-1}\oplus f^{-1}y)R_g^{-1}\oplus zL_f^{-1}$$

$$= (xg^{-1} \oplus f^{-1}y)g^{-1} \oplus f^{-1}z = xg^{-1} \oplus f^{-1}yg^{-1} \oplus f^{-1}z.$$

$$x \circ (y \circ z) = x \circ (yR_g^{-1} \oplus zL_f^{-1}) = x \circ (yg^{-1} \oplus f^{-1}z) = xR_g^{-1} \oplus (yg^{-1} \oplus f^{-1}z)L_f^{-1}$$

$$= xg^{-1} \oplus f^{-1}(yg^{-1} \oplus f^{-1}z) = xg^{-1} \oplus f^{-1}yg^{-1} \oplus f^{-1}z.$$

Thus, $(S, \circ)$ is an S-subquasigroup of $(G, \circ)$ hence, $(G, \circ)$ is a S-quasigroup. By Theorem 2.1, for any isotope $(H, \otimes)$ of $(G, \oplus)$, there exists a $(G, \circ)$ such that $(H, \otimes) \cong (G, \circ)$. So we can now choose the isomorphic image of $(S, \circ)$ which will now be an S-subquasigroup in $(H, \otimes)$. So, $(H, \otimes)$ is an S-quasigroup. This conclusion can also be drawn straight from Corollary 2.1.

Theorem 3.2 *A Smarandache loop is universal if all its f, g-principal isotopes are Smarandache f, g-principal isotopes. Conversely, if a Smarandache loop is universal then*

$$(I, L_f R_g^{-1} R_{f^\rho} L_f^{-1}, R_g^{-1} R_{f^\rho})$$

is an autotopism of an S-subloop of the S-loop such that f and g are S-elements.

Proof

Every loop is a quasigroup. Hence, the first claim follows from Theorem 3.1. The proof of the converse is as follows. If a Smarandache loop $(G, \oplus)$ is universal then every isotope $(H, \otimes)$ is an S-loop i.e there exists an S-subloop $(S, \otimes)$ in $(H, \otimes)$. Let $(G, \circ)$ be the f, g-principal isotope of $(G, \oplus)$, then by Corollary 2.1, $(G, \circ)$ is an S-loop with say an S-subloop $(S, \circ)$. So,

$$(x \circ y) \circ z = x \circ (y \circ z) \ \forall \ x, y, z \in S$$

where

$$x \circ y = xR_g^{-1} \oplus yL_f^{-1} \ \forall \ x, y \in S.$$

$$(xR_g^{-1} \oplus yL_f^{-1})R_g^{-1} \oplus zL_f^{-1} = xR_g^{-1} \oplus (yR_g^{-1} \oplus zL_f^{-1})L_f^{-1}.$$

Replacing xR_g^{-1} by x', yL_f^{-1} by y' and taking $z = e$ in $(S, \oplus)$ we have;

$$(x' \oplus y')R_g^{-1}R_{f^\rho} = x' \oplus y'L_f R_g^{-1} R_{f^\rho} L_f^{-1} \Rightarrow (I, L_f R_g^{-1} R_{f^\rho} L_f^{-1}, R_g^{-1} R_{f^\rho})$$

is an autotopism of an S-subloop $(S, \oplus)$ of the S-loop $(G, \oplus)$ such that f and g are S-elements.

Universality of Smarandache Bol, Moufang and Extra Loops

Theorem 3.3 *A Smarandache right(left)Bol loop is universal if all its f, g-principal isotopes are Smarandache f, g-principal isotopes. Conversely, if a Smarandache right(left)Bol loop is universal then*

$$\mathcal{T}_1 = (R_g R_{f^\rho}^{-1}, L_{g^\lambda} R_g^{-1} R_{f^\rho} L_f^{-1}, R_g^{-1} R_{f^\rho})\Big(\mathcal{T}_2 = (R_{f^\rho} L_f^{-1} L_{g^\lambda} R_g^{-1}, L_f L_{g^\lambda}^{-1}, L_f^{-1} L_{g^\lambda})\Big)$$

is an autotopism of an SRB(SLB)-subloop of the SRBL(SLBL) such that f and g are S-elements.

Proof
Let $(G,\oplus)$ be a SRBL(SLBL) with a S-RB(LB)-subloop $(S,\oplus)$. If $(G,\circ)$ is an arbitrary f,g-principal isotope of $(G,\oplus)$, then by Lemma 2.1, $(S,\circ)$ is a subloop of $(G,\circ)$ if $(S,\circ)$ is a Smarandache f,g-principal isotope of $(S,\oplus)$. Let us choose all $(S,\circ)$ in this manner. So,

$$x\circ y = xR_g^{-1}\oplus yL_f^{-1}\ \forall\ x,y\in S.$$

It is already known from [24] that RB(LB) loops are universal, hence $(S,\circ)$ is a RB(LB) loop thus an S-RB(LB)-subloop of $(G,\circ)$. By Theorem 2.1, for any isotope $(H,\otimes)$ of $(G,\oplus)$, there exists a $(G,\circ)$ such that $(H,\otimes)\cong(G,\circ)$. So we can now choose the isomorphic image of $(S,\circ)$ which will now be an S-RB(LB)-subloop in $(H,\otimes)$. So, $(H,\otimes)$ is an SRBL(SLBL). This conclusion can also be drawn straight from Corollary 2.1.

The proof of the converse is as follows. If a SRBL(SLBL) $(G,\oplus)$ is universal then every isotope $(H,\otimes)$ is an SRBL(SLBL) i.e there exists an S-RB(LB)-subloop $(S,\otimes)$ in $(H,\otimes)$. Let $(G,\circ)$ be the f,g-principal isotope of $(G,\oplus)$, then by Corollary 2.1, $(G,\circ)$ is an SRBL(SLBL) with say an SRB(SLB)-subloop $(S,\circ)$. So for an SRB-subloop $(S,\circ)$,

$$[(y\circ x)\circ z]\circ x = y\circ[(x\circ z)\circ x]\ \forall\ x,y,z\in S$$

where

$$x\circ y = xR_g^{-1}\oplus yL_f^{-1}\ \forall\ x,y\in S.$$

Thus,

$$[(yR_g^{-1}\oplus xL_f^{-1})R_g^{-1}\oplus zL_f^{-1}]R_g^{-1}\oplus xL_f^{-1} = yR_g^{-1}\oplus[(xR_g^{-1}\oplus zL_f^{-1})R_g^{-1}\oplus xL_f^{-1}]L_f^{-1}.$$

Replacing yR_g^{-1} by y', zL_f^{-1} by z' and taking $x=e$ in $(S,\oplus)$ we have

$$(y'R_{f^\rho}R_g^{-1}\oplus z')R_g^{-1}R_{f^\rho} = y'\oplus z'L_{g^\lambda}R_g^{-1}R_{f^\rho}L_f^{-1}.$$

Again, replace $y'R_{f^\rho}R_g^{-1}$ by y'' so that

$$(y''\oplus z')R_g^{-1}R_{f^\rho} = y''R_gR_{f^\rho}^{-1}\oplus z'L_{g^\lambda}R_g^{-1}R_{f^\rho}L_f^{-1}\Rightarrow(R_gR_{f^\rho}^{-1},L_{g^\lambda}R_g^{-1}R_{f^\rho}L_f^{-1},R_g^{-1}R_{f^\rho})$$

is an autotopism of an SRB-subloop $(S,\oplus)$ of the S-loop $(G,\oplus)$ such that f and g are S-elements.
On the other hand, for a SLB-subloop $(S,\circ)$,

$$[x\circ(y\circ x)]\circ z = x\circ[y\circ(x\circ z)]\ \forall\ x,y,z\in S$$

where

$$x\circ y = xR_g^{-1}\oplus yL_f^{-1}\ \forall\ x,y\in S.$$

Thus,

$$[xR_g^{-1}\oplus(yR_g^{-1}\oplus xL_f^{-1})L_f^{-1}]R_g^{-1}\oplus zL_f^{-1} = xR_g^{-1}\oplus[yR_g^{-1}\oplus(xR_g^{-1}\oplus zL_f^{-1})L_f^{-1}]L_f^{-1}.$$

Replacing yR_g^{-1} by y', zL_f^{-1} by z' and taking $x = e$ in $(S, \oplus)$ we have

$$y'R_{f^\rho}L_f^{-1}L_{g^\lambda}R_g^{-1} \oplus z' = (y' \oplus z'L_{g^\lambda}L_f^{-1})L_f^{-1}L_{g^\lambda}.$$

Again, replace $z'L_{g^\lambda}L_f^{-1}$ by z'' so that

$$y'R_{f^\rho}L_f^{-1}L_{g^\lambda}R_g^{-1} \oplus z''L_fL_{g^\lambda}^{-1} = (y' \oplus z'')L_f^{-1}L_{g^\lambda} \Rightarrow (R_{f^\rho}L_f^{-1}L_{g^\lambda}R_g^{-1}, L_fL_{g^\lambda}^{-1}, L_f^{-1}L_{g^\lambda})$$

is an autotopism of an SLB-subloop $(S, \oplus)$ of the S-loop $(G, \oplus)$ such that f and g are S-elements.

Theorem 3.4 *A Smarandache Moufang loop is universal if all its f, g-principal isotopes are Smarandache f, g-principal isotopes. Conversely, if a Smarandache Moufang loop is universal then*

$$(R_gL_f^{-1}L_{g^\lambda}R_g^{-1}, L_fR_g^{-1}R_{f^\rho}L_f^{-1}, L_f^{-1}L_{g^\lambda}R_g^{-1}R_{f^\rho}), (R_gL_f^{-1}L_{g^\lambda}R_g^{-1}, L_fR_g^{-1}R_{f^\rho}L_f^{-1}, R_g^{-1}R_{f^\rho}L_f^{-1}L_{g^\lambda}),$$

$$(R_gL_f^{-1}L_{g^\lambda}R_g^{-1}R_{f^\rho}R_g^{-1}, L_fL_{g^\lambda}^{-1}, L_f^{-1}L_{g^\lambda}), (R_gR_{f^\rho}^{-1}, L_fR_g^{-1}R_{f^\rho}L_f^{-1}L_{g^\lambda}L_f^{-1}, R_g^{-1}R_{f^\rho}),$$

$$(R_gL_f^{-1}L_{g^\lambda}R_g^{-1}, L_{g^\lambda}R_g^{-1}R_{f^\rho}L_{g^\lambda}^{-1}, R_g^{-1}R_{f^\rho}L_f^{-1}L_{g^\lambda}), (R_{f^\rho}L_f^{-1}L_{g^\lambda}R_{f^\rho}^{-1}, L_fR_g^{-1}R_{f^\rho}L_f^{-1}, L_f^{-1}L_{g^\lambda}R_g^{-1}R_{f^\rho})$$

are autotopisms of an SM-subloop of the SML such that f and g are S-elements.

Proof
Let $(G, \oplus)$ be a SML with a SM-subloop $(S, \oplus)$. If $(G, \circ)$ is an arbitrary f, g-principal isotope of $(G, \oplus)$, then by Lemma 2.1, $(S, \circ)$ is a subloop of $(G, \circ)$ if $(S, \circ)$ is a Smarandache f, g-principal isotope of $(S, \oplus)$. Let us choose all $(S, \circ)$ in this manner. So,

$$x \circ y = xR_g^{-1} \oplus yL_f^{-1} \ \forall \ x, y \in S.$$

It is already known from [24] that Moufang loops are universal, hence $(S, \circ)$ is a Moufang loop thus an SM-subloop of $(G, \circ)$. By Theorem 2.1, for any isotope $(H, \otimes)$ of $(G, \oplus)$, there exists a $(G, \circ)$ such that $(H, \otimes) \cong (G, \circ)$. So we can now choose the isomorphic image of $(S, \circ)$ which will now be an SM-subloop in $(H, \otimes)$. So, $(H, \otimes)$ is an SML. This conclusion can also be drawn straight from Corollary 2.1.

The proof of the converse is as follows. If a SML $(G, \oplus)$ is universal then every isotope $(H, \otimes)$ is an SML i.e there exists an SM-subloop $(S, \otimes)$ in $(H, \otimes)$. Let $(G, \circ)$ be the f, g-principal isotope of $(G, \oplus)$, then by Corollary 2.1, $(G, \circ)$ is an SML with say an SM-subloop $(S, \circ)$. For an SM-subloop $(S, \circ)$,

$$(x \circ y) \circ (z \circ x) = [x \circ (y \circ z)] \circ x \ \forall \ x, y, z \in S$$

where

$$x \circ y = xR_g^{-1} \oplus yL_f^{-1} \ \forall \ x, y \in S.$$

Thus,

$$(xR_g^{-1} \oplus yL_f^{-1})R_g^{-1} \oplus (zR_g^{-1} \oplus xL_f^{-1})L_f^{-1} = [xR_g^{-1} \oplus (yR_g^{-1} \oplus zL_f^{-1})L_f^{-1}]R_g^{-1} \oplus xL_f^{-1}.$$

Replacing yR_g^{-1} by y', zL_f^{-1} by z' and taking $x = e$ in $(S, \oplus)$ we have

$$y'R_gL_f^{-1}L_{g^\lambda}R_g^{-1} \oplus z'L_fR_g^{-1}R_{f^\rho}L_f^{-1} = (y' \oplus z')L_f^{-1}L_{g^\lambda}R_g^{-1}R_{f^\rho} \Rightarrow$$

$$(R_gL_f^{-1}L_{g^\lambda}R_g^{-1}, L_fR_g^{-1}R_{f^\rho}L_f^{-1}, L_f^{-1}L_{g^\lambda}R_g^{-1}R_{f^\rho})$$

is an autotopism of an SM-subloop $(S, \oplus)$ of the S-loop $(G, \oplus)$ such that f and g are S-elements.

Again, for an SM-subloop $(S, \circ)$,

$$(x \circ y) \circ (z \circ x) = x \circ [(y \circ z) \circ x] \ \forall \ x, y, z \in S$$

where

$$x \circ y = xR_g^{-1} \oplus yL_f^{-1} \ \forall \ x, y \in S.$$

Thus,

$$(xR_g^{-1} \oplus yL_f^{-1})R_g^{-1} \oplus (zR_g^{-1} \oplus xL_f^{-1})L_f^{-1} = xR_g^{-1} \oplus [(yR_g^{-1} \oplus zL_f^{-1})R_g^{-1} \oplus xL_f^{-1}]L_f^{-1}.$$

Replacing yR_g^{-1} by y', zL_f^{-1} by z' and taking $x = e$ in $(S, \oplus)$ we have

$$y'R_gL_f^{-1}L_{g^\lambda}R_g^{-1} \oplus z'L_fR_g^{-1}R_{f^\rho}L_f^{-1} = (y' \oplus z')R_g^{-1}R_{f^\rho}L_f^{-1}L_{g^\lambda} \Rightarrow$$

$$(R_gL_f^{-1}L_{g^\lambda}R_g^{-1}, L_fR_g^{-1}R_{f^\rho}L_f^{-1}, R_g^{-1}R_{f^\rho}L_f^{-1}L_{g^\lambda})$$

is an autotopism of an SM-subloop $(S, \oplus)$ of the S-loop $(G, \oplus)$ such that f and g are S-elements.

Also, if $(S, \circ)$ is an SM-subloop then,

$$[(x \circ y) \circ x] \circ z = x \circ [y \circ (x \circ z)] \ \forall \ x, y, z \in S$$

where

$$x \circ y = xR_g^{-1} \oplus yL_f^{-1} \ \forall \ x, y \in S.$$

Thus,

$$[(xR_g^{-1} \oplus yL_f^{-1})R_g^{-1} \oplus xL_f^{-1}]R_g^{-1} \oplus zL_f^{-1} = xR_g^{-1} \oplus [yR_g^{-1} \oplus (xR_g^{-1} \oplus zL_f^{-1})L_f^{-1}]L_f^{-1}.$$

Replacing yR_g^{-1} by y', zL_f^{-1} by z' and taking $x = e$ in $(S, \oplus)$ we have

$$y'R_gL_f^{-1}L_{g^\lambda}R_g^{-1}R_{f^\rho}R_g^{-1} \oplus z' = (y' \oplus z'L_{g^\lambda}L_f^{-1})L_f^{-1}L_{g^\lambda}.$$

Again, replace $z'L_{g^\lambda}L_f^{-1}$ by z'' so that

$$y'R_gL_f^{-1}L_{g^\lambda}R_g^{-1}R_{f^\rho}R_g^{-1} \oplus z''L_fL_{g^\lambda}^{-1} = (y' \oplus z'')L_f^{-1}L_{g^\lambda} \Rightarrow (R_gL_f^{-1}L_{g^\lambda}R_g^{-1}R_{f^\rho}R_g^{-1}, L_fL_{g^\lambda}^{-1}, L_f^{-1}L_{g^\lambda})$$

is an autotopism of an SM-subloop $(S, \oplus)$ of the S-loop $(G, \oplus)$ such that f and g are S-elements.

Furthermore, if $(S,\circ)$ is an SM-subloop then,

$$[(y\circ x)\circ z]\circ x = y\circ[x\circ(z\circ x)]\ \forall\ x,y,z\in S$$

where

$$x\circ y = xR_g^{-1}\oplus yL_f^{-1}\ \forall\ x,y\in S.$$

Thus,

$$[(yR_g^{-1}\oplus xL_f^{-1})R_g^{-1}\oplus zL_f^{-1}]R_g^{-1}\oplus xL_f^{-1} = yR_g^{-1}\oplus[xR_g^{-1}\oplus(zR_g^{-1}\oplus xL_f^{-1})L_f^{-1}]L_f^{-1}.$$

Replacing yR_g^{-1} by y', zL_f^{-1} by z' and taking $x=e$ in $(S,\oplus)$ we have

$$(y'R_{f^\rho}R_g^{-1}\oplus z')R_g^{-1}R_{f^\rho} = y'\oplus z'L_fR_g^{-1}R_{f^\rho}L_f^{-1}L_{g^\lambda}L_f^{-1}.$$

Again, replace $y'R_{f^\rho}R_g^{-1}$ by y'' so that

$$(y''\oplus z')R_g^{-1}R_{f^\rho} = y''R_gR_{f^\rho}^{-1}\oplus z'L_fR_g^{-1}R_{f^\rho}L_f^{-1}L_{g^\lambda}L_f^{-1} \Rightarrow (R_gR_{f^\rho}^{-1}, L_fR_g^{-1}R_{f^\rho}L_f^{-1}L_{g^\lambda}L_f^{-1}, R_g^{-1}R_{f^\rho})$$

is an autotopism of an SM-subloop $(S,\oplus)$ of the S-loop $(G,\oplus)$ such that f and g are S-elements.

Lastly, $(S,\oplus)$ is an SM-subloop if and only if $(S,\circ)$ is an SRB-subloop and an SLB-subloop. So by Theorem 3.3, $\mathcal{T}_1$ and $\mathcal{T}_2$ are autotopisms in $(S,\oplus)$, hence $\mathcal{T}_1\mathcal{T}_2$ and $\mathcal{T}_2\mathcal{T}_1$ are autotopisms in $(S,\oplus)$.

Theorem 3.5 *A Smarandache extra loop is universal if all its f,g-principal isotopes are Smarandache f,g-principal isotopes. Conversely, if a Smarandache extra loop is universal then* $(R_gL_f^{-1}L_{g^\lambda}R_g^{-1}, L_fR_{f^\rho}^{-1}R_gL_f^{-1}, L_f^{-1}L_{g^\lambda}R_{f^\rho}^{-1}R_g)$,

$$(R_gR_{f^\rho}^{-1}R_gL_f^{-1}L_{g^\lambda}R_g^{-1}, L_{g^\lambda}L_f^{-1}, L_f^{-1}L_{g^\lambda}), (R_{f^\rho}R_g^{-1}, L_fL_{g^\lambda}^{-1}L_fR_g^{-1}R_{f^\rho}L_f^{-1}, R_g^{-1}R_{f^\rho})$$

$$(R_gL_f^{-1}L_{g^\lambda}R_g^{-1}, L_fR_g^{-1}R_{f^\rho}L_f^{-1}, L_f^{-1}L_{g^\lambda}R_g^{-1}R_{f^\rho}), (R_gL_f^{-1}L_{g^\lambda}R_g^{-1}, L_fR_g^{-1}R_{f^\rho}L_f^{-1}, R_g^{-1}R_{f^\rho}L_f^{-1}L_{g^\lambda}),$$

$$(R_gL_f^{-1}L_{g^\lambda}R_g^{-1}R_{f^\rho}R_g^{-1}, L_fL_{g^\lambda}^{-1}, L_f^{-1}L_{g^\lambda}), (R_gR_{f^\rho}^{-1}, L_fR_g^{-1}R_{f^\rho}L_f^{-1}L_{g^\lambda}L_f^{-1}, R_g^{-1}R_{f^\rho}),$$

$$(R_gL_f^{-1}L_{g^\lambda}R_g^{-1}, L_{g^\lambda}R_g^{-1}R_{f^\rho}L_{g^\lambda}^{-1}, R_g^{-1}R_{f^\rho}L_f^{-1}L_{g^\lambda}), (R_{f^\rho}L_f^{-1}L_{g^\lambda}R_{f^\rho}^{-1}, L_fR_g^{-1}R_{f^\rho}L_f^{-1}, L_f^{-1}L_{g^\lambda}R_g^{-1}R_{f^\rho}),$$

are autotopisms of an SE-subloop of the SEL such that f and g are S-elements.

Proof

Let $(G,\oplus)$ be a SEL with a SE-subloop $(S,\oplus)$. If $(G,\circ)$ is an arbitrary f,g-principal isotope of $(G,\oplus)$, then by Lemma 2.1, $(S,\circ)$ is a subloop of $(G,\circ)$ if $(S,\circ)$ is a Smarandache f,g-principal isotope of $(S,\oplus)$. Let us choose all $(S,\circ)$ in this manner. So,

$$x\circ y = xR_g^{-1}\oplus yL_f^{-1}\ \forall\ x,y\in S.$$

In [9] and [17] respectively, it was shown and stated that a loop is an extra loop if and only if it is a Moufang loop and a CC-loop. But since CC-loops are G-loops(they are isomorphic to all loop isotopes) then extra loops are universal, hence $(S,\circ)$ is an extra loop thus an SE-subloop of $(G,\circ)$. By Theorem 2.1, for any isotope $(H,\otimes)$ of $(G,\oplus)$, there exists a $(G,\circ)$ such that $(H,\otimes)\cong(G,\circ)$. So we can now choose the isomorphic image of $(S,\circ)$ which will now be an SE-subloop in $(H,\otimes)$. So, $(H,\otimes)$ is an SEL. This conclusion can also be drawn straight from Corollary 2.1.

The proof of the converse is as follows. If a SEL $(G,\oplus)$ is universal then every isotope $(H,\otimes)$ is an SEL i.e there exists an SE-subloop $(S,\otimes)$ in $(H,\otimes)$. Let $(G,\circ)$ be the f,g-principal isotope of $(G,\oplus)$, then by Corollary 2.1, $(G,\circ)$ is an SEL with say an SE-subloop $(S,\circ)$. For an SE-subloop $(S,\circ)$,

$$[(x\circ y)\circ z]\circ x = x\circ[y\circ(z\circ x)]\ \forall\ x,y,z\in S$$

where

$$x\circ y = xR_g^{-1}\oplus yL_f^{-1}\ \forall\ x,y\in S.$$

Thus,

$$[(xR_g^{-1}\oplus yL_f^{-1})R_g^{-1}\oplus zL_f^{-1}]R_g^{-1}\oplus xL_f^{-1} = xR_g^{-1}\oplus[yR_g^{-1}\oplus(zR_g^{-1}\oplus xL_f^{-1})L_f^{-1}]L_f^{-1}.$$

Replacing yR_g^{-1} by y', zL_f^{-1} by z' and taking $x=e$ in $(S,\oplus)$ we have

$$(y'R_gL_f^{-1}L_{g^\lambda}R_g^{-1}\oplus z')R_g^{-1}R_{f^\rho} = (y'\oplus z'L_fR_g^{-1}R_{f^\rho}L_f^{-1})L_f^{-1}L_{g^\lambda}.$$

Again, replace $z'L_fR_g^{-1}R_{f^\rho}L_f^{-1}$ by z'' so that

$$y'R_gL_f^{-1}L_{g^\lambda}R_g^{-1}\oplus z''L_fR_{f^\rho}^{-1}R_gL_f^{-1} = (y'\oplus z'')L_f^{-1}L_{g^\lambda}R_{f^\rho}^{-1}R_g \Rightarrow$$

$$(R_gL_f^{-1}L_{g^\lambda}R_g^{-1}, L_fR_{f^\rho}^{-1}R_gL_f^{-1}, L_f^{-1}L_{g^\lambda}R_{f^\rho}^{-1}R_g)$$

is an autotopism of an SE-subloop $(S,\oplus)$ of the S-loop $(G,\oplus)$ such that f and g are S-elements.

Again, for an SE-subloop $(S,\circ)$,

$$(x\circ y)\circ(x\circ z) = x\circ[(y\circ x)\circ z]\ \forall\ x,y,z\in S$$

where

$$x\circ y = xR_g^{-1}\oplus yL_f^{-1}\ \forall\ x,y\in S.$$

Thus,

$$(xR_g^{-1}\oplus yL_f^{-1})R_g^{-1}\oplus(xR_g^{-1}\oplus zL_f^{-1})L_f^{-1} = xR_g^{-1}\oplus[(yR_g^{-1}\oplus xL_f^{-1})R_g^{-1}\oplus zL_f^{-1}]L_f^{-1}.$$

Replacing yR_g^{-1} by y', zL_f^{-1} by z' and taking $x=e$ in $(S,\oplus)$ we have

$$y'R_gL_f^{-1}L_{g^\lambda}R_g^{-1}\oplus z'L_{g^\lambda}L_f^{-1} = (y'R_{f^\rho}R_g^{-1}\oplus z')L_f^{-1}L_{g^\lambda}.$$

Again, replace $y'R_{f^\rho}R_g^{-1}$ by y'' so that

$$y''R_gR_{f^\rho}^{-1}R_gL_f^{-1}L_{g^\lambda}R_g^{-1}\oplus z'L_{g^\lambda}L_f^{-1} = (y''\oplus z')L_f^{-1}L_{g^\lambda} \Rightarrow (R_gR_{f^\rho}^{-1}R_gL_f^{-1}L_{g^\lambda}R_g^{-1}, L_{g^\lambda}L_f^{-1}, L_f^{-1}L_{g^\lambda})$$

is an autotopism of an SE-subloop $(S,\oplus)$ of the S-loop $(G,\oplus)$ such that f and g are S-elements.

Also, if $(S,\circ)$ is an SE-subloop then,

$$(y\circ x)\circ(z\circ x)=[y\circ(x\circ z)]\circ x\ \forall\ x,y,z\in S$$

where

$$x\circ y=xR_g^{-1}\oplus yL_f^{-1}\ \forall\ x,y\in S.$$

Thus,

$$(yR_g^{-1}\oplus xL_f^{-1})R_g^{-1}\oplus(zR_g^{-1}\oplus xL_f^{-1})L_f^{-1}=[(yR_g^{-1}\oplus(xR_g^{-1}\oplus zL_f^{-1})L_f^{-1}]R_g^{-1}\oplus xL_f^{-1}.$$

Replacing yR_g^{-1} by y', zL_f^{-1} by z' and taking $x=e$ in $(S,\oplus)$ we have

$$y'R_{f^\rho}R_g^{-1}\oplus z'L_fR_g^{-1}R_{f^\rho}L_f^{-1}=(y'\oplus z'L_{g^\lambda}L_f^{-1})R_g^{-1}R_{f^\rho}.$$

Again, replace $z'L_{g^\lambda}L_f^{-1}$ by z'' so that

$$y'R_{f^\rho}R_g^{-1}\oplus z''L_fL_{g^\lambda}^{-1}L_fR_g^{-1}R_{f^\rho}L_f^{-1}=(y'\oplus z')R_g^{-1}R_{f^\rho}\Rightarrow(R_{f^\rho}R_g^{-1},L_fL_{g^\lambda}^{-1}L_fR_g^{-1}R_{f^\rho}L_f^{-1},R_g^{-1}R_{f^\rho})$$

is an autotopism of an SE-subloop $(S,\oplus)$ of the S-loop $(G,\oplus)$ such that f and g are S-elements.

Lastly, $(S,\oplus)$ is an SE-subloop if and only if $(S,\circ)$ is an SM-subloop and an SCC-subloop. So by Theorem 3.4, the six remaining triples are autotopisms in $(S,\oplus)$.

Universality of Smarandache Inverse Property Loops

Theorem 3.6 *A Smarandache left(right) inverse property loop in which all its f,g-principal isotopes are Smarandache f,g-principal isotopes is universal if and only if it is a Smarandache left(right) Bol loop in which all its f,g-principal isotopes are Smarandache f,g-principal isotopes.*

Proof

Let $(G,\oplus)$ be a SLIPL with a SLIP-subloop $(S,\oplus)$. If $(G,\circ)$ is an arbitrary f,g-principal isotope of $(G,\oplus)$, then by Lemma 2.1, $(S,\circ)$ is a subloop of $(G,\circ)$ if $(S,\circ)$ is a Smarandache f,g-principal isotope of $(S,\oplus)$. Let us choose all $(S,\circ)$ in this manner. So,

$$x\circ y=xR_g^{-1}\oplus yL_f^{-1}\ \forall\ x,y\in S.$$

$(G,\oplus)$ is a universal SLIPL if and only if every isotope $(H,\otimes)$ is a SLIPL. $(H,\otimes)$ is a SLIPL if and only if it has at least a SLIP-subloop $(S,\otimes)$. By Theorem 2.1, for any isotope $(H,\otimes)$ of $(G,\oplus)$, there exists a $(G,\circ)$ such that $(H,\otimes)\cong(G,\circ)$. So we can now choose the isomorphic image of $(S,\circ)$ to be $(S,\otimes)$ which is already a SLIP-subloop in $(H,\otimes)$. So, $(S,\circ)$ is also a SLIP-subloop in $(G,\circ)$. As shown in [24], $(S,\oplus)$ and its f,g-isotope(Smarandache f,g-isotope) $(S,\circ)$ are SLIP-subloops if and only if $(S,\oplus)$ is a left Bol subloop(i.e a SLB-subloop). So, $(G,\oplus)$ is SLBL.

Conversely, if $(G, \oplus)$ is SLBL, then there exists a SLB-subloop $(S, \oplus)$ in $(G, \oplus)$. If $(G, \circ)$ is an arbitrary f, g-principal isotope of $(G, \oplus)$, then by Lemma 2.1, $(S, \circ)$ is a subloop of $(G, \circ)$ if $(S, \circ)$ is a Smarandache f, g-principal isotope of $(S, \oplus)$. Let us choose all $(S, \circ)$ in this manner. So,

$$x \circ y = xR_g^{-1} \oplus yL_f^{-1} \; \forall \; x, y \in S.$$

By Theorem 2.1, for any isotope $(H, \otimes)$ of $(G, \oplus)$, there exists a $(G, \circ)$ such that $(H, \otimes) \cong (G, \circ)$. So we can now choose the isomorphic image of $(S, \circ)$ to be $(S, \otimes)$ which is a SLB-subloop in $(H, \otimes)$ using the same reasoning in Theorem 3.3. So, $(S, \circ)$ is a SLB-subloop in $(G, \circ)$. Left Bol loops have the left inverse property(LIP), hence, $(S, \oplus)$ and $(S, \circ)$ are SLIP-subloops in $(G, \oplus)$ and $(G, \circ)$ respectively. Thence, $(G, \oplus)$ and $(G, \circ)$ are SLBLs. Therefore, $(G, \oplus)$ is a universal SLIPL.

The proof for a Smarandache right inverse property loop is similar and is as follows. Let $(G, \oplus)$ be a SRIPL with a SRIP-subloop $(S, \oplus)$. If $(G, \circ)$ is an arbitrary f, g-principal isotope of $(G, \oplus)$, then by Lemma 2.1, $(S, \circ)$ is a subloop of $(G, \circ)$ if $(S, \circ)$ is a Smarandache f, g-principal isotope of $(S, \oplus)$. Let us choose all $(S, \circ)$ in this manner. So,

$$x \circ y = xR_g^{-1} \oplus yL_f^{-1} \; \forall \; x, y \in S.$$

$(G, \oplus)$ is a universal SRIPL if and only if every isotope $(H, \otimes)$ is a SRIPL. $(H, \otimes)$ is a SRIPL if and only if it has at least a SRIP-subloop $(S, \otimes)$. By Theorem 2.1, for any isotope $(H, \otimes)$ of $(G, \oplus)$, there exists a $(G, \circ)$ such that $(H, \otimes) \cong (G, \circ)$. So we can now choose the isomorphic image of $(S, \circ)$ to be $(S, \otimes)$ which is already a SRIP-subloop in $(H, \otimes)$. So, $(S, \circ)$ is also a SRIP-subloop in $(G, \circ)$. As shown in [24], $(S, \oplus)$ and its f, g-isotope(Smarandache f, g-isotope) $(S, \circ)$ are SRIP-subloops if and only if $(S, \oplus)$ is a right Bol subloop(i.e a SRB-subloop). So, $(G, \oplus)$ is SRBL.

Conversely, if $(G, \oplus)$ is SRBL, then there exists a SRB-subloop $(S, \oplus)$ in $(G, \oplus)$. If $(G, \circ)$ is an arbitrary f, g-principal isotope of $(G, \oplus)$, then by Lemma 2.1, $(S, \circ)$ is a subloop of $(G, \circ)$ if $(S, \circ)$ is a Smarandache f, g-principal isotope of $(S, \oplus)$. Let us choose all $(S, \circ)$ in this manner. So,

$$x \circ y = xR_g^{-1} \oplus yL_f^{-1} \; \forall \; x, y \in S.$$

By Theorem 2.1, for any isotope $(H, \otimes)$ of $(G, \oplus)$, there exists a $(G, \circ)$ such that $(H, \otimes) \cong (G, \circ)$. So we can now choose the isomorphic image of $(S, \circ)$ to be $(S, \otimes)$ which is a SRB-subloop in $(H, \otimes)$ using the same reasoning in Theorem 3.3. So, $(S, \circ)$ is a SRB-subloop in $(G, \circ)$. Right Bol loops have the right inverse property(RIP), hence, $(S, \oplus)$ and $(S, \circ)$ are SRIP-subloops in $(G, \oplus)$ and $(G, \circ)$ respectively. Thence, $(G, \oplus)$ and $(G, \circ)$ are SRBLs. Therefore, $(G, \oplus)$ is a universal SRIPL.

Theorem 3.7 *A Smarandache inverse property loop in which all its f, g-principal isotopes are Smarandache f, g-principal isotopes is universal if and only if it is a Smarandache Moufang loop in which all its f, g-principal isotopes are Smarandache f, g-principal isotopes.*

Proof

Let $(G, \oplus)$ be a SIPL with a SIP-subloop $(S, \oplus)$. If $(G, \circ)$ is an arbitrary f, g-principal isotope of $(G,\oplus)$, then by Lemma 2.1, $(S,\circ)$ is a subloop of $(G,\circ)$ if $(S,\circ)$ is a Smarandache f,g-principal isotope of $(S,\oplus)$. Let us choose all $(S,\circ)$ in this manner. So,

$$x\circ y = xR_g^{-1}\oplus yL_f^{-1}\ \forall\ x,y\in S.$$

$(G,\oplus)$ is a universal SIPL if and only if every isotope $(H,\otimes)$ is a SIPL. $(H,\otimes)$ is a SIPL if and only if it has at least a SIP-subloop $(S,\otimes)$. By Theorem 2.1, for any isotope $(H,\otimes)$ of $(G,\oplus)$, there exists a $(G,\circ)$ such that $(H,\otimes)\cong(G,\circ)$. So we can now choose the isomorphic image of $(S,\circ)$ to be $(S,\otimes)$ which is already a SIP-subloop in $(H,\otimes)$. So, $(S,\circ)$ is also a SIP-subloop in $(G,\circ)$. As shown in [24], $(S,\oplus)$ and its f,g-isotope(Smarandache f,g-isotope) $(S,\circ)$ are SIP-subloops if and only if $(S,\oplus)$ is a Moufang subloop(i.e a SM-subloop). So, $(G,\oplus)$ is SML.

Conversely, if $(G,\oplus)$ is SML, then there exists a SM-subloop $(S,\oplus)$ in $(G,\oplus)$. If $(G,\circ)$ is an arbitrary f,g-principal isotope of $(G,\oplus)$, then by Lemma 2.1, $(S,\circ)$ is a subloop of $(G,\circ)$ if $(S,\circ)$ is a Smarandache f,g-principal isotope of $(S,\oplus)$. Let us choose all $(S,\circ)$ in this manner. So,

$$x\circ y = xR_g^{-1}\oplus yL_f^{-1}\ \forall\ x,y\in S.$$

By Theorem 2.1, for any isotope $(H,\otimes)$ of $(G,\oplus)$, there exists a $(G,\circ)$ such that $(H,\otimes)\cong(G,\circ)$. So we can now choose the isomorphic image of $(S,\circ)$ to be $(S,\otimes)$ which is a SM-subloop in $(H,\otimes)$ using the same reasoning in Theorem 3.3. So, $(S,\circ)$ is a SM-subloop in $(G,\circ)$. Moufang loops have the inverse property(IP), hence, $(S,\oplus)$ and $(S,\circ)$ are SIP-subloops in $(G,\oplus)$ and $(G,\circ)$ respectively. Thence, $(G,\oplus)$ and $(G,\circ)$ are SMLs. Therefore, $(G,\oplus)$ is a universal SIPL.

Corollary 3.1 *If a Smarandache left(right) inverse property loop is universal then*

$$(R_gR_{f^\rho}^{-1}, L_{g^\lambda}R_g^{-1}R_{f^\rho}L_f^{-1}, R_g^{-1}R_{f^\rho})\Big((R_{f^\rho}L_f^{-1}L_{g^\lambda}R_g^{-1}, L_fL_{g^\lambda}^{-1}, L_f^{-1}L_{g^\lambda})\Big)$$

is an autotopism of an SLIP(SRIP)-subloop of the SLIPL(SRIPL) such that f and g are S-elements.

Proof
This follows by Theorem 3.6 and Theorem 3.1.

Corollary 3.2 *If a Smarandache inverse property loop is universal then*

$$(R_gL_f^{-1}L_{g^\lambda}R_g^{-1}, L_fR_g^{-1}R_{f^\rho}L_f^{-1}, L_f^{-1}L_{g^\lambda}R_g^{-1}R_{f^\rho}), (R_gL_f^{-1}L_{g^\lambda}R_g^{-1}, L_fR_g^{-1}R_{f^\rho}L_f^{-1}, R_g^{-1}R_{f^\rho}L_f^{-1}L_{g^\lambda}),$$

$$(R_gL_f^{-1}L_{g^\lambda}R_g^{-1}R_{f^\rho}R_g^{-1}, L_fL_{g^\lambda}^{-1}, L_f^{-1}L_{g^\lambda}), (R_gR_{f^\rho}^{-1}, L_fR_g^{-1}R_{f^\rho}L_f^{-1}L_{g^\lambda}L_f^{-1}, R_g^{-1}R_{f^\rho}),$$

$$(R_gL_f^{-1}L_{g^\lambda}R_g^{-1}, L_{g^\lambda}R_g^{-1}R_{f^\rho}L_{g^\lambda}^{-1}, R_g^{-1}R_{f^\rho}L_f^{-1}L_{g^\lambda}), (R_{f^\rho}L_f^{-1}L_{g^\lambda}R_{f^\rho}^{-1}, L_fR_g^{-1}R_{f^\rho}L_f^{-1}, L_f^{-1}L_{g^\lambda}R_g^{-1}R_{f^\rho})$$

are autotopisms of an SIP-subloop of the SIPL such that f and g are S-elements.

Proof
This follows from Theorem 3.7 and Theorem 3.4.

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