The origin of Gravity

An attempt to answer this question.

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Contents.

Paragraph.	page
Symbols used.	2
1. Introduction.	4
2. Specific Relativity Theory.	5
3. Quantum rules.	8
4. Groups of particles and sub-spaces.	13
5. The sub-space in more detail.	16
6. Summary of the general treatment and further steps to be taken.	21
7. Relativistic interaction.	24
8. Law of gravity.	28
9. Transfer of mass and energy.	30
10. The hydrogen atom and the planet Mercury.	33
11. The Bohr-Einstein controversy.	36
12. Summary of the preceding paragraphs and some remarks for completeness.	38
13. Gravity in dependence on dynamical mass.	41
14. Cosmological consequences. Dark matter and energy?	44
15. References.	49

Symbols used.

$x, y. z, t, x', y', z', t'$	spatial coordinates and time in Σ, and Σ'
Σ, Σ'	space of observer, space of observed
r	spherical coordinate
r_p	particle radius
r_{loc}	local spherical coordinate
r_{inf}	global spherical coordinate
r_i	spherical coordinate of i^{th} particle
r_{ij}	distance from i^{th} particle in group (ij) in sub-space
i	complex number ($\sqrt{-1}$)
i, j, k, g	index i, j, k, g
c	velocity of light (3x10^8 m/sec)
v	velocity
g	group index g
R, $\boldsymbol{R}$	distance between particles and vectorial distance between particles
R_u	radius of universe
m, m_0	mass, rest mass
m_i, m_j	mass of particle i and j
m_p, m_e	proton mass (1.7x10^{-27}kg), electron mass (0.91x10^{-30}kg)
$\boldsymbol{m(x, y, z, t)}$	expectation value of mass in sub-space
M_S, M_M	mass of sun (1.99x10^{30} kg) and mass of Mercury (3.3x10^{25} kg)
M_{10}, M_{20}	rest masses of body 1 and 2
$M(R_u)$	mass of the universe
p_x, p_y, p_z	momentum in $x, y,$ and z direction
$\widehat{p_x}.\ \widehat{p_y}.\ \widehat{p_z}$	momentum operators
$\widehat{p_i}.\ \widehat{p_i}.\ \widehat{p_{ij}}$	momentum operator for particle i, j and particle i in group (ij)
$\boldsymbol{P}$	expectation value of momentum
$\hat{\mathrm{H}}(\widehat{p_x}, \widehat{p_y}, \widehat{p_z}, x, y. z)$	Hamilton operator
E	energy
E_{loc}	energy due to local interactions
E_{inf}	energy in open space

$\hat{E}(t)$	energy operator
E_{ij}	energy of particle i in group (ij)
$V(x,y,z)$	potential energy
V_{ij}	potential energy of particle i in group (ij)
$\varepsilon_{ij} = E_{ij} - V_{ij}$	kinetic energy of particle i in group (ij)
$\gamma_k = 1/\sqrt{1 - v_k^2/c^2}$	relativistic transformation factor for particle $k=i,j$ or group $k=g$
$\psi_i(x,y,z,t)$	wave function of particle i
$\Psi_{loc}(r_{loc},t)$	local wave function
$\Psi_{inf}(r_{inf},t)$	wave function in free space
$\Psi_{ij,t}$	wave function of particle i in group (ij)
$\varphi_{ij,t}, \varphi_{ij}$	time dependent and time independent wave function of particle i in group ij as solution of the KG-equation
$\phi_{kl} = \varphi_{1k}\varphi_{2l}$	interaction potential
δ_k	angle
α_{ij}	amplitude x r_{ij} of particle i in group (ij)
β_{ji}	2π x inverse wave length of particle i in group (ij)
$\hbar$	Planck's constant/2π (1.054×10^{-34} Jsec)
h	Hubble constant (2.3×10^{-18} sec^{-1})
γ_{ij}	amplitude of relativistic particle i in group (ij) in sub-space
σ	connection factor (2.7×10^{2} Jm/kg^4)
G	gravity constant (6.673×10^{-11} m3/kg.sec^2)
$\boldsymbol{F}_{12}$	vectorial force between particles
N	number of particles
$\mathcal{N}$	number of groups
Ze	electric charge (Zx1.6×10^{-19} Coulomb)
ε_0	dielectric constant ($10^7/4\pi c^2$)
T, T_k	period (paragraph 10), kinetic energy (paragraph 13 and14).
ω	angular velocity
ρ_0	average rest mass density of universe (10^{-24} kg/m^3)

1. Introduction.

In our daily life, gravity is experienced everywhere and at all moments. Without gravity the world as an entity would not exist, the Sun would not shine, water waves would not run, etc. Even if we would evaluate the consequences of a small change in the gravitational interaction, the universe would look different from how it is now. It is accepted as an inescapable force that keeps our existence together. However, where we have some basic understanding of the processes around us, there does not exist a suitable explanation for this force at a microscopic level.
Gravitation interaction manifests itself where other forces are not the determining factor. Therefore, in our real world, we see that our direct vicinity has structures of forms that are changing over short distances like mountains, cities, sky scrapers, boats, forests etc. At larger distances, of the order of 100 kilometers, the gravity becomes the dominant factor and bodies begin to take spherical shapes. Obviously, the smaller the gravity is, so to speak at smaller planets than earth, the structural variability will become larger. That the electromagnetic interaction becomes insignificant in shaping the environment is not due to the form of the electrostatic interaction, which has basically the same shape as the gravitational interaction, but it is due to the fact that positive and negative charges balance and compensate for their interaction, the influence of electromagnetism is becoming insignificant already at short distances.
Now the general belief is that any suitable theory should include, or will be, a merger of classical quantum theory and relativity, but until now no theory that is widely accepted has been proposed. In the present document a new scheme of analysis for the mutual interaction between particles that have some exchange with respect to time and space will be presented. The remarkable thing is that, apparently for more than one reason, particles will be interacting in groups of two and only two and can give rise to gravitational exchange. This pair formation is described quantum mechanically. Either starting from the classical Schrödinger equation or the relativistic Einstein energy equation, but this latter formulated in a quantum mechanical setting known as the “Klein Gordon” (KG) equation, results in the same wave function describing pairs of particles. Since this wave function represents a pair potential, a relativistic mass can be attributed to it which is used in the KG-equation to derive an interaction field between the members that form the ensemble. It is found that the right form of Newton’s gravity law emerges by consequently working through the proposed schemes of both quantum mechanics and the basic equations of relativity theory

as expressed by the quantum mechanical equivalent of the Einstein energy equation.

This document is based earlier papers by H.J. Veringa, published in 2016 in the Journal of Modern Physics, [1] and [2], but at places data, interpretations and conclusions are improved.

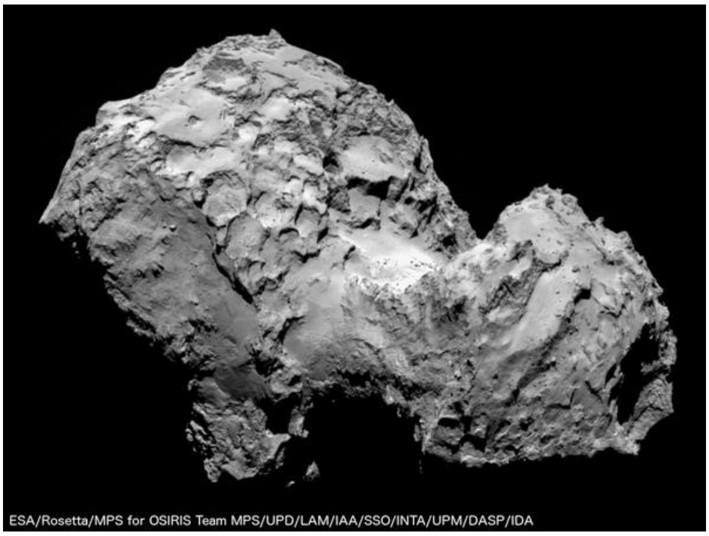

Figure 1: *The satellite Rosetta was launched in 2004 and arrived at Comet 67P/Churyumov-Gerasimenko on 6 August 2014. It is the first mission in history to rendezvous with a comet, escort it as it orbits the sun, and deploy a lander to its surface. The mission ended on 30 September 2016. The comet is an irregular object roughly 3 kilometers wide and 5 kilometers across*

Figure 2: *The earth moon is the biggest object in our night sky, but that is only because it is the closest celestial body. The moon is about 27% of the size of the earth. If equal densities are assumed, it would be 2% of the mass of the earth, but it is about 1.2%. The earth moon is the fifth largest moon in our solar system. The gravitational acceleration at the moon's surface is about 17% of that of the earth (1.7 m/sec^2).*

2. The Special Relativity Theory.

At the beginning of the previous century it became apparent that the basic rules of mechanics shows some discrepancies when speeds are increasing. In daily life this was, however, not serious because the speeds at which discrepancies occur are far beyond the speeds which we are used to work with, but some remarkable facts, particularly when our understanding of cosmology increased, were observed which did not allow an explanation on the basis of classical Newtonian mechanics. Particularly when it became possible to measure with good accuracy the speed of light the peculiarities became even bigger. It was thought that the earth and light are moving through a stationary cosmological substance, the "ether frame", so that any measurement of the speed of light would depend on the direction at which we would measure it. It was the Michelson-Morley experiment [3] that showed that, whatever we try, we will always find the same value of the speed of light: c = 3×10^8 m/sec.
This unexpected result remained puzzling for some years but later it was realized that also the laws of electromagnetic are entirely independent of the speed and place of the observer.
So if we have a frame of reference, a coordinate system, in which the observer is situated, *Σ,* and a moving one travelling with speed *v*, which we call *Σ'*, the distance between points in either system remains the same. As the information about this distance is based on visual observation of things which are happening, we can conclude that if the coordinates, including time, in *Σ* are x, y, z and t and in *Σ'* they are x', y', z' and t', the following relation must hold:

$$x^2 + y^2 + z^2 - c^2t^2 = x'^2 + y'^2 + z'^2 - c^2t'^2. \tag{2.1}$$

Lorentz proposed on the basis of this invariance a scheme of transformations between the coordinates in Σ and Σ' to guarantee that this invariance is always valid. These transformations, known as the Lorentz transformations, can be derived from this equation (2.1) and read:

$$x = \gamma(x' + vt'),\ y = y',\ z = z', ct = \gamma(ct' + vt'), \tag{2.2}$$

with: $\gamma = 1/\sqrt{1 - v^2/c^2}$.

Einstein realized that there is more in this than just a few transformation rules and generalized this idea by proposing the concept of "four-vectors". Four-vectors are mathematical objects in four dimensional space which transform according to the Lorentz rules and therefore have the Lorentz invariance as in equation (2.1). Such a four vector is symbolized as $\{x,y,z,ct\}$. There are many different four-vectors, particularly in electrodynamics, but for the present theory we will only need, next to the one mentioned, the momentum-energy vector $\{p_x,p_y,p_z,E/c\}$ with the invariance:

$$p_x^2 + p_y^2 + p_z^2 - E^2/c^2 = C(\text{constant}). \tag{2.3}$$

With the help of the famous mass-energy equation of Einstein: $E = \gamma m_0 c^2$ this invariance is readily rewritten as an equation that will be of great importance for the rest of this document:

$$E^2 - p^2c^2 = m_0^2c^4. \tag{2.4}$$

The gradient operator $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{1}{c}\frac{\partial}{\partial t}\}$ is a special one with the invariance, but applied to a field φ:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\varphi = C\varphi, \tag{2.5}$$

which is the well known equation describing the movement of a wave through a medium. If this medium is the vacuum, we can set C = 0, and signals move at the speed of light as the general solution is:

$$\varphi = f(x - ct)g(y - ct)h(z - ct).$$

In which f, g and h are arbitrary functions in the arguments as indicated.

The famous Lectures on Physics [4] gives a very good introduction into four-vectors as they are dealt with in this paragraph. Further reading Special Relativity is recommended in Ney [3].

3. Quantum rules.

In quantum theory the behaviour of a microscopic particle in space and time is described by a wave function, denoted by ψ. This wave function is a mathematical expression in space and time, usually in complex notation. The product of this wave function and its complex conjugated ψ*: ψ ψ* is the probability that a particle can be found at that particular place and moment. Quantum mechanics relies largely on operators. Operators are mathematical abstractions that do something with a wave function. In the first place a wave function is a solution of the operator equation like, as an example for the momentum:

$$\widehat{p_x}\psi = \frac{\hbar}{i}\frac{\partial}{\partial x}\psi = p\psi. \tag{3.1}$$

The expectation value of the momentum in this example then is given by:

$$\boldsymbol{P} = \int \psi^* \, \widehat{p_x}\psi \mathrm{dV}, \tag{3.2}$$

with the integration over the entire space where the operator is active. As a consequence of this the normalisation of a wave function is:

$$\int \psi^* \, \psi \mathrm{dV} = 1.$$

The famous Schrödinger equation derives from the basic rule based on the Hamiltonian and energy operator acting on this wave function and gives the energy as a function of momentum and space as a solution. In short it reads::

$$\hat{\mathrm{E}}(t)\psi(x,y,z,t) = \hat{\mathrm{H}}(\widehat{p_x},\widehat{p_y},\widehat{p_z},x,y.z)\psi(x,y,z,t). \tag{3.4}$$

In this equation the time dependence is allocated to the energy operator by $\hat{\mathrm{E}}(t) = i\hbar\frac{\partial}{\partial t}$, while the dependences on momentum and place are allocated to the Hamiltonian. The momentum in the Hamiltonian is, obviously, also an operator and given by $\widehat{p_x} = \frac{\hbar}{i}\frac{\partial}{\partial x}$, and similarly for the *y*- and *z*- coordinates. It is worthwhile to note that the conjugated wave function is a solution of the conjugated Schrödinger equation and not the result of an operation on the wave function itself.

For the development of the present model we can write the Hamiltonian more explicitly as: $H = (p_x^2 + p_y^2 + p_z^2)/2m + V(x,y,z)$ or as operators:

$$\hat{\mathrm{H}}(\widehat{p_x},\widehat{p_y},\widehat{p_z},x,y.z) = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) + V(x,y,z). \tag{3.5}$$

This equation has been very successful in describing the behaviour of microscopic particles in their local environment. Such a local environment can be anywhere: in open space as well as in structures like a solid. But once speeds are getting higher, it is found that the validity of this equation breaks down.
For this reason another energy equation which better fulfills the relativistic behaviour is proposed based on the equation (2.4), but now in the form of operators:

$$-\hbar^2\left(\frac{\partial^2}{\partial t^2}-c^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\right)\varphi(x,y,z,t) = m_0^2c^4\varphi(x,y,z,t). \quad (3.6)$$

This equation is the relativistic alternative for the Schrödinger equation. It is called the "Gordon Klein" equation. From the form it can be seen that the wave function has the character of a travelling wave in open space. If there is no mass, $m_o = 0$, it will propagate with the speed of light c. With mass the propagation speed will always be lower than the speed of light.
Both wave equations (3.4) and (3.6) with the Hamilton operator in (3.5) will be used in the further development of the theory.
If we bring together the two equations in their basic form:

$$E = (p_x^2+p_y^2+p_z^2)/2m + V(x,y,z), \text{ and:} \quad (3.7a)$$

$$E^2 = (p_x^2+p_y^2+p_z^2)c^2 + m_0^2c^4, \quad (3.7b)$$

we see immediately that in the second equation all the parameters show up as squares whereas in the first one it is a mixed representation. This means that the quantum mechanical rules and the relativistic rules are incompatible. But there are ways to circumvent this problem. For instance, Dirac found a way to fulfill both equations and arrived at the magnetic moment of particles and its peculiarities.
In respect of the development of the theory of gravity one important remark has to be made about the use of operators. The expectation value for the mass is found by considering the mass as an operator. Therefore the mass distribution due to the wave function $\psi(x,y,z,t)$ is given by:

$$\boldsymbol{m(x,y,z,t)} = \psi^*(x,y,z,t)m(x,y,z,t)\psi(x,y,z,t). \quad (3.8)$$

Although this looks self-evident, it however is not. The equation (3.7b) concerns the square of the rest mass distribution $m_0^2(x,y,z.t)$ and the expectation value is:

$$\boldsymbol{m_0^2(x.y,z,t)} = \psi^*(x,y,z,t)m_0^2(x,y,z,t)\psi(x,y,z,t). \tag{3.9}$$

To finish these more general rules in this paragraph one more thing has to be done. As we are dealing with particles which do not have a geometrical structure we can adopt spherical symmetry throughout the development of the theory. For this reason the coordinates x, y, and z are not so practical. It would be more easy to work with coordinates which have the same symmetry as the particles and their environment. The main issue therefore is to cast the operator $\widehat{p_x}^2 + \widehat{p_y}^2 + \widehat{p_z}^2$ into a more appropriate form involving spherical symmetry so that the only coordinate r is necessary.
This is first done for a system of one particle, starting from the x-, y-, z- component of the gradient operator $(grad)_x\psi$, $(grad)_y\psi$, $(grad)_z\psi$, only taking r- dependences:

$(grad)_x\psi = \frac{\partial\psi}{\partial x} = \frac{\partial\psi}{\partial r}\frac{\partial r}{\partial x} = \frac{x}{r}\frac{\partial\psi}{\partial r}$, and differentiating again:

$$(grad)_x(grad)_x\psi = \frac{\partial^2\psi}{\partial x^2} = \frac{1}{r}\frac{\partial\psi}{\partial r} - \frac{x^2}{r^3}\frac{\partial\psi}{\partial r} + \frac{x^2}{r^2}\frac{\partial^2\psi}{\partial r^2},$$

and the same for the y- and z- coordinates we find, remembering that $x^2 + y^2 + z^2 = r^2$:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\psi = \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right)\psi = \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial\psi}{\partial r}. \tag{3.10a}$$

But for a system of two particles, labeled i and j we have to use vector notation for the gradients working on the wave function. This wave function describes in one expression the set of the i and j particle. We first take the gradient for particle i as before:

$$(grad)_{xi}\psi = \frac{\partial\psi}{\partial x_i} = \frac{\partial\psi}{\partial r_i}\frac{\partial r_{i.}}{\partial x_i} = \frac{x_i}{r_i}\frac{\partial\psi}{\partial r_i},$$

and the same for the y- and z- component of the coordinates of particle i and j. So we have in the r_i- direction:

$$(grad)_{ri}\psi = \frac{\partial\psi}{\partial r_{i.}}.$$

Applying the same procedure for gradient in the r_j- direction we get the final result:

$$(grad)_{ri}\psi(grad)_{rj}\psi = \frac{\partial\psi}{\partial r_i}\frac{\partial\psi}{\partial r_j}. \tag{3.10b}$$

Although these rules are not particularly quantum rules, they will be important for the development of the quantum-based theory.

For the further development of the theory, however, one other aspect has to be paid attention to. That is how a particle, which is part of a specific structure like a solid, can be treated. The operators mentioned until now have particular emphasis for freely moving particles, but it is obvious that also particles constituting larger entitles do show gravitational interaction.
From the basic concept of quantum mechanics we know that particles do have a non zero probability to show up anywhere in space. This probability can be very small and for the determination of its behaviour in its local environment it is so small that it is usually neglected. But for gravitational interaction, which in essence is extremely small compared to any other force, it is of relevance.
To investigate this problem we therefore construct a wave function that combines its local behaviour with its global one,
The total wave function describing a particle under its local influences, $\Psi_{loc}(r_{loc}, t)$, and its extension in free space, $\Psi_{inf}(r_{inf}, t)$, is given by: $\Psi_{tot} = \Psi_{loc}.\Psi_{inf}$. The coordinate r_{loc} is the position of the centre-of-mass of the particle inside the atom or nucleus or a solid object and the coordinate r_{inf} is the position of the particle from the point of view of an outside observer. They therefore are mutually independent. In the same way we define, as before, the Hamilton operator as: $\widehat{H}_{tot} = \{\hat{p}^2\}_{loc}/2m_{loc} + \{\hat{p}^2\}_{inf}/2m_{inf} + V_{loc}(r_{loc}) + V_{inf}(r_{inf})$. The masses m_{loc} and m_{inf} are not necessarily the same. The m_{inf} is the mass to be connected to the particle as it can move freely around whereas m_{loc} is the mass of the particle under the influence of the local interactions, sometimes called “reduced mass”. It follows that:

$$\widehat{H}_{tot}\Psi_{tot} = (\{\hat{p}^2\}_{loc}/2m_{loc} + \{\hat{p}^2\}_{inf}/2m_{inf} + V_{loc} + V_{inf})(\Psi_{loc}\Psi_{inf}) =$$

$$= (\{\hat{p}^2\}_{loc}/2m_{loc} + V_{loc})\Psi_{loc}\Psi_{inf} + (\{\hat{p}^2\}_{inf}/2m_{inf} + V_{inf})\Psi_{loc}\Psi_{inf}. \tag{3.11}$$

Separating the local effect from the surroundings we can set:

$(\{\hat{p}^2\}_{loc}/2m_{loc} + V_{loc})\,\Psi_{loc} = E_{loc}\Psi_{loc}$ and: (3.12a)

$(\{\hat{p}^2\}_{inf}/2m_{inf} + V_{inf})\,\Psi_{inf} = E_{inf}\Psi_{inf}$. (3.12b)

The first equation (3.12a) is the Schrödinger equation describing the behaviour of the particle in its local environment like in the nucleus or a solid where it has its individual interactions. The second equation (3.12b) describes its movement or presence in the free space in which the particle, or as part of a larger entity, can move around. By taking V_{inf} as a constant it is assumed that the behaviour out of its local influences is taken into consideration. This second equation is the starting point in the development of the theory in the next paragraphs. The splitting up as in equation (3.12a) and (3.12b) disconnects the local interaction of separate particles, as is normally done in quantum mechanics, from the movement or presence of the particle individually or as part of a larger entity. In what follows we will only consider the second equation as this gives the generator for the gravitational interaction. As we are interested in the effects of masses outside the local interactions we will from now on take for the mass m_{inf} the quantity m, as it will also be the case for the coordinate.

More on this subject can be read in Ney [3], Messiah [5] and [6] and Heitler [7].

4. Groups of particles and sub-spaces.

Gravity is an attractive force between two bodies, or, at a microscopic level, two particles and therefore any theory will have to account for multi-particle systems. The previous paragraph has shown that the Hamiltonian will be the central operator but in that representation it only accounts for single entities. For the development of the theory we have to modify the Hamiltonian for multi-particle systems.

The most simple expression for the kinetic energy in the Hamiltonian for a group of particles numbered by k is given by:

$$\hat{p}^2/2m = \sum_k \widehat{p_k}^2/2m_k \ . \tag{4.1}$$

This expression does, however, not clearly enough describe the behaviour of particle interaction as members of a group, but it will be shown that an alternative representation is possible in which still the total kinetic energy remains the same. The first step is to write equation slightly different:

$$\frac{\hat{p}^2}{2m} = \sum_k \widehat{p_k}^2/2m_k = \sum_k \left(\widehat{p_k}/\sqrt{2m_k}\right)^2. \tag{4.2}$$

This equation does not look so special, but it shows that, if we want to modify the kinetic energy in the Hamiltonian, we will have to perform our analysis in the $\widehat{p_k}/\sqrt{2m_k}$ –space.

For reasons that will become clear later we will now modify the Hamiltonian for the two-particle ensemble (*ij*) and refer to Figure 3.

In Figure 3 particles m_i and m_j are moving with momenta p_i and p_j. But we are interested in their behaviour in the space as seen from point O_2 and therefore we apply the cosine-rule to both triangles [1] and [2]. Knowing that:

$\cos\delta_2 = -\cos(180-\delta_2)$ and taking $p_{ij}/\sqrt{2m_i} = p_{ji}/\sqrt{2m_j}$, it follows that:

$$\widehat{p_i}^2/2m_i + \widehat{p_j}^2/2m_j = \widehat{p_g}^2/2m_g + \widehat{p_{ij}}^2/2m_i + \widehat{p_{ij}}^2/2m_j. \tag{4.3}$$

In this modified kinetic energy part of the Hamiltonian the first term at the right hand is the kinetic energy of the group, identified with label g, consisting of m_i and m_j with mass $m_g = m_i + m_j$ and moving as one single entity. The second term is the kinetic energy in the sub-space. The group momentum vector $\widehat{p_g}/\sqrt{2m_g}$ is not equal to any of the other ones so that $\widehat{p_g}$ has to be defined separately. As the interaction between the two particles is only within the sub-space r_{ij}, we will not have to bother about this first term at the right hand. This is very fortunate because it depends on the angle between $\widehat{p_i}/\sqrt{2m_i}$ and $\widehat{p_j}/\sqrt{2m_j}$ which would severely complicate the problem.

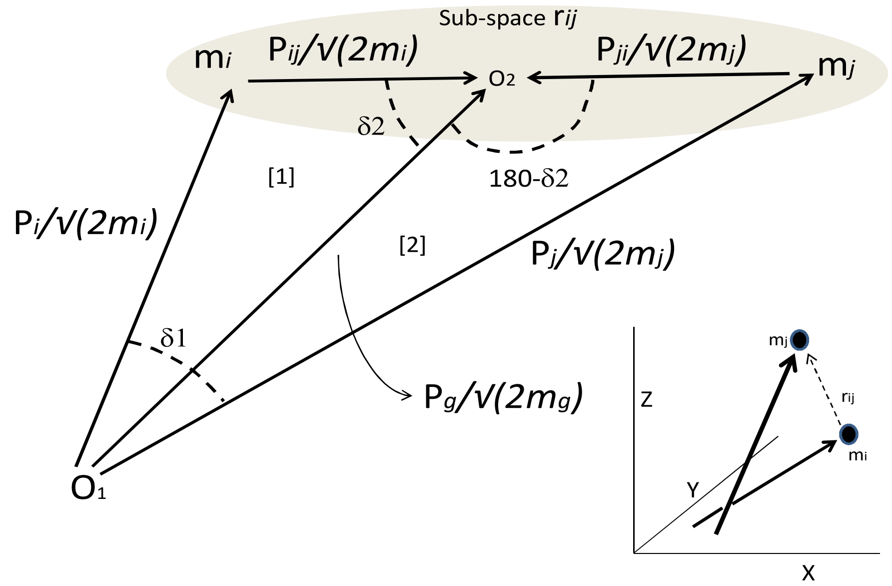

Figure 3: *The relation (4.3) found by applying the cosine-rule to both triangles [1] and [2] if the lengths of the arrows $p_{ij}/\sqrt{2m_i}$ and $p_{ji}/\sqrt{2m_j}$ are the same. In this view vectors and operators are treated as equivalent. Note that, differently from the situation in the real x-, y-, and z- space, in the representation in the two-dimensional momentum space the velocity or the momentum vectors for particles always have the same origin.*

It can also be seen that this modification of the Hamiltonian only works well for two particles as the geometrical argument is confined to one plane. More particles would compel us to perform the analysis in many more different planes and would not give a tractable solution.

Another important observation is that because $p_{ij}/\sqrt{2m_i} = p_{ji}/\sqrt{2m_j}$ the sub-space is symmetric from the point of view of an observer in O_2. This issue of symmetry will come back in the solution of the Schrödinger equation with the modified Hamiltonian in the sub-space.

We will now extend the modified Hamiltonian equation for more than two particles, but all of them interacting in groups of two and only two:

$$\sum_k \widehat{p_k}^2/2m_k = 1/N(\sum_g \widehat{p_g}^2/2m_g + 1/2\sum_{i\neq j}(\widehat{p_{ij}}^2/2m_i + \widehat{p_{ij}}^2/2m_j). \quad (4.4)$$

The N-factor, the number of particles, is necessary as in the summation each particle is counted N times. The pairs are counted by the g-index. Later, when the analysis brings us to the final result, we will come back to the group momentum and evaluate the consequence of its dependence on the momenta of m_i and m_j.

For completeness we will now derive this dependence but come back to it later. For this we apply the cosine-rules from the corner O_1 for the triangles [1] and [2] separately and together: [1+2]. the equations are:

$$\left(\widehat{p_{ij}}/\sqrt{2m_i}+\widehat{p_{ji}}/\sqrt{2m_j}\right)^2=\widehat{p_i}^2/2m_i+\widehat{p_j}^2/2m_j-$$

$$-2\widehat{p_i}\widehat{p_j}cos\ _1/\sqrt{4m_im_j}, \tag{4.5a}$$

$$\widehat{p_g}^2/2m_g=\widehat{p_i}^2/2m_i+\widehat{p_j}^2/2m_j-\widehat{p_{ij}}^2/2m_i-\widehat{p_{ij}}^2/2m_j. \tag{4.5b}$$

This cosine factor showing up complicates the analysis, but in the end it will not trouble our analysis as it can be circumvented.

Now also the analysis will have to be repeated starting from the equation (2.4), but as it is only dealing with the momenta, the result of the previous analysis can be used if we simply replace the vector $p_k/\sqrt{2m_k}$ by $cm_k\sqrt{\gamma_k^2-1}$ with the k-label representing i, j, g and p_{ij}, p_{ji} unchanged. The analysis will be continued in paragraph 13.

5. The sub-space in more detail.

Starting from the unmodified Hamiltonian, the general solution of a wave equation describing independent particles in spherical symmetry is initiated by the operator:

$\hat{p}^2/2m = \sum_k \hat{p}_k^2/2m_k$, and reads:

$$\Psi = ..\Psi_i\Psi_j \ldots \Psi_l = ..\left(\frac{\alpha_i}{r_i}\right) e^{i\beta_i r_i} \text{x} \left(\frac{\alpha_j}{r_j}\right) e^{i\beta_j r_j} \text{x....x} \left(\frac{\alpha_l}{r_l}\right) e^{i\beta_l r_l} \text{x} \ldots =$$

$$= \prod_k \left(\frac{\alpha_k}{r_k}\right) e^{i\beta_k r_k}. \qquad (5.1)$$

For this equation the equations (3.4), (3.5), and (3.10) have been used. But we have regrouped the kinetic contribution to the Hamiltonian for the same set of particles as:

$$\frac{\hat{p}^2}{2m} = \sum_k \hat{p}_k^2/2m_k \ = 1/N(\sum_g \hat{p}_g^2/2m_g \ + 1/2\sum_{i\neq j}(\hat{p}_{ij}^2/2m_i \ + \hat{p}_{ji}^2/2m_j)),$$

and first we will only consider the second part of it at the right hand side, to start with the group (ij) of two particles only, thus we restrict ourselves to the sub-space with coordinates r_{ij}.

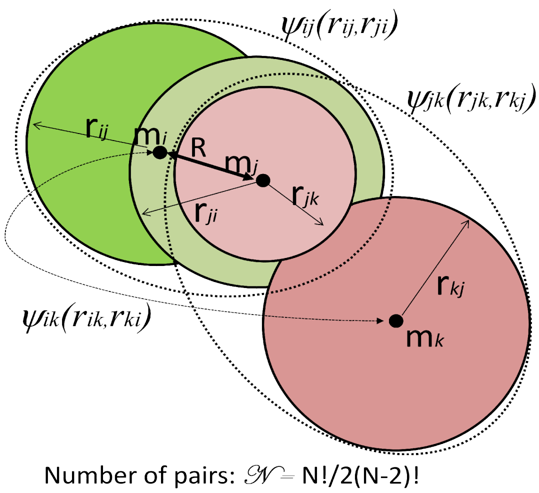

***Figure 4**: Forming and describing of $\mathcal{N}$=N!/2(N-2)! Pairs. In this example the number of groups is three.*

Per group there are two independent particles, for the group under consideration like in Figure 4, it is indicated by the masses m_i and m_j and they experience some force reflected by the potential V_i and V_j. Spherical symmetry is next adopted and the only boundary condition is that the wave function is zero at infinity. An observer at m_i at a distance r_{ij} from particle m_j and another on m_j at r_{ji} from particle m_i will see that the total wave equation of the individual pair (ij) is defined as follows [5]:

$$\widehat{H_{ij}}\Psi_{ij,t} = i\hbar\frac{\partial}{\partial t}\Psi_{ij,t} = -\left(\frac{\hbar^2}{2m_i}\frac{1}{r_{ij}^2}\frac{\partial}{\partial r_{ij}}r_{ij}^2\frac{\partial}{\partial r_{ij}} + \frac{\hbar^2}{2m_j}\frac{1}{r_{ji}^2}\frac{\partial}{\partial r_{ji}}r_{ji}^2\frac{\partial}{\partial r_{ji}}\right)\Psi_{ij,t} +$$
$$+\left(V_i + V_j\right)\Psi_{ij,t}. \tag{5.2}$$

$\Psi_{ij,t}$ is the time and space dependent wave function. The time dependence can be removed by replacing the time dependent wave function $\Psi_{ij,t}$ by $\Psi_{ij}e^{iE_{ij}t/\hbar}$. Further, define $V_i + V_j$ by $V_{ij}$and we get:

$$(E_{ij} - V_{ij})\Psi_{ij} + \frac{\hbar^2}{2m_i}\frac{1}{r_{ij}^2}\frac{\partial}{\partial r_{ij}}r_{ij}^2\frac{\partial}{\partial r_{ij}}\Psi_{ij} + \frac{\hbar^2}{2m_j}\frac{1}{r_{ji}^2}\frac{\partial}{\partial r_{ji}}r_{ji}^2\frac{\partial}{\partial r_{ji}}\Psi_{ij} = 0. \tag{5.3}$$

To simplify the equation replace $E_{ij} - V_{ij}$ by ε_{ij} to propose a solution that is valid in areas where the V_{ij} is not of great influence anymore as follows:

$$\Psi_{ij} = \left(\frac{\alpha_{ij}}{r_{ij}} + \frac{\alpha_{ji}}{r_{ji}}\right)e^{i\beta_{ij}r_{ij}+i\beta_{ji}r_{ji}}, \tag{5.4}$$

where $\alpha_{ij}$and β_{ij} are constants independent of space coordinates and time. This solution means that we consider the wave function outside the surroundings where the potential energy with all its peculiarities has a very minor effect on the shape of the wave function. The only interaction that can play a role will then be based solely on gravitational interaction. By substituting the solution in equation (5.3) the following relation is found:

$$-\frac{\hbar^2 i}{r_{ij}r_{ji}}\left(\frac{\alpha_{ij}\beta_{ji}}{m_j} + \frac{\alpha_{ji}\beta_{ij}}{m_i}\right)e^{i\beta_{ij}r_{ij}+i\beta_{ji}r_{ji}} - \frac{\hbar^2}{2}\left(\frac{\beta_{ij}^2}{m_i} + \frac{\beta_{ji}^2}{m_j}\right)x$$
$$x\left(\frac{\alpha_{ij}}{r_{ij}} + \frac{\alpha_{ji}}{r_{ji}}\right)e^{i\beta_{ij}r_{ij}+i\beta_{ji}r_{ji}} + \varepsilon_{ij}\left(\frac{\alpha_{ij}}{r_{ij}} + \frac{\alpha_{ji}}{r_{ji}}\right)e^{i\beta_{ij}r_{ij}+i\beta_{ji}r_{ji}} = 0. \tag{5.5}$$

The complex first term at the left hand side is to be set to zero and in a pair-wise process $\alpha_{ij}\beta_{ji}/m_j + \alpha_{ji}\beta_{ij}/m_i = 0$ and $\beta_{ij}^2\hbar^2/2m_i + \beta_{ji}^2\hbar^2/2m_j = \varepsilon_{ij} = \sigma(m_i + m_j)$ so that for every value of the energy there will be a value for σ and the β's can adapt themselves. Therefore, whatever is the situation in which m_i and m_j find themselves, there is always a β_{ji} and a β_{ij} and they have no influence on the $\alpha's$ as long as $\alpha_{ij} = \alpha_{ji}$. It means, that the interaction occurs in the sub-space with a pair to be considered as one single entity with a mass of $(m_i + m_j)$ and, apart from the separation between the members of the pair (R), independent of the situation these members are in. Further, it has to be noticed that the Schrödinger equation based on the modified Hamiltonian only is possible for groups of two and only two particles. This conclusion has already been drawn in

a slightly different way in the previous paragraph where the geometrical argument in momentum space is only possible for two partices with momenta vectors in one plane.
We already came across the fact that the sub-space r_{ij} in memontum space for the observer in O_2 in Figure 3 is symmetric and therefore the solution $\Psi(\alpha_{ij}, \alpha_{ji})$ is symmetric, meaning, again, that $\alpha_{ij} = \alpha_{ji}$.
At the moment not much is known about the $\alpha's$, but one requirement to be imposed on the wave function is that it represents a pair of particles. For the time being it can be said that:
i. The $\alpha's$ cannot depend on the running variables in the wave equation: r_{ij} or t. It will be a constant that can only depend on fundamental nature constants and the particle masses.
ii. It should make no difference for the outside world how one member sees its partner or whether and how we see the two members of the pair. It means that we can say: $\alpha_{ij} = f(m_i)f(m_j)$.
iii. There is no pair if either m_i or m_j equals zero so that $f(m_i) = 0$ for $m_i = 0$ and the pair potential should increase linearly with both participating masses in the pair.

To sum up also the movement of the group as one entity and the fact that there are N particles and $\mathcal{N}=N!/2(N-2)!$ pairs, the total wave function is:

$$\Psi = \prod_{ij}\left(\frac{\alpha_{ij}}{r_{ij}} + \frac{\alpha_{ji}}{r_{ji}}\right) e^{i\beta_{ij}r_{ij}+i\beta_{ji}r_{ji}} \prod_g \left(\frac{\alpha_g}{r_g}\right) e^{i\beta_g r_g}. \tag{5.6}$$

The second product is due to the first contribution to the momentum-based energy term in equation (4.4) and, as already mentioned, it generates no gravitational interaction. The index g is identified by the pair (ij) as indicated in figure 3. This term gives no gravitational interaction whereas the first one in the product (5.6) does in the case of two, and only two members in an ensemble where the sum is taken over all possible and unique pairs (ij). As the pairs are to be considered in their own unique coordinate system r_{ij}, there is no reason to consider all the pairs together but only the behaviour of a single pair. In the end we will add up all the contributions of the pairs as shown schematically in Figure 7 in paragraph 8.
There is freedom in the choice of the particles m_i, m_j, ---, m_l ---. It can actuallly be anything like elementary particles, nuclei or even larger entities if, at least, we can describe such an entity by a single wave function in its own coordinate

system and solve the equation to form a pair with another entity.

Later it will be confirmed that, as before and for the sake of symmetry in the mutual gravitational interaction, the two $\alpha's$ should be equal. It also means that the β's have opposite signs and fixed values and, by taking the $\alpha's$ equal, we make their values independent of the masses and the energies of the members of the pair. The ε_{ij} could have been split into two separate quantities as ε_{ij} and ε_{ji} to dedicate the β_{ij}^2 and β_{ji}^2-values to the separate energies of the two particles.
It is also interesting to notice that the solution of the wave equation for the pairs like in equation (5.6) looks different from a solution for a single particle on the basis of the unmodified Hamiltonian as in equation (5.1). For instance, if we take a look at the r_i dependence in the solution (5.6), we see that there is an extra r_i dependent factor in the exponential term. This latter term is insufficient to make such a solution applicable for the operator working on r_i. For it to be sufficient we need the total pre-exponential factor as given in equation (5.6).

The second approach is taking the KG-equation as the starting point. In this way we guarantee full co-variance throughout the entire analysis. The equation has been given already and reads:

E^2 - $p^2c^2 = m_0^2c^4$ or expressed alternatively: $E^2/m_0^2c^4$ - $p^2/m_0^2c^2 = 1$,

and translated into quantum mechanical language for an ensemble of two particles:

$$(E_{ij}^2/m_{i,0}^2c^4 - E_{ji}^2/m_{j,0}^2c^4)\Psi_{ij} - ((\widehat{p_{ij}})^2/m_{i,0}^2c^2 - (\widehat{p_{ji}})^2/m_{j,0}^2c^2))\Psi_{ij} = 0. \quad (5.7)$$

Where $\widehat{p_{ij}}^{\,2}$ is the square of the momentum operator in spherical coordinates as in equation (5.2) or (5.3) and $m_{i,o}$ the rest mass of the particle i in the ensemble (ij). Also in this case it immediately can be seen that, with the solution of the form as in equation (5.4), the same interpretation as before can be given. So there is not much news in this alternative, but a wave equation with zero masses starting from:

$$(E_{ij}^2 + E_{ji}^2)\Psi_{ij} - c^2((\widehat{p_{ij}})^2 - (\widehat{p_{ji}})^2)\Psi_{ij} = 0 \quad (5.8)$$

has a non constant solution in space and time coordinates. This is remarkable as a zero mass particle like a photon can result in a mass-like presence in open space. It may well be that this is the basis for the fact that in the Friedmann cosmological equations also energy related gravitational pull has to be adopted [9].

6. Summary of the general treatment and further steps to be taken.

In the preceding paragraphs some of the basic rules of special relativity theory and quantum mechanics are given insofar they are of relevance for developing a theory for the attractive force: gravitational attraction between two bodies. It is not specified what kind of bodies we are talking about and actually it is not even relevant. There is absolutely no speculation about the validity of these rules. They have shown their validity over and over, but working with both concepts should be done with care as the two concepts, although they have shown their validity, are not completely compatible. In paragraph 3, equation (3.7a) and 3.7b) the basic equations are compared and it can be seen that the basic quantum equation has both linear and squared parameters whereas the relativistic one has only squares. Mathematically we say that the quantum equation is not co-variant and the relativistic one is co-variant. The consequence of co-variance is that the laws of physics are the same no matter the coordinate system, in which it is observed, is moving or not. With this latter determination in mind we can use both but we should be aware of the dangers involved.
When the two concepts are being applied to particles with the aim to describe their movement and connected interaction, we have seen in two different ways that the movement of particles can be described for all the particles together, but individually, and also as in groups of two and only two members. These groups are then described together to arrive at a complete description of the movement of all groups together but mutually independent.

The grouping results in the description in a momentum based sub-space for the individual groups. In this momentum space the sub-spaces appear to be symmetric from the observer's point of view in O_2, as shown in Figure 3. This symmetry will also be reflected in the symmetry of the gravitational interaction which will be shown in what follows.

It has been shown that particles, which can have high probability to be present in a physical entity like a solid material, or whatever, do have some probability to show up in open space outside the entity where it normally is, or stated in quantum mechanical language: where it has highest probability. The probability to show up in open space can be so low that for normal practices it is of no relevance and in usual quantum mechanical considerations it is neglected. If we, however, want to come to a theory for gravity, we cannot ignore its probability in open space. This low probability already has the consequence that for individual particles the gravitational interaction is small but definitely not zero. The other consequence is that the extent where this probability is manifesting itself extends to the whole space. Gravity is a force that is present even at cosmological distances.

Now we come to the central transition point from quantum mechanics to quantum-based relativity.
The wave function as derived gives the presence of an entity for which it is derived. In this case it is the pair potential so that a mass can be dedicated to this potential defined as $\Psi_{ij}^{*} m_o^2 \Psi_{ij}$ and which becomes, based on the operator rules, equal to:

$$\Psi_{ij}^{*} c^4 m_0^2 \Psi_{ij} = c^4 m_0^2 \left(\frac{\alpha_{ij}}{r_{ij}} + \frac{\alpha_{ji}}{r_{ji}}\right)^2. \tag{6.1}$$

As said before, in this expression the m_0 which occurs in the KG-equation can be identified as a quantity that represents the presence of a pair of particles. It should therefore be linearly dependent on the masses of the participating particles in the pair. The same applies for both the amplitude of the wave function and its complex conjugated. Apart from its proportionality with $m_i m_j$, it follows from $\alpha_{ij}\beta_{ji}/m_j$+$\alpha_{ji}\beta_{ij}/m_i$= 0 with $\alpha_{ij} = \alpha_{ji}$ that there is for the α-values some freedom in choosing its dependence on relativistic parameters such that the right hand side of equation (3.6) becomes an invariant as it should be.

In Figure 5 the comparison is made between the effect of two particles separately according to equation (5.1) and the same particles in the group as an entity according to equation (6.1). The overlap in the factor $\left(\alpha_{ij}/r_{ij} + \alpha_{ji}/r_{ji}\right)^2$ is due to an increased amplitude between the particles in the group. In this respect it is worthwhile to refer to the analogy in the preface.

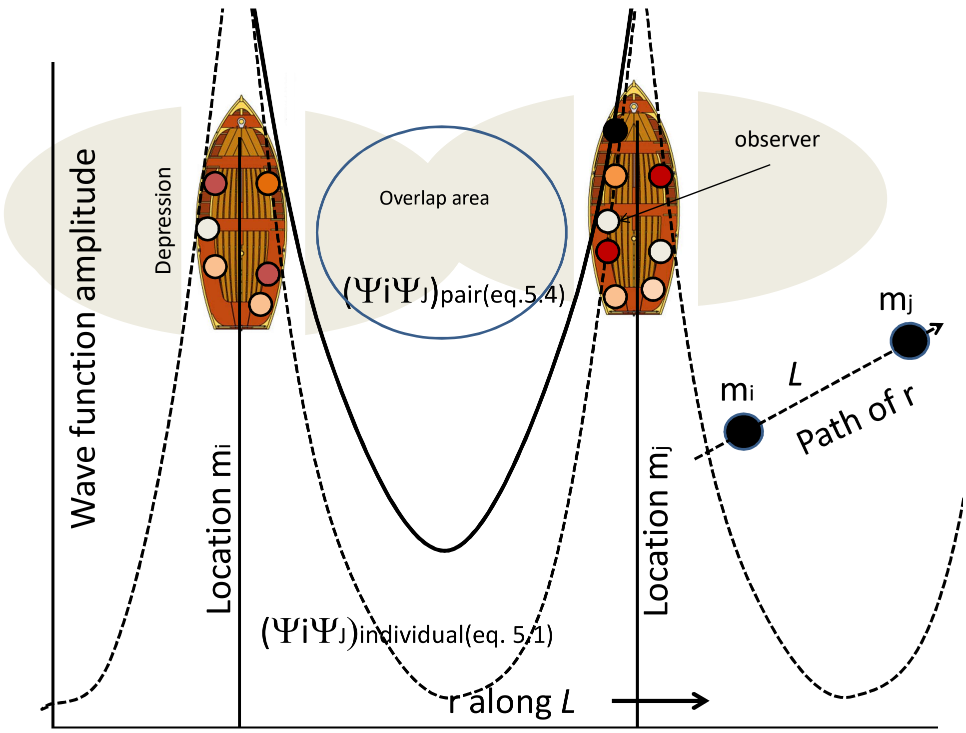

Figure 5: *Overlap of amplitudes in a group of two particles in a group. Compare this with the analogy of the two boats sailing as described in the Preface.*

7. Relativistic interaction.

Now, as a next step, the pair is considered as essentially one entity and the problem can be analysed in the relativistic four dimensional space where the KG-equation is the appropriate starting point. But the most important difference from the treatment before is that we will be entirely working in the momentum based sub-space r_{ij} where the group is seen as one single entity. The energy reflects the energy of the two particles together as well as masses and momenta like: $p^2 = (\widehat{p_{ij}} + \widehat{p_{ji}})^2$ and: $E^2 = \left(\widehat{E_i} + \widehat{E_j}\right)^2$ with:

$E^2 - p^2c^2 = m_0^2c^4$ or: $-p^2c^2 = m_0^2c^4 - E^2$.

Again we will have to translate this equation into the appropriate quantum mechanical language for pairs as one entity and therefore make the following transformations:

$$-p^2c^2\varphi_{ij,t}\varphi_{ji,t} = (m_0^2c^4 - E^2)\varphi_{ij,t}\varphi_{ji,t},\quad E^2 = \left(\widehat{E_i} + \widehat{E_j}\right)^2 = -\hbar^2\frac{\partial^2}{\partial t^2} \text{ and:}$$

$$p^2 = (\widehat{p_{ij}} + \widehat{p_{ji}})^2 = -\hbar^2\left(\frac{1}{r_{ij}^2}\frac{\partial}{\partial r_{ij}}r_{ij}^2\frac{\partial}{\partial r_{ij}} + \frac{1}{r_{ji}^2}\frac{\partial}{\partial r_{ji}}r_{ji}^2\frac{\partial}{\partial r_{ji}} + \frac{\partial}{\partial r_{ij}}\frac{\partial}{\partial r_{ji}} + \frac{\partial}{\partial r_{ji}}\frac{\partial}{\partial r_{ij}}\right).$$

The last expression is, as different from earlier, a mixed sum of the momenta. This representation is a consequence of the fact that the particles have been treated only in pairs and that spherical symmetry remains to be adopted. The energy operators involving the momenta, which are used here, are given in equations (3.10a) and (3.10b).

Referring to Figure 4 the total relativistic KG-equation for a number of pairs ($\mathcal{N}$) now will be set up. There are N particles which make a total of $\mathcal{N}=N!/(2(N-2)!$ pairs, each of which are described by a wave function as a solution of the initial Schrödinger equation. As before, the α-values accommodate all necessary multiplication factors. Adding up for all pairs and treating them as mutually independent and taking into account the basic rules of quantum mechanics, lead to:

$$c^2\hbar^2\sum_{ij}\left(\frac{1}{r_{ij}^2}\frac{\partial}{\partial r_{ij}}r_{ij}^2\frac{\partial}{\partial r_{ij}} + \frac{1}{r_{ji}^2}\frac{\partial}{\partial r_{ji}}r_{ji}^2\frac{\partial}{\partial r_{ji}} + \frac{\partial}{\partial r_{ij}}\frac{\partial}{\partial r_{ji}} + \frac{\partial}{\partial r_{ji}}\frac{\partial}{\partial r_{ij}}\right)\prod_{ij}\varphi_{ji,t}\varphi_{ij,t} =$$

$$= \sum_{ij} m_{0ij}^2 \left(\frac{\alpha_{ij}^2}{r_{ij}^2} + 2\frac{\alpha_{ij}}{r_{ij}}\frac{\alpha_{ji}}{r_{ji}} + \frac{\alpha_{ji}^2}{r_{ji}^2}\right) \prod_{ij} \varphi_{ji,t}\varphi_{ij,t} - \sum_{ij}(\widehat{E}_i + \widehat{E}_j)^2 \prod_{ij} \varphi_{ij,t}\varphi_{ji,t} \quad (7.1)$$

with:

$$\prod_{ij} \varphi_{ij,t}\varphi_{ji,t} = F(t)\prod_{ij} \varphi_{ij}\varphi_{ji} = \prod_{ij} \varphi_{ij}\varphi_{ji} \prod_{1}^{\mathcal{N}} \left(\frac{\alpha_g}{r_g}\right) e^{i(k_g r_g - \omega_g t)}. \quad (7.2)$$

m_{0ij} is the rest mass to be dedicated to the interaction field created by the masses m_i and m_j. This factor also accommodates the c^2 as in equation (6.1) .The pairs in both products, in total $\mathcal{N}=N!/(2(N-2)!$ are numbered by g, if there are N particles. The term $e^{i(k_g r_g - \omega_g t)}$ expresses a wave propagating in radial direction representing the moving of individual groups, but with reducing amplitude, or, rather probability, as it progresses. If there is no interaction between members of the pairs ($\alpha_{mn} = 0$) we get the movement of the individual particles outside their local influence.

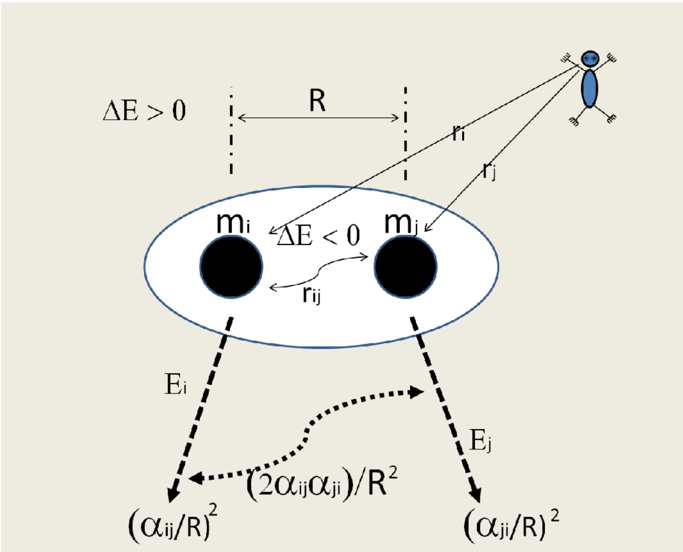

Figure 6*: Energy transfer from the pair to the surroundings and the sub-space (white area) with internal exchanges as observed from far away.*

This set-up has a very delicate interpretation. It shows that an observer from outside sees a pair creating a sub-space but cannot determine its structure inside. In the space inside, expressed by the coordinates r_{ij} and r_{ji}, gravitational interactions are occurring. Our observer only sees the separate interacting members of the pair with an energy due to this interaction as is shown schematically in Figure 6. It is as if we see two persons who have made a secret agreement and are, by acting as a pair, exchanging information. We can see both persons but we cannot explain why they behave as they behave.

As before the time dependences can be removed by setting:

$$\varphi_{ij,t}\varphi_{ji,t} = \varphi_{ij}\varphi_{ji}e^{i(E_{ij}+E_{ji})t/\hbar}, \quad (7.2)$$

so that:

$\sum_{ij}(\widehat{E_\iota}+\widehat{E_J})^2\varphi_{ij,t}\varphi_{ji,t}=\sum_{ij}(E_{ij}+E_{ji})^2\varphi_{ij}\varphi_{ji}.$ (7.3)

If all $\alpha's$ would have been equal to zero, a propagating wave $\varphi_{ij,t}\varphi_{ji,t}$ extending in the radial direction with the light velocity would have resulted. Non zero values of α reduce this speed and, as a consequence, give mass to the field $\varphi_{ij,t}\varphi_{ji,t}$. The proposed solution will be:

$$\varphi_{ij} = \gamma_{ij} r_{ij}^{\,m_{0ij}\alpha_{ij}/\hbar c}, \tag{7.4}$$

which is inserted into:

$$\sum_{ij}(E_{ij}+E_{ji})^2\prod_{ij}\varphi_{ji}\varphi_{ij} +$$

$$+c^2\hbar^2\sum_{ij}\left(\frac{1}{r_{ij}^2}\frac{\partial}{\partial r_{ij}}r_{ij}^2\frac{\partial}{\partial r_{ij}} + \frac{1}{r_{ji}^2}\frac{\partial}{\partial r_{ji}}r_{ji}^2\frac{\partial}{\partial r_{ji}} + \frac{\partial}{\partial r_{ij}}\frac{\partial}{\partial r_{ji}} + \frac{\partial}{\partial r_{ji}}\frac{\partial}{\partial r_{ij}}\right)\prod_{ij}\varphi_{ji}\varphi_{ij} =|$$

$$= \sum_{ij} m_{0ij}^2\left(\frac{\alpha_{ij}^2}{r_{ij}^2} + 2\frac{\alpha_{ij}}{r_{ij}}\frac{\alpha_{ji}}{r_{ji}} + \frac{\alpha_{ji}^2}{r_{ji}^2}\right)\prod_{ij}\varphi_{ji}\varphi_{ij}. \tag{7.5}$$

From the boundary condition that $\varphi_{ij}(r_{ji}, \alpha_{ij}) = 0$ for r_{ji} to infinity a fifth condition on the $\alpha's$ can be derived:
v. α_{ij} is negative under all circumstances.
From equation (5.5) it followed that the energy in the interaction field is given by $\varepsilon_{ij} = \sigma(m_i + m_j)$. From this we can derive the m_{0ij}^2-dependence by subtracting from ε_{ij}^2 the self-energies: $\varepsilon_{ij}^2 - \varepsilon_{ii}^2 - \varepsilon_{jj}^2$.
Putting all five conditions on α_{ij} together with m_{0ij} we can conclude that the explicit expression for $m_{0ij}\,\alpha_{ij}$ becomes:
vi. $m_{0i}\ \alpha_{ij} = m_{0j}\ \alpha_{ji} = -\sigma'(m_i m_j)^2$.

Now some algebra needs to be done and it will be found that many terms on the left hand side are equal to the ones at the right hand side and therefore disappear. We get:

$$(E_{ij}^2 + 2E_{ij}E_{ji} + E_{ji}^2)\varphi_{ij}\varphi_{ji} + c\hbar m_{0ij}\left(\frac{\alpha_{ij}}{r_{ij}^2} + \frac{\alpha_{ji}}{r_{ji}^2}\right)\varphi_{ji}\varphi_{ij} = 0 \tag{7.6}$$

At this point a remark has to be made: removing the term α_{kl}^2/r_{kl}^2 means that some basic interaction occurs between the gravitational field and the particle. Obviously, for this separate term, a KG-equation can be formulated that shows

that an entity with some relativistically derived mass operates and leaves behind a contribution to the interaction energy in the equation (7.6). So already at this point there is direct interaction between the pair and the field around. Also removing term with $\alpha_{ij}\alpha_{ji}/r_{ji}r_{ij}$ means that there is a third interaction between the fields and the pair. It is schematically represented in Figure 6.
Taking all these interactions into account it is seen that all α-terms in equation (7.5) have disappeared. This has a profound meaning: in this model gravity is due to second order effects of the peculiarities of the spherical symmetry in a relativistic setting. The effect is weak and operates over a long range.

The contributions can now be redistributed, but first multiply all terms by $r_{ij}r_{ji}$ and observe that the proposed solution is the only one that gives a sharp value for the quantity $E_{ij}\, r_{ij}$ and $E_{ji}\, r_{ji}$:

$$(E_{ij}^2 r_{ji}r_{ij} + E_{ij}E_{ji}r_{ji}r_{ij})\varphi_{ij}\varphi_{ji} + c\hbar m_{0ij}\alpha_{ij}\frac{r_{ji}}{r_{ij}}\varphi_{ij}\varphi_{ji} = 0, \tag{7.7a}$$

$$(E_{ji}^2 r_{ji}r_{ij} + E_{ij}E_{ji}r_{ji}r_{ij})\varphi_{ij}\varphi_{ji} + c\hbar m_{0i}\ \alpha_{ji}\ \frac{r_{ij}}{r_{ji}}\varphi_{ij}\varphi_{ji} = 0. \tag{7.7b}$$

Cutting the equation (7.6) into two separate ones as given in equations (7.7a) and (7.7b) looks like arbitrary, as any cut between terms can be made. But if we now come back to the original suggestion as made in *vi,* we see that the gravitational interaction becomes symmetric. The gravitational energy of particle *i* is equal to the gravitational energy of particle *j*. It also reflects the point that a pair has to be seen one entity. The observer cannot distinguish between the separate members of the pair.
Although not touched upon in paragraph 3, it is also important to notice that the operators $\widehat{E_k}$ and r_l commute. It means that "*Er*" is the quantity that has a sharp value, meaning that *E* has sharp value if *r* is well defined.

8. Law of gravity.

Most important for finding out how the members of a pair see each other is to look at the equations (7.7a) and (7.7b) by an observer on m_i who sees the particle m_j at a distance of r_{ij} and an observer on particle m_j looking at m_i from a distance r_{ji}. Both see each other from the same distance $r_{ij} = r_{ji} = R$ and they already know that $\alpha_{ij} = \alpha_{ji} = -\alpha'$. There are no operators anymore in equation (7.7a) and (7.7b) and they can conclude that $E_{ij} = E_{ji} = E$. This is an important conclusion. In the interpretation of the equations care has to be taken to the viewpoint from where the equations have been defined. This is the point O_2 in Figure 3 so that we will have to take *R/2* as distances. Obviously an electron and a proton forming a pair will have mutual interaction which are the same although their masses differ by some factor of about 1800. The result is a simple relation:

$$2E^2(R/2)^2 = c\hbar\alpha' m_0. \tag{8.1}$$

The boundary condition is that $\varphi_{ij}\varphi_{ji}$ goes to zero for r to infinity so that $\alpha' > 0$, and because both particles in the pair change their energy by the same amount. It follows for the two members of the ensemble together that:

$$ER = \sqrt{2c\hbar\alpha' m_0}, \tag{8.2}$$

and the gravitational force is given by: $-\partial E/\partial R = constant/R^2$.

Now it is important to see how pairs consisting of particles of different masses present themselves in α'. In view of equation (8.2) we conclude that also the gravitational interaction is proportional to both masses of the participating particles in the pair. As a consequence, the attractive force between two particles is proportional to the product of the two interacting masses. It also follows that, due to gravitational interaction which carries energy and for which a separate KG-equation can be set up, some mass, although not much, is attributed to the pairs. Taking into account the argument *v* and *vi* in the previous paragraph, the final result is:

$$E_{ij} = \sqrt{2\sigma' c\hbar}.(m_i m_j)/R, \tag{8.3}$$

All the work done to describe the total gravitational force, or rather the potential energy, has been based on the idea that all pairs that have been formed are acting independently so that we can add all the contributions of different masses constituting bodies in the real world without any interference. Referring to Figure 7 where two masses M_1 and M_2 have particles numbered as m_{1j} and m_{2k} from N_1xN_2 pairs described by $\phi_{kl}= \varphi_{1k}\varphi_{2l}$ in which each *kl*-combination contributes separately to the interaction energy.

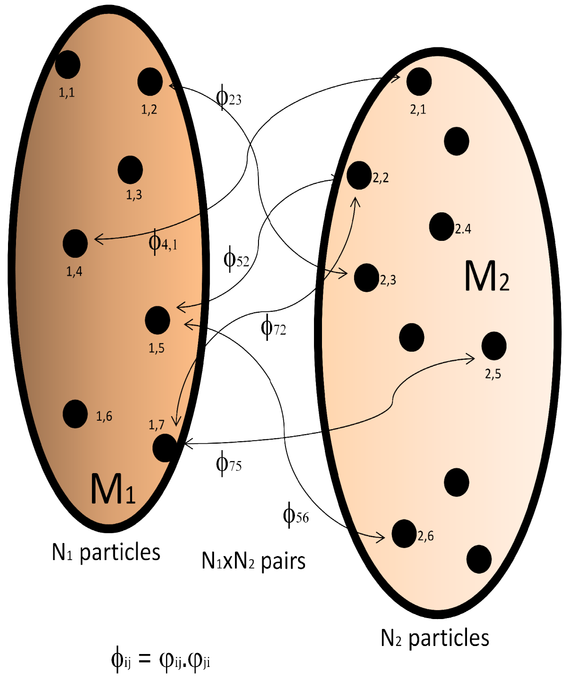

Figure 7: *Interaction between masses.*

Adding up all the interactions between particles, which in principle see each other at different distances is a problem that has already been solved in the formulation of the classical theory of electrostatics [8], finally Newton's gravitation law is obtained which, in vector notation reads: $div\boldsymbol{g} = 4\pi\rho G$ in which $\boldsymbol{g}$ is defined as a gravitational field around an entity constituting g space coordinates dependent mass density ρ. G is the well known gravitational constant equal to: $6.673x10^{-11}$ $m^3kg^{-1}sec^{-2}$ [9].

In accordance with the theory of electrostatics the gravity law can also be given in vector representation for bodies M_1 and M_2 which have their centres of gravity at a separation of R:

$$\boldsymbol{F_{12}} = (GM_1M_2/R^3)\,\boldsymbol{R}. \tag{8.4}$$

From the equations (8.3) and (8.4) an explicit expression for the parameter σ' can be derived and also, with the help of these equations the small mass to be attributed to the gravitational interaction can be found. This σ' parameter is equal to $2.7x10^2$ Jm/kg^4.

9. Transfer of energy and mass.

In the analysis going from equation (7.5) to (7.7) terms are disappearing due to the solution proposed in equation (7.4). But this has to be interpreted with caution. The pair probability density $\varphi_{ij}\varphi_{ji}$. in equation (7.1) represents a field carrying the gravitational energy. Therefore, the disappearance of the generator at the right hand side of equation (7.5), $(\alpha_{ij}/r_{ij} + \alpha_{ji}/r_{ji})^2$, involves exchange of energy from the pair to the surrounding space which is equal to the energy given in equation (8.3). As a consequence, when the positive value for the energy is taken, the energy of the pair itself is reduced by the same amount. In that case the interaction between the members of the pair is attractive. The process is schematically shown in Figure 6. The transferred energy is the difference between the energy levels shown Figure 8.

The opposite situation in which the energy of the pair is positive, which in principle is allowed by the Einstein energy equation (2.4), is not possible when we assume that the energy of the vacuum, to be taken as the reference point, is zero. In this interpretation the interaction between mass and the surroundings is a means to transfer mass related energy (mc^2) to gravitational energy. This transfer changes the rest masses of the pair but does not create new mass. The consequences at a larger scale are worked later in this paragraph.

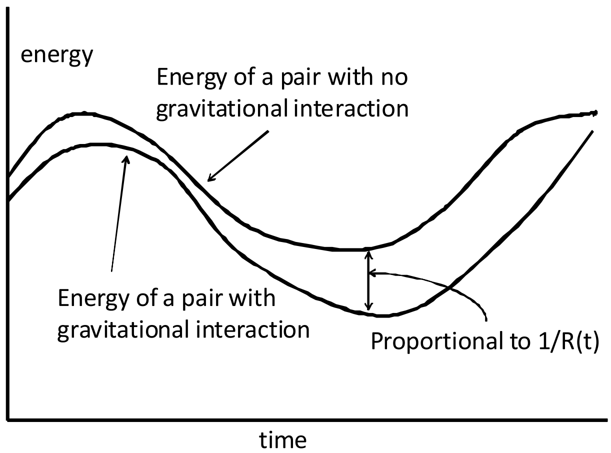

Figure 8*: Energy difference between local states and incorporating gravitational interaction.*

If, however the vacuum state is, as it is generally believed, a non-zero energy state there might be energy available which increases with the interaction area, the white area in Figure 6, that can be transferred to the pair. The situation could be such that, when the distance between the members of the pair increases, the energy needed is reducing whereas the energy, or number of fluctuations carrying sufficient energy is increasing. It means that at some separation distance of the members of the pair the interaction can become repulsive as the Einstein equation allows both negative and positive values for the interaction energy.

A solution for the Schrödinger equation of a pair of particles for an observer at distances r_{ij} and r_{ji} from particle *i* and *j* is given in equation (5.4). Now if we put our observer close by particle *i*, the second term in equation (5.4) becomes negligible against the first term:

$$\Psi_{ij} = \left(\frac{\alpha_{ij}}{r_{ij}} + \frac{\alpha_{ji}}{r_{ji}}\right) e^{i\beta_{ij}r_{ij}+i\beta_{ji}r_{ji}} \cong \left(\frac{\alpha_{ij}}{r_{ij}}\right) e^{i\beta_{ij}r_{ij}+i\beta_{ji}r_{ji}} \text{ and: } \Psi_{ij}^{*}\Psi_{ij} \cong \left(\frac{\alpha_{ij}}{r_{ij}}\right)^2 \quad (9.1)$$

The KG-equation in operator language now reads:

$$-\hbar^2\left(\frac{\partial^2}{\partial t^2} - c^2\frac{1}{r_{ij}^2}\frac{\partial}{\partial r_{ij}}r_{ij}^2\frac{\partial}{\partial r_{ij}}\right)\varphi_{ij,t} = m_{0ij}^2\left(\frac{\alpha_{ij}}{r_{ij}}\right)^2\varphi_{ij,t}. \quad (9.2)$$

Setting the right hand side to zero, a mass-less particle, we see an equation for a travelling wave at the speed of light. To get rid of the singularity we set $\alpha_{ij}/r_{ij} = \alpha_{ij}/r_{ip}$ for $r_{ij} < r_{ip}(= r_p)$, and removing the first term on the left hand side gives the London Equation that explains the shielding of the inside of a superconducting material from the outside magnetic field: the "Meissner" effect [10]. A similar thing can be imagined in this case with the $\varphi_{ij,t}$-field for $r_{ij} < r_p$. The distance r_p can be identified as the distance from the centre to where local influences have no impact.

We can solve the equation (9.2) with in the right hand term r_p for r_{ij}, but it is not necessary as it can immediately be seen that it dedicates mass to the field in the vicinity of the particle which is equal to $m_p = m_{0ij}\alpha_{ij}/r_p c^2$. As this is the mass to be attributed to the i^{th} particle, due to another particle somewhere in the surroundings, we will have to add up over all particles which can make a pair with our particle, so with $m_p = m_i$:

$$m_p = \sum_j m_{0ij}\alpha_{ij}/r_p c^2 = (\sigma'/_{r_p c^2})\sum_j m_p^2 m_j^2. \quad (9.3)$$

The consequence is that either $m_p = 0$, a mass-free particle, or: $m_p = r_p c^2/\sigma' \sum_j m_j^2$, with, as shown, $m_{0ij}\alpha_{ij} = -\sigma' m_i^2 m_j^2$. First the equation allows that there are mass-free particles like a photon which makes no pairs according the theorem based on the Schrödinger equation, but it can, according to the KG-equations in paragraph 5, equation (5.8). It could generate gravity as it is argued in Chapter 9: Cosmography of W.D. Heacox's book on the expanding Universe [9]. Second, the other solution is that there is a mass carrying particle whose mass becomes higher when r_p increases and, most important, it is all the mass in the surroundings that generate the mass of the i^{th}

particle. It is actually mass due to the field, but since the singularity moves with the particle the observer nearby can only interpret it as a mass contribution to the particle he is looking at. The conclusion taken here corresponds to Mach's ideas about the effect of all physical entities in the universe.
It would be tempting to evaluate m_p but, as we know already from observation, it is better to estimate the size or the extension of the particle if only this effect is responsible for the mass. The analysis concerns incredibly large and small numbers but leads to a surprising outcome.
Starting from $m_p = r_p c^2/\sigma' \sum_j m_j^2$ and assuming that the mass of the universe is basically due to protons and neutrons with almost the same mass, so $m_p = m_j$, and assuming there are *N* particles in the whole universe giving it a total mass of M_u we can set:

$$M_u = Nm_j = Nr_p c^2/\sigma' \sum_j m_j^2 = Nr_p c^2/\sigma' N m_j^2 = r_p c^2/\sigma' m_j^2. \qquad (9.4)$$

Estimates of the size of the universe on the basis of the inverse Hubble constant and the fact that the average intergalactic density is 1000 hydrogen atoms per cubic tells us that the total mass of the universe is of the order of 10^{55} kg. σ' is calculated in paragraph 8 at $2.7\text{x}10^{2}$ Jm/kg^4 and the proton mass is $1.7\text{x}10^{-27}$ kg [11]. It leads to an estimate for the r_p-value in the order of 10^{-15} m, which is about the size of a proton (0.8 femtometers) [12]. An electron which is 1840 times lighter than the proton will, according to equation (9.3), see the same surrounding as the proton, so its size would be smaller by the same factor. Although the correspondence with measured data is surprisingly good, it is still a rough estimate and not without speculation.
Even a discrepancy by a factor of 10 would already be acceptable for the outcome of this analysis. For instance, the sub-space due to the generator $m_{0ij}^2(\alpha_{ij}/r_{ij} + \alpha_{ji}/r_{ji})^2$ would be a quantum-mechanical reality, but it says nothing about its internal structure and interactions. The mass of the universe is rather uncertain in view of the discussion about dark matter, and the proton size, or how to define it, is not so obvious.

10. The hydrogen atom and the planet Mercury.

Consider the hydrogen atom with one outer electron of mass m_e that is circulating at some distance from the nucleus of mass m_p and experiencing a net charge *Ze*. The Schrödinger equation which incorporates both electrostatic interaction, given by $Ze^2/4\pi\varepsilon_0 r$ and gravitational interaction, $Gm_p m_e/r$ reads as follows:

$$\left(\frac{ħ^2}{2m_e}\left(\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}\right)+E+\frac{Ze^2}{4\pi\varepsilon_0 r}+\frac{Gm_p m_e}{r}\right)\varphi=0, \tag{10.1}$$

from which it can be seen that the gravitational contribution is completely insignificant for any value of r. The general solution is:

$$\varphi = Ae^{-ar}P_n(ar), \tag{10.2}$$

where $P_n(ar)$ is a polynomial, for instance for n = 1, $P_n = (2-ar)$, and: $a = Zm_e e^2/4\pi ħ^2\varepsilon_0 + Gm_p m_e^2/ħ^2$. But now, as a "thought-experiment" (Gedanken Experiment), we ignore the electrostatic contribution and take the general solution. It will be found that: $E_n = -m_p m_e^2 G^2/2ħ^2 n^2$ and $r = ħ^2 n^2/Gm_p$ leading to: $E = -Gm_p m_e/r$ in the limit of n to infinity. According to the correspondence principle the electron, when moving in electro-statically determined orbits (s-, p-, d-states), has very high quantum numbers in the reference frame of gravity. Returning to the thought- experiment we conclude that, if the atom is held together by gravitational forces only, the electron would have orbits at distances from the nucleus many orders of magnitude smaller than is actually the case. The argument above would, however, suggest that the electromagnetic force, although much larger than the gravitational interaction, manifests itself similarly in the entire space, but at some distance of the order of *1/a* opposite charges start to compensate and electromagnetic forces due to the charge of the nucleus will be suppressed and only gravitational forces start to dominate.

Now we will shortly investigate the precession of the planet Mercury. Mercury is the planet closest to the Sun and it has an almost circular orbit. But the point nearest to the sun also, to some extent, moves around the Sun. This is called "precession". It is a small effect and actions have been taken to explain it due to the existence of the other planets nearby with the help of Newtonian mechanics. But the result was that it could only partly explain the rate of precession.

Figure 9: *The planet Mercury: Duration of one Mercury year: 88 earth days, mass: 5.5 % of earth mass, radius: 2440 km and gravity at the surface: 3.7 m/sec^2.*

However, with the help of general relativity the last pieces of this puzzle were found and actually it was one of the major breakthroughs in the acceptance of the general relativity theory. To give the most simple argument based upon the findings in the document we can consider a piece of matter m_0 in a gravitational field generated by a larger mass M_o, and an observer far away out of this gravitational field, like on earth. This observer will interpret the real rest mass of m_o after he has taken it from a distance of R from the center of gravity to his free space. For the observer the rest mass is given by $m_0(1 + GM_0/Rc^2)$. Similarly the observer sees the other mass M_o, generating the gravity field, as $M_0(1 + Gm_0/Rc^2)$. If the two masses are the Sun-Mercury system: $M_S - M_M$ are encircling each other at a distance r, the observer will conclude that the force balance is given by:

$$M_M\omega^2 r^3 = GM_M M_s(1 + GM_M/Rc^2)(1 + GM_S/Rc^2). \tag{10.3}$$

Actually we should not have taken the rest mass for the dynamic mass of the planet Mercury, as the Mercury-Sun system has to be considered as a group, but since the mass m_0 shows up in the left and right hand side of equation (10.3) the effect cancels out and we have not to bother about it. It is important to note here that, for the observer outside the gravity field of the two masses, the experience of equality of the gravitational mass and inertial mass is not valid anymore. Also the observer sees that the time lapse T in the $M_S - M_M$ system is changed to:
$T\,(1 + GM_M/Rc^2)(1 + GM_S/Rc^2)$and the length r is changed to $r/((1 + GM_M/Rc^2)(1 + GM_S/Rc^2))$.
When the observer on the planet Mercury, knowing that his mass is much smaller than the mass of the Sun, sees that he has made one complete revolution around the sun, so: $\omega T = 2\pi$, the observer in outer space will see in accordance

with equation (10.3) that the planet Mercury has made a round trip of $2\pi(1 + 3\,GM_S/Rc^2)$. Only the first order terms in GM_S/Rc^2, with $r = R$ at the end, have been taken in the calculation. The outcome corresponds with the original analysis given by Einstein as the result of a more lengthy calculation [13].

11. The Bohr-Einstein controversy.

The model describing the gravitational interaction between particles has a some relation with the classical Bohr-Einstein dispute [14]. This dispute has been dealt with in many sessions between 1913 and 1930 as a subject of the Solvay Conferences. The issue was Einstein's belief that the quantum theory is an incomplete theory as he rejected the idea that positions in space-time could never be completely known. Einstein did not want to allow the uncertainty principle to necessitate an apparently random non-deterministic mechanism by which the law of physics would be operating.

The controversy culminated in the well known Einstein-Podolsky-Rosen Paradox, (EPR) of 1935 [15] which comes close to the ideas presented in this document. Two particles have a common source, like two photons originating from one process in terms of space and time. From a quantum mechanical point of view the set of the two particles are represented by a joint wave function. One particle has orientation up (U) and the other down (D) and we do not know on beforehand which of the two is up and which or down:

$$\Psi_{12} = \Psi_1(U)\Psi_2(D) + \Psi_1(D)\Psi_2(U). \tag{11.1}$$

This is a superposition of two states of the ensemble. At some moment we do an experiment and find out that one of the particles is specified as "up". From quantum theory we conclude that the other should be "down". It might be that the system is influenced by the measurement so that the result "up" emerged, but the other particle is definitely not influenced and we know that it is characterized as "down". It appears that the wave function has selected the option $\Psi_1(U)\Psi_2(D)$ out of the superposition. From quantum mechanical point of view the process occurs independently of where in space and at which moment it takes place.

For Einstein this was unacceptable and he suggested that the particle might have some "hidden variables" which we do not know and which decide the choice of the system. Niels Bohr could, however, justify his result by working out the situation in a more statistical way as quantum mechanics is basically a theory of probabilities which has been experimentally confirmed on several occasions [16].

We can now identify the solution for the pair potential in equation (5.4) or, more generally, equation (5.6), for a multi group particle pair system, in a similar way as the "up/down" combination given above. As Max Born pointed out in a letter to Einstein [17]: "There is a wholeness to a quantum events that persists over time and space and makes linkages possible”. These linkages, leading to the definition of the invariant in the KG-equation, apparently, give rise to the gravitational interaction. Apparently a single particle sees an environment and makes pairs with all of the particles around it. Suppose that at the other side of our galaxy two particles *k,l* annihilate. Suddenly the number of pairs reduces and this is seen by our particle. This change in the wave function:

$$\prod_{ij}\left(\frac{\alpha_{ij}}{r_{ij}}+\frac{\alpha_{ji}}{r_{ji}}\right)e^{i\,\beta_{ij}r_{ij}+i\beta_{ji}r_{ji}}e^{i(E_{ij}+\,E_{ji})t/\hbar}\text{ to:}$$

$$\left\{\prod_{ij}\left(\frac{\alpha_{ij}}{r_{ij}}+\frac{\alpha_{ji}}{r_{ji}}\right)e^{i\,\beta_{ij}r_{ij}+i\beta_{ji}r_{ji}}e^{i(E_{ij}+\,E_{ji})t/\hbar}\right\}_{i,j\neq k,l,}$$

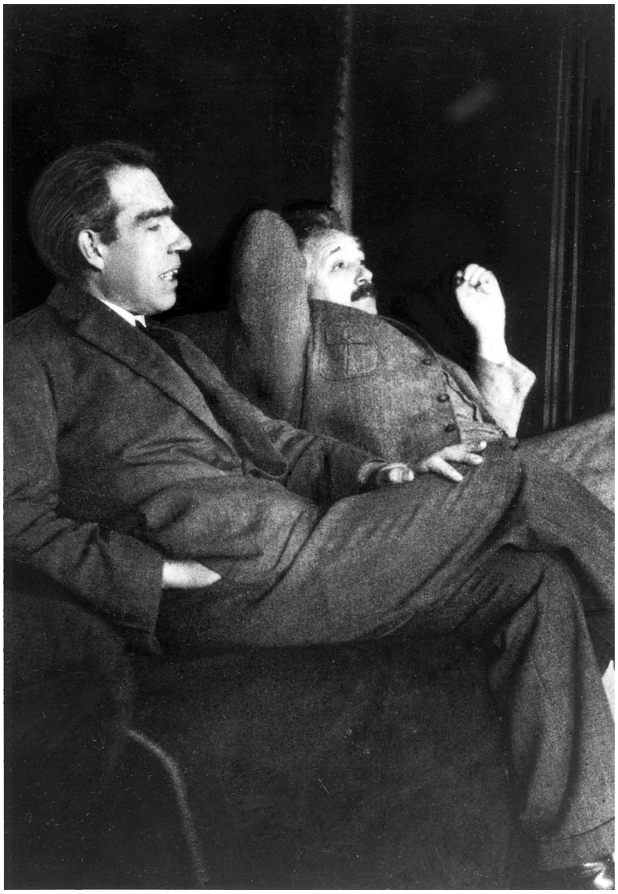

***Figure 10**: Niels Bohr and Albert Einstein in Ehrenfest’s home in Leiden in December 1925.*

produces a gravitational wave travelling through empty space at the speed of light and that adjusts to the new situation. But the information that the gravitational wave must start has already been exchanged between our particle with the observer and the annihilating pair. Again we end up in the same controversy as between Bohr and Einstein.

It is like a person somewhere far away sends me a message that something will be put into water so that I can go to the beach in The Netherlands and observe that much later there is some small rise of the sea level. If I will see anything, I have no idea where and when and in which way it originated, only that something has happened. Most likely the signal will be too small compared with the random disturbances. This analogy, however, is different from gravitational waves in that I have at least the sand of the beach as a reference level, whereas with gravitational waves such a thing does not exist.

12. Summary of preceding paragraphs and some remarks for completeness.

An important first conclusion in an earlier summary was that particles with a mass can be described as a single non-interaction pair containing only two members. The individual members can make pairs with all other mass in its surroundings. This, already peculiar pair effect, is used in the KG-equation which, in a quantum-mechanical representation, describes a field around these members. The second conclusion made is that energy is subtracted from the pair and gives rise to an attractive force between the two members of a pair. By setting this force equal to the well-known parameters of Newton's third law, numerical values can be given to the main parameters found with the KG-equation. It is then found that an observer watching the pair will see that the pair has two members, but he cannot see how they interact or exchange information, only that it leads to a force between them. This force is independent of the movement of the members in the sub-space as shown in Figure 3.

The field that occurs due to the KG-equation is not only present outside the particle but must also have its influence in areas where the particle mass density manifests itself. Not much is known about what this field inside the particle looks like and its local interactions, but the most simple approach would be to assume that the amplitude of the generator of this field is constant. The dependence on space coordinates of this field inside the outer boundary of the particle leads to the attribution of mass. This mass is then found to be a consequence of all the interactions which the single particle has with the surrounding mass in which the distance, apparently, plays no role. If we start from the values of the parameters derived from the gravitational interaction, and the known mass of a proton, its outer boundaries can be calculated which agree surprisingly well with the data found experimentally. However, it must be stressed that this last reasoning is speculative.

Going from small to large the theory that is developed has its consequences. One is that mass is to be attributed to a rest mass in a gravity field which in normal cases is merely negligible, but it has consequences on, for instance, the movement of the perihelion of Mercury during its encircling of the Sun. It is also shown how mass and gravity are intimately connected and that the description of the cosmos at large distances is governed by the specific gravitational interaction between bodies constituting the universe.

One point, difficult to accept from logical point of view, is that members of a pair seem to have instantaneous contact no matter how far they are apart and therewith generate the interaction field that gives rise to gravity. Gravity waves move at the speed of light or slightly less, depending on the mass density it is moving through, but its generator works apparently without delay. The situation is the same as the classical debate in the previous century between Einstein and Bohr and have remained to be an issue which is hard to believe but more than once shown to be true.

Another point to remark here is the occurrence of a generator creating a sub-space. It follows unambiguously from the Schrödinger equation but nothing can be said about its internal structure where particles are entangled and apparently exchange information. This might be close by the idea of Einstein about "hidden variables".

The gravitational constant in Newton's law, G, is expressed by $G = \sqrt{2\sigma' c\hbar}$ in which the parameter σ', equal to $2.7\text{x}10^2$ mJ/kg^4, can be seen as a universal constant that connects relativity with quantum mechanics.

The surprising, and at the same time bizarre, conclusion of the analysis given is that, apparently, each single particle has interaction with all other particles in the cosmos. It means that in the universe an unimaginable number of pair-wise interactions exists with greatly varying intensity and extensions and which depend on the masses of the members of the pair. It is difficult to comprehend, but it follows unambiguously from the equations describing the behaviour of the pairs.

An important aspect to mention is the fact that the right hand side in equation (7.5) should be invariant under Lorentz transformation. However, the r_{kl} transforms as a member of a four-vector. Therefore, the parameters α_{kl} should transform in the same way as r_{kl}, but apparently it would make left and right hand side in equation (8.2) transform differently, which cannot be the case. We should however notice that the Planck's constant, h, is invariant, but $\hbar = h/2\pi$ is not.

Make the following "thought-experiment". Consider a pair flying away from us at a speed v such that the separation vector of the members of the pair is aligned in the direction of v. Due to the fact that π transforms just like $1/r_{kl}$ the result is that the interaction energy of the pair we measure becomes invariant. There is invariance throughout if the alignment perpendicular to the speed. So the conclusion is that the interaction energy in the pair is invariant and independent of the alignment towards the observer, as it has to be.

As a last remark for this paragraph, causality is of importance to keep in mind. The model starts from the fact that there are masses, and it is seen that they can form pairs and generate gravity. It yields numerical data about the masses following gravitational parameters. The strength of the model is the consistency of the data with what we observe in reality. On the other hand one can say that the mass can be introduced into the Schrödinger equation as an unknown quantity and the theory comes back with a numerical value for it if the size of the particle is known.

13. Gravity depending on dynamical mass.

Mentioned in paragraph 4 at the end is that a group as a whole, identified with the label *l,* has kinetic energy and therefore a relativistic mass equal to $\gamma_g m_g c^2$. Although the present theory is only concerned about the situation in the sub-space r_{ij} where gravity originates, it still is of interest to know the dynamic mass of the group because of the fact that the Hamiltonian operator has been modified. The dynamic mass is influenced through the group-momentum $\widehat{p_g}/\sqrt{2m_g}$.

The equations (4.5a) and (4.5b) can, in principle, give the value for this group momentum if the replacement of the vectors $\widehat{p_k}/\sqrt{2m_k}$ by their relativistic equivalents has been done. However, there remains a disturbing $cos\delta_1$-term making a general solution inappropriate. But the purpose of an endeavour in which such a group related dynamic mass is significant makes only sense where gravity is important and speeds are approaching the speed of light. So it is not relevant outside the realm of cosmology.

In this respect the main problem of the incompatibility between quantum theory and relativity comes to the surface.

We therefore have to carefully replace the vectors in Figure 3 by the relativistically relevant ones which are to be derived from the equations (4.5a) and (4.5b) leading to the transitions:

$p_a/\sqrt{2m_a} \mapsto cm_a\sqrt{\gamma_a^2 - 1}$ with $a = i, j$ and g.

Now we can put our observer on one of the interacting particles, say m_i in the group (ij), and consider the surroundings from this point of view so that $p_i = 0$. In this case $cos\delta_1 = -1$. but because $p_i = 0$ the $cos\delta_1$- factor has no influence anymore. We end up in a rather complicated situation if we want to know the mass and γ_g- values for the group and after a tedious lot of algebra we find:

non relativistic:

$\gamma_g{}^2 - 1 = \frac{m_j}{2(m_i+m_j)}\left(\gamma_j{}^2 - 1\right)$, and relativistic:

$$\gamma_g{}^2 - 1 = \frac{m_j^2}{2\eta^2\left(m_i+m_j\right)^2}\left(\gamma_j{}^2 - 1\right). \qquad (13.1a, 13.1b)$$

The extra parameter η complicates the situation. If our observer is on mass m_i which is much smaller than m_j: η=1, and both equations are identical. If our observer is on a mass $m_i = m_j$, like in the case of direct proton-proton-neutron-

neutron interaction as the most common one in the universe: $\eta = \sqrt{1/2}$, the result is:

$$\gamma_g{}^2 - 1 = (\gamma_j{}^2 - 1)/4. \tag{13.2}$$

With the aid of the definition of γ this result is easily changed into the relation:

$$v_g^2/c^2 = v_j^2/(4c^3 - 3v^2). \tag{13.3}$$

This gives the mass to be allotted to both members of the group. At low velocities (v << c), The mass of the group particles is determined by half the speed of the moving particle. When the speed of the moving particle approaches the light velocity, both speeds become equal. So at low speeds we have to dedicate dynamic mass to both particles and the Newton's equation will read:

$$\boldsymbol{F}_{\boldsymbol{12}} = \boldsymbol{R}G\left(M_{1o}(1 - v_j^2/4c^2)^{-1/2} M_{20}(1 - v_j^2/4c^2)^{-1/2}\right)/R^3. \tag{13.3}$$

It is like the two rest masses, M_{1o} and M_{2o}, are moving away with opposite speeds but equal to $v_j/2$ from the observer, in the middle.
When speeds are approaching the speed of light, of course, the speeds of both particles are still the same and opposite, but at the value $v_{j.}$ This result is similar to the velocity addition rule for relativistic velocities on the basis of standard relativity theory [3], but in this case arrived at in way involving gravity.
An alternative way of interpreting equation (13.3) is to place the observer in the sub-space in the middle between the two particles so that their speeds are opposite and equal to *v* as seen from the observer's point of view. In that case the equation becomes:

$$\boldsymbol{F}_{\boldsymbol{12}} = \boldsymbol{R}G\left(M_{1o}/\sqrt{(1 - v^2/c^2} M_{20}/\sqrt{(1 - v^2/c^2)}\right)/R^3. \tag{13.4}$$

As a side step: In the interpretation of equation (13.2): even if the two particles with the observer in the middle have speeds approaching the speed of light, the observer on one of these sees the other one not moving at almost *2c*, but at nearly the speed of light, *c*.

In conclusion it can be said that particles in a group have gravitational interaction as if they have in the sub-space their rest masses. These rest masses must be corrected with the relativistic transformation factor γ_g as defined by equation (13.1).

The kinetic energy of the group remains to be defined by the value

$$E_{kin} = T = M_{1o}(\gamma_1 - 1)c^2 + M_{2o}(\gamma_2 - 1)c^2. \tag{13.5}$$

One may say that this allocation of dynamic mass to the masses in the Newton equation started from the assumption that we only have particles with equal masses, but we can, as a "thought-experiment" build up the parts out of separate but equal masses. After adding up all the effects of them in the separate groups we arrive finally at the same result as in equation (13.3).

14. Cosmological consequences. Dark matter and energy?

The model presented here started from the Schrödinger equation where no boundary conditions are imposed on a system of particles. The subsequent finding of the pair formation as a result of this has consequences. First it means that the entire space, in which the particle pairs are embedded, is necessary for the interaction. Second, it is only the pairs that create the forces between them such that in Newton's law the product of the masses gives rise to a total gravitational force. Also, the particular form of the solution of the Schrödinger equation, in which the wave function amplitude itself is used as an operator in the KG equation, leads to the R^2- dependence. It is also important to remark that the gravitational interaction, leading to an interaction energy, is, for the observer in the sub-space, dependent on the product of the rest masses and independent of the speed in any direction in which one of the members of the pair or both are moving, even though the total mass of the pair changes relativistically.
We will calculate the energy balance of the universe assuming that, where matter is manifesting itself, the mass is distributed homogeneously. This assumption ignores any clustering of matter that will influence the energy, but it can be shown that this contribution is negligible against the energy that is dedicated to the relativistically defined masses (mc^2). To start with, relativistic masses are not taken into consideration. If the density of the rest mass is given by ρ_0 there are two contributions: potential energy, V, as the masses feel their gravitational pull to all other surrounding masses, and the kinetic energy, T_k, as the masses are moving relatively to each other. Now:

$$V = -\int_0^{R_u} \frac{GM(r)}{r} dM = -\int_0^{R_u} (4/3)G\pi\rho_0^2 r^2 4\pi r^2 dr =$$
$$= -(16/15)G\pi^2\rho_0^2 R_u^5, \tag{14.1a}$$

$$T = \int_0^{R_u} 2\pi\rho_0 r^2 h^2 r^2 dr = (2/5)\pi\rho_0 h^2 R_u^5, \tag{14.1b}$$

where h is the so-called "Hubble constant". It connects the expansion speed, v, of the matter with the distance, r, from the observer so that: $v=hr$.
We will take the sum as zero which means that, in the case of a "flat, matter only" universe, ultimately the universal expansion comes to rest, and we find:

$$h^2 = (8/3)\pi\rho_0 G = {c^2}/{R_u^2} = 8.37\rho_0 G. \tag{14.2}$$

This value of $8\pi/3$ occurs as a proportionality constant in the Friedmann expansion equations which are based on the geodesic equations as derived from the Einstein field equation [9]. The last step is based on the assumption that the expansion speed is equal to the speed of light at the outer boundary of our cosmos.
It is interesting to evaluate this last equation, knowing that R_u/c is the radius of the universe in light seconds, that the average intergalactic density is about 1000 hydrogen atoms per cubic meter so that $R_u/c = 0.3\text{x}10^{17}$ light seconds or about 10^{10} light years and that the generally accepted value of h is $2.3\text{x}10^{-18}\ \text{sec}^{-1}$. These values are in the right order of the values assumed on the basis of telescopic observations.

Now we want to play the same game as above, but in a relativistic context. For this we should first, again as a "thought-experiment", place an observer somewhere in space and make him or her look around. He or she will see a universe in which all objects are moving away from him or her. and independent of the place where we have put him. In other words: the universe is isotropic and approximately homogeneous as seen in a cosmological context.
We will first for our observer calculate how much dynamic mass $M(R_u)$ is around in a volume bounded by the radius R_u:

$$M(R_u) \ = \int_0^{R_u} \rho_0 \frac{4\pi r^2}{\sqrt{1 - h^2 r^2/c^2}} dr = \frac{4\pi\rho_0 c^3}{h^3} \int_0^{hR_u/c} \frac{z^2 dz}{\sqrt{1-z^2}} =$$

$$= \frac{4\pi\rho_0 c^3}{h^3}\left[-\frac{z}{2}\sqrt{1-z^2} + \frac{1}{2}arcsin(z)\right]_0^{1=hR_u/c} = \pi^2\rho_0 R_u^3, \tag{14.3}$$

In this equation the factor $\sqrt{1 - h^2 r^2/c^2}$ is connected to the mass increase due to the expansion speed.
Just for the moment we can calculate the mass of the "observable" universe when $hR_u/c = 1,\ M(R_u) \ = \pi^2\rho_0 R_u^3$ and the apparent volume is: $M(R_u)\,/\rho_o$.
This is significantly larger than the non-relativistic number.
The kinetic energy cannot be evaluated by "M$v^2/2$" but more simply by:

$$T_k = M(R_u)c^2 - M_0(R_u)c^2 = \pi^2\rho_0 R_u^3 c^2 - \frac{4}{3}\pi\rho_0 R_u^3 c^2. \tag{14.4}$$

For the potential energy we need a second "thought-experiment". Suppose the observer is sitting on a mass m and this interacts gravitationally will all masses around. We take the universe as isotropic and dedicate the relativistic mass only to the moving matter. Then we find:

$$V(m) = -\int_0^{R_u} mG\rho_0 \frac{4\pi r}{\sqrt{1-\frac{h^2r^2}{c^2}}} dr = -\frac{4\pi\rho_0 mGc^2}{h^2}\left[\sqrt{1-z^2}\,\right]_0^{hR_u/c} =$$

$$= -4\pi\rho_0 mGc^2/h^2 \tag{14.5}$$

We can then replace *m* by $(4/3)\pi\rho_0 R_u^3$ and obtain the result:

$$\mathrm{V(M)} = -(16/3)G\pi^2\rho_0^2 R_u^5. \tag{14.6}$$

Again, setting the sum equal to zero, it gives:

$$h^2 = (\tfrac{16\pi/3}{\pi-4/3)})\rho_0 G = {c^2}/{R_u^2} = 9.5\rho_0 G. \tag{14.7}$$

This result is not so different from the non-relativistic value. This is because the relativistic kinetic and potential energy both depend on the dynamic masses in almost, but not entirely, the same way.
There is more than one way of interpreting this result. First we can say that from our observer's point of view the energy due to expansion speeds in the outer areas is so high that it will never balance the gravitational energy, whereas for the observer in the outer areas, however, it could. From another point of view our observer sees that time is progressing more and more slowly in the outer areas and even, if the expansion speed will slow down, he will never be able to see it.
This interpretation already indicates that there is a problem:
We have placed our observer on a mass which consists of particles where all are members of groups with the masses that are moving away from him, or her. So we have, due to the arguments given in paragraph 3, to dedicate a relativistic mass also to the mass on which we have placed the observer. If we take this effect into account, like as was done in equation (14.5), we see that, when performing the integration up to the Hubble limit of $hR_u/c = 1,$ there is no convergence: infinities are showing up .
There have been speculations that, by unknown mechanisms, the creation of mass is balanced by a loss in potential energy [9], [18]. In the model expressed by the equations (14.4) and (14.5) it would mean that in equation (14.5) we only use the Mc^2 – term:

$$h^2 = (16/3)\rho_0 G = {c^2}/{R_u^2} = 5.3\rho_0 G. \tag{14.8}$$

In that case, the universe should be significantly larger, or heavier, than on the basis of the analysis of the "non relativistic" universe. But this latter argument should be considered with scepticism. Rest mass m_0 seen by an observer in free space changes from its value in a local gravity field by a factor of $1 + GM/Rc^2$, in which M is the mass with which m_0 is interacting and R the distance, but mass is not newly created. The argument suggests that there is a mechanism by which gravitational energy can be changed into new rest mass. The factor mentioned is a small correction that plays no significant role in the analysis here, but it has been shown in paragraph 10 that it is of significance in calculating the precession of the perihelion of the planet Mercury around the Sun.

Returning to the argument where we also have to take the relativistic value for the mass on which we have placed the observer into account, there is by far enough of potential energy available to balance for the creation of new mass. No conclusion can, however, be connected to these considerations. It may well be that the universe cannot be considered as infinite and isotropic. It can also be that the gravity loses its significance beyond a certain distance of cosmological scales.

Having done all the bookkeeping for the masses and wave function in the KG-equation it is worthwhile, as a "thought-experiment", to concentrate all masses of the interacting pieces of material into one point at the centre of gravity, so that the system can be represented in the one-plane subspace, as in Figure 3. In this way the attractive force between macroscopic bodies can be calculated. In reality this is however not precisely right. The $(\alpha_{ij}/r_{ij} + \alpha_{ji}/r_{ji})$-value varies from place to place and therefore have to be averaged out taking the clustering of masses into account. This clustering can be thought of as having occurred in two steps. First, basic particles carrying mass are clustered into the nuclei of atoms, and, subsequently, into crystalline structures, or a liquid or a concentrated gases.

Therefore we average out the α_{ij}/r_{ij} -factor by assuming that in a certain region, R_1, there are a number of α_{ij}/r_{ij} -carrying particles:

$$\left(\frac{\alpha_{ij}}{r_{ij}}\right)_{av_1} = \frac{3\alpha_{ij}}{4\pi R_1^3}\int_0^{R_1} 4\pi\, r_{ij} dr_{ij} = \frac{3\alpha_{ij}}{2R_1}, \tag{14.9}$$

and again we cluster into a macroscopic structure confined into R_2:

$$\left(\frac{\alpha_{ij}}{r_{ij}}\right)_{av_2} = \frac{9\alpha_{ij}}{8\pi R_2^3}\int_0^{R_2} 4\pi\, R_1 dR_1 = \frac{9\alpha_{ij}}{4R_2}. \tag{14.10}$$

In our world of material, clustered into solids, liquid and molecular gases, we have related our gravitational interaction to the gravity law on the basis of α_{ij}/r_{ij} -values for which no clustering is assumed. This leads, however, to a gravity law that corresponds to Newton's third law, but one in which the α_{ij}-value is taken too low, compared to a situation in which we have no clustering. For our daily earth-bound life it makes no difference as our direct neighbourhood, our solar system, consists almost entirely of clustered material. At the scale of a galaxy, however, the situation is different. There we have to a high extent non-clustered and almost non-interaction single particles like hydrogen atoms. For them an α_{ij} -value of more than twice higher than the one in our environment has to be taken. So, apparently, there is more gravity interaction than we can expect on the basis of our earth-bound interpretation.

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