# The Repeated Divisor Function and Possible Correlation with Highly Composite Numbers

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**Abstract**

Let n be a non-null positive integer and $d(n)$ is the number of positive divisors of n, called the divisor function. Of course, $d(n) \leq n$. $d(n) = 1$ if and only if $n = 1$. For $n > 2$ we have $d(n) \geq 2$ and in this paper we try to find the smallest $k$ such that $d(d(...d(n)...)) = 2$ where the divisor function is applied $k$ times. At the end of the paper we make a conjecture based on some observations.

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# 1 Introduction

We found this problem in a paper by Florentin Smarandache, see [3]. This is the 18th unsolved problem in his paper.

We start with some trivial observations. $d(d(...d(n)...)) = 2$ implies $d^{k-1}(n) = p$ where $p$ is a prime. If $p = 2$ then the chain continues infinitely long without any significance.

Otherwise suppose p is odd, $p = 2\alpha + 1$. We know that only perfect squares have odd number of factors and since that odd number $2\alpha + 1$ is prime the only choice for the perfect square is $q^{2\alpha}$ where $q$ is a prime. Now this $q$ can be arbitrarily large.

Going back one step more, we see that a number with number of divisors equal to $q^{2\alpha}$ will be of the form $\prod_{i=1}^{2\alpha} p_i^{q-1}$ where $p_i$ are distinct primes. Now this number can be arbitrarily large since though fixing $\alpha$ will fix the number of $p_i$'s, still $q$ can be arbitrarily large.

# 2 The Values K can Attain

From introduction we clearly observe that $n$ can be arbitrarily large while $k = 3$ remains fixed and we get $d^k(n) = 2$ at the end. But computer programming reveals that if we plot $k$ with respect with $n$, the frequency with which $k = 3$ or $k = 4$ occurs is far above any other frequency for at least numbers up to numbers like 5000000. $k = 5$ first occurs at 60 and $k = 6$ first occurs at 5040. $k = 7$ first occurs when $n = 2^6$ x $3^4$ x $5^2$ x 72 x 11 x 13 x 17 x 19 which is more than 10 digit number. We observe that k increases very slowly compared to n. But what is interesting is that $k = 3$ or $k = 4$ occurs with same frequency almost in every sufficiently large interval. $k = 1$ also occurs sometimes due to the distribution of primes and the presence of twin primes.

But we can clearly see here that $k$ attains every integer $m \in \mathbb{N}$. Observe that given $n = \prod_{i=1}^{m} p_i^{a_i}$ and $k = r$ we just construct $n_1$ such that $d(n_1) = n$, then for $n_1$ we have $k = r + 1$. Just put $n_1 = \prod_i^m q_i^{p_i^{a_i}-1}$ where $q_j$ is the $j^{th}$ prime starting from 2. So k is unbounded.

## 3 The least $n$ for a given $k$

After the previous section, he we give an algorithm for which given $n$ for which $k = r$, we give the smallest $n_1$ for which $k = r+1$. Since we know that 60 is the smallest number where $k = 5$ the first time, by induction we can consequently find the $n_1's$ for which $k = 6, 7, 8$...... Look at the following image on the next page to get an idea of the variation of $k$ with respect to n when n is taken in the range (0,350). We plot the n along the x axis and the corresponding k along the y axis.

## 3.1 The Algorithm

Given $k = k_0$ for a particular $n \in \mathbb{N}$, we give an explicit construction of minimal integer $L \in \mathbb{N}$ such that $d(L) = k_0 + 1$. Assume an ordering of primes $2 = p_1 < p_2 < p_3 <.....$
Say $n = \prod_{i=1}^{m} p_i^{a_i}$ and we assume $L = 2^{a_1}.3^{a_2}.5^{a_3}....$

**Case 1:** $a_i = 1$ **for some** $i$
In order to construct the minimum $L$, we need to make sure that the largest prime should be put as the index on the smallest possible prime. So if $a_i = 1$ for some $i$, clearly it goes to power of single prime because if $a_m = 1$ without loss of generality, then $a_1 = p - 1$ because otherwise $L$ will not me minimal.

**Case 2:** $a_i \geq 2$ **for some** $i$
Here we say that for a generic term in prime decomposition say $p_j^{a_j}$, it can be distributed like $2^{p_j^{a_j}-1}$ or $2^{p_j-1}.3^{p_j-1}......p_{a_j}^{p_j-1}$ two ways.We will prove that to achieve the minimal $L$, the second choice is better. Similarly we can argue $3^{p_j^{a_j}-1} > 3^{p_j-1}...p_{a_{j+1}}^{p_j-1}$. This will lead to the conclusion that each generic coupe, say without loss of generality $p_m^{a_m}$ will give $(2^{p_m-1}.3^{p_m-1}....p_{a_m}^{p_m-1})$ contribution in the prime factorization of $L$.
**Example:** If we put $n = 5040 =$ then we get $L = 2^6.3^4.5^2.7^2.11.13.17.19$ which is a 13 digit number. Observe how we use the algorithm.
$5040 = 2^4.3^2.5.7$. So according to our algorithm since 5 and 7 have index 1, they will correspond to a single prime number each. We have to construct $L$ such that $d(L) = 5040$. So the prime factorization of $L$ will begin with $2^6.3^4$ for sure. Now to get $3^2$ as a factor of $d(L)$ we need to distribute it in such a way that our obtained $L$ is minimum.
So we have $L = 2^6.3^4.5^2.7^2....$ and by similar reasoning we finish the construction of $L$ as $L = 2^6.3^4.5^2.7^2.11.13.17.19$.
It is noticeable that the algorithm shows it is always better to distribute the indexes over as many primes as possible to minimize the outcome.

## 3.2 The Proof of the Algorithm

*Proof.* We will use induction on $a_j$.
For $a_j = 2$, without loss of generality let $j = m$. If $j = k(< m)$ then instead of 2, our

decomposition will start with $p_{a_m+a_{m-1}+...+a_{k-1}+1}$ and argument for that will be similar.
If $a_j = 2$ we have to show:

$$2^{p_m^2-1} > 2^{p_m-1}.3^{p_m-1} \tag{1}$$

$$\implies 2^{p_m} > 3 \tag{2}$$

**Induction Step:** Assuming $a_m = k$ we will prove for $a_m = k+1$
$2^{p_m^{k+1}-1} > (2^{p_m-1}.3^{p_m-1}......p_{a_m-1}^{p_m-1})(p_{a_m}^{p_m-1})$

Now $(2^{p_m-1}.3^{p_m-1}......p_{a_m-1}^{p_m-1}) < 2^{p_m^k-1}$ by the hypothesis.
So it is enough to check if

$$2^{p_m^{k+1}-1} > 2^{p_m^k-1}.p_{a_m}^{p_m-1} \tag{3}$$

$$\implies 2^{p_m^{k+1}-p_m^k} > p_{a_m}^{p_m-1} \tag{4}$$

$$\implies 2^{p_m^k} > p_{a_m} \tag{5}$$

Now it is clearly true that $p_n \leq 2^n$ and so enough to show
$2^{p_m^k} \geq 2^{k+1}$. But clearly $p_m^k > k+1$, and so we are done.

□

## 4 An estimate of $k$ for all $n$

Here we return to our original problem of finding the smallest $k$ such that $d(d(...d(n)...)) = 2$. Constructing $n_1$ from $n$ according to our algorithm, we see that if n has prime decomposition of the form $p_1^{a_1}.p_2^{a_2}.....p_m^{a_m}$ then the same for $n_1$ will be
$n_1 = \left(2^{p_m-1}.3^{p_m-1}....p_{a_m}^{p_m-1}\right)\left(p_{a_m+1}^{p_{m-1}-1}.p_{a_m+2}^{p_{m-1}-1}......p_{a_m+a_{m-1}}^{p_{m-1}-1}\right)\left(p_{a_m+a_{m-1}+1}^{p_{m-2}-1}.....\right)$.

So $\log n = \sum_{i=1}^{m} a_i \log p_i$ and also
$\log n_1 = (p_m - 1)\log[2.3...p_{a_m}] + (p_m - 1)\log(p_{a_m+1}....p_{a_m+a_{m-1}}) + ....$

Now we will use a well known fact that product of first n primes is asymptotically $e^{n \log n}$. Using this above result changes the above equation

$\log n_1 = (p_m - 1)a_m \log a_m + (p_{m-1} - 1)\Big[(a_m + a_{m-1})\log(a_m + a_{m-1}) - a_m \log a_m\Big] + (p_{m-2} - 1)\Big[(a_m + a_{m-1} + a_{m-2})\log(a_m + a_{m-1} + a_{m-2}) - (a_m + a_{m-1}\log(a_m + a_{m-1})\Big] + ....$

Now to compare $\log n_1$ to $\log n$ we will investigate the increment for each $a_i$'s. We have to begin with the coefficient for $a_m$ in $\log n_1$ .

Observe that $(p_m - 1)a_m \log a_m$ serves as the main term since except this term, others involve decreasing functions which can be arbitrarily small but all these terms are clearly non-negative.

This follows because

$a_i \geq 2$ and $\log(n+2) - \log n = \log(1 + \frac{2}{n}) \to 0$ as $n \to \infty$.

The assumption that $a_i \geq 2$ will be justified shortly.

So the main contribution is due to $(p_m - 1)a_m \log a_m$. And similarly main term related to increment for the co-efficient of $a_{m-1}$ will be $(p_{m-1} - 1)(a_m + a_{m-1})\log(a_m + a_{m-1})$ which is greater than $(p_m - 1)a_m \log a_m$. An interesting thing to observe is that the above inequality cannot be considerably made better since $a_m$ can be as small as 2 and $\log(n+2) \sim \log n$. So all we have got is the generic main term for increment corresponding to the co-efficient $a_i$ will be $p_i \log a_i$.

For measuring the increase from $\log n$ to $\log n_1$ we try to estimate the increase for each $a_i$.

Now

$[(p_m - 1)\log a_m - \log p_m] \sim [(p_m - 1)\log 2 - \log p_m] \sim [m \log m \log 2 - \log m - \log\log m]$ (using $p_n \sim n \log n$).

Now for the function

$f(x) = x \log x \log 2 - \log x - \log\log x$ we seek to find its minimum and for that we solve for its derivative.

This clearly is the solution of the equation

$(\log 2)x(\log x)^2 + (\log 2x - 1)\log x = 1.$

$\implies x = 0.130488$ or $2.39604$.

So from here we get that the minimum increase will be at-least

$(p_m - 1)\log a_m - \log p_m \sim 2\log\log 2 - \log 2 - \log 2 \geq 0.634.$

So $a_m((p_m - 1)\log a_m - \log p_m) \geq 2$ x $0.634 = 1.268$ So evidently we have

$\log n_1 - \log n \geq m.(1.26)$

$\implies \log_{10} n_1 - \log_{10} n \geq 0.545\ \nu(n)$

where $\nu(n)$ is the number of distinct prime divisors of n. Since there are at least 2 distinct prime divisors with $a_i \geq 2$, we are done.

So by inductive argument we have the minimum size of n for which $d^k(n) = 2$ occurs is at-least $10^k$.
Correspondingly $\forall$ n, k has size $O(\log n)$.
The bound for k can be considerably improved for large n using a well known result due to Wigart. See [4] for more information.

$$\limsup_n \frac{\log d(n) \log\log n}{\log n} = \log 2$$

which translates to: given $\epsilon > 0$, $\exists N_0$ such that $\forall n \geq N_0$ we have

$$d(n) < n^{\frac{\log 2(1+\epsilon)}{\log\log n}} \tag{6}$$

$$\implies \log n > \frac{\log\log n}{\log 2(1+\epsilon)} \log d(n) \tag{7}$$

This clearly improves the bound on k. Assuming $d(n_1) = n$, we have to choose $n \geq max\big(N_0, \frac{N_1}{10}\big)$ where $N_1$ is the least integer such that $\log\log N_1 \geq \log 2(1+\epsilon)(1+c)$

$$\log n_1 > \frac{\log\log n_1}{\log 2(1+\epsilon)} \log d(n) \tag{8}$$

$$\implies \log n_1 \geq (1+c)\log n \tag{9}$$

here $c > 0$ is a constant.
So we have by iteration $\log n_1 \geq (1+c)^k \log 2$
So $k = O(\log\log n)$ for large enough n.
We observe that :
k : 1 2 3 4 5 6 ...
n : 2 4 6 12 60 5040 ...
Here given $k$ we have listed the least $n = n_k$ for which $d^k(n) = 2$. Now we make the following conjecture.
**Conjecture:** All the $n_k$'s which are produced by our algorithm are highly composite numbers. For a complete idea about what highly composite numbers are we refer [1].
From a well known result(for more information about the source see [2]) we have:

$$\max_{n\le x} d(n) = exp\Big(\log 2\frac{\log x}{\log\log x} + O(\frac{\log x \log\log\log x}{(\log x)^2})\Big)$$

So for large $n_k$ we expect that $\log n_{k-1} \sim \log 2\frac{\log n_k}{\log\log n_k}$

$$\max_{n\le n_k} d(n) = exp\Big(\log 2\frac{\log n_k}{\log\log n_k} + O(\frac{\log n_k \log\log\log n_k}{(\log n_k)^2})\Big)$$

$$\implies \max_{n\le n_k} d(n) \sim exp\big(\log 2\frac{\log n_{k-1}}{\log 2}\big)$$
$\implies \max_{n\le n_k} d(n) \sim n_{k-1} \implies n_k$ is highly composite.

# 5 Acknowledgement

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