On the Mass Quantization of Black Holes

Black holes are relatively simple cosmic objects that are characterized by their mass, their angular momentum and their electric charge. However, the laws that govern them are laws that we do not yet fully know. We can only sketch what really happens inside or around them. This paper tries to discover some of its secrets such as its minimum size and the law of the quantification of its mass. Finally, the "myth" of the Planck mass is busted.

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***Keywords**: Planck's constant, Planck time, Planck length, Planck mass, Planck mass density, reduced Planck mass density, quantum gravitational relativity, density-time squared cosmological equation.*

Contents

1. Postulates

This theory, which deals with Schwazschild black holes (non-rotating black holes), rests on three postulates. The first two postulates were introduced in an article entitled: *Quantum Gravitational Relativity Part I* [1] while the third postulate was introduced in the article entitled: *The Quantum Gravitational Cosmological Model without Singularity* [2]. These postulates are:

Postulate 1: Time quantization postulate

Time is discrete. This means that there is a time, T_P, which is the minimum time with physical meaning. In other words there is no time or time interval smaller than T_P. This time is known as the Planck time.

The Planck time is defined as follows

$$T_P = \sqrt{\frac{h\,G}{2\,\pi\,c^5}} \tag{1.1}$$

Postulate 2: Space quantization postulate

Space is discrete. This means that there is a length, L_P, which is the minimum length with physical meaning. In other words there is no length or distance smaller than L_P. This length is known as the Planck length. The Planck length is defined as follows

$$L_P = \sqrt{\frac{h\,G}{2\,\pi\,c^3}} \tag{1.2}$$

Postulate 3: Density-time squared cosmological postulate

The density-time squared cosmological postulate is an equation I formulated in 2011 [2]. This equation allow us to calculate the average mass density of the universe at the present time knowing three quantities:

a) the average mass density at the beginning of time
b) the age of the universe (means its age now), and
c) the Planck time

The density-time squared cosmological equation may be applied to any time T and is given by the following relation

Density-time squared cosmological equation

$$\rho\,T^2 = \rho_P\,T_P^2 \tag{1.3}$$

Where

$\rho = \rho(T) =$ average mass density of the universe at time T
$T =$ universal time
$\rho_P = \rho_P(T_P) =$ average mass density of the universe at the beginning of time (T_P)
(This turned out to be the reduced Planck density, $\rho_{P\text{-}red} = \rho_{P\text{-}bar}$)
$T_P =$ This is the Planck time or, equivalently, the beginning of time

The reader may ask: how do we know that equation (1.3) is correct? The answer is that in the referenced article [2] I proved that the mass density of the universe we observe today (ρ_{now}) can be calculated from equation (1.3). This was done by solving this equation for $\rho = \rho_{now}$. This yielded

$$\rho_{now} = \frac{\rho_P T_P^2}{T_{now}^2} \tag{1.4}$$

Where

$\rho_{now} = \rho(T_{now}) =$ average mass density of the universe at the present time
$T_{now} =$ the age of the universe (this is the present time)

The value this equation yielded was

Theoretical value $$\rho_{now} = 9.399\,462\,317 \times 10^{-27} \frac{Kg}{m^3} \tag{1.5}$$

This means that the average number of protons, n_{p-now} , per cubic meter (in the observable universe, now) is

Theoretical value $$n_{p\text{-}now} = \frac{\rho_{now}}{m_p} = 5.62 \frac{protons}{m^3} \tag{1.6}$$

The WMAP spacecraft measured the following value for the average number of protons, n_{p-WMAP} , per cubic meter

Observed value $$n_{p\text{-}WMAP} = 5.9 \frac{protons}{m^3} \tag{1.7}$$

The close agreement between the theoretical and the experimental values supports the model based on the above cosmological equation and on a spherical model of the universe. Because at the beginning of time the size of the universe was identical to the size of the minimum black hole, we shall use this equation for selecting those black holes models that agree with the observed values of mass density and we shall discard those that do not agree with the observations. We shall see that the spherical model is not the only one that explains the experiments. In the next two sections we shall derive the minimum mass a black hole can have. We need to know the minimum mass to be able to compare it with the formula for the quantized mass of a black hole for the minimum quantum number (n=1). This way we shall know that the quantization equation we shall derive in section 4 will be the correct formula.

2. Derivation of the Minimum Mass of a Black Hole from the Density-Time Squared Cosmological Equation

Quantum mechanics has proven time and time again that particles have properties that contradict the common sense. For this reason we must have an open mind to all possibilities even if they seem strange. Following this philosophy, I will present two major geometries for the smallest black hole produced by nature: (a) a cubic black hole model (which I suppose are very unpopular) and (b) a spherical black hole model (which are more popular).

Model 1: Cubic Black Hole with Side equal to the Planck Length and Mass equal to the Planck Mass Divided by Two

The mass density of a cube is

$$\rho_{cube} = \frac{M}{V} \tag{2.1.1}$$

Where M is the mass of a cube whose side is the Planck length, L_P , and whose volume is $V = L_P^3$. If $M = M_P/2$ we the density turns out to be

$$\rho_{cube} = \frac{M_P}{2\,L_P^3} \tag{2.1.2}$$

$$\rho_{cube} = \frac{1}{2}\sqrt{\frac{h\,c}{2\,\pi\,G}}\sqrt{\left(\frac{2\,\pi\,c^3}{h G}\right)^3} \tag{2.1.3}$$

$$\rho_{cube} = \frac{1}{2}\frac{2\,\pi\,c^5}{h G^2} = \frac{\pi\,c^5}{h\,G^2} \tag{2.1.4}$$

$$\rho_{cube} = \frac{\pi\,c^5}{h\,G^2} \tag{2.1.5}$$

According to this model the average mass density of the universe at the present time is

$$\rho_{now} = \frac{\pi\,c^5}{h\,G^2}\frac{T_P^2}{T^2} \tag{2.1.6}$$

$$\rho_{now} = \frac{\pi\,c^5}{h\,G^2}\frac{h\,G}{2\,\pi\,c^5}\frac{1}{T_{now}^2} \tag{2.1.7}$$

$$\rho_{now} = \frac{1}{2\,G\,T_{now}^2} \tag{2.1.8}$$

The value of this density is

$$\rho_{now} = 3.937\times 10^{-26}\,\frac{Kg}{m^3} \tag{2.1.9}$$

And the number of protons per cubic meter is

Does not agree with the observations

$$n_{now} = \frac{\rho_{now}}{m_p} = 23.54 \frac{protons}{m^3} \tag{2.1.10}$$

Model 2: Cubic Black Hole with Side equal to the Planck Length and Mass equal to the Planck Mass Divided by Eight

The mass density of a cube is

$$\rho_{cube} = \frac{M}{V} \tag{2.2.1}$$

Where M is the mass of a cube whose side is the Planck length, L_P , and whose volume is $V = L_P^3$. If $M = M_P/8$ we the density turns out to be

$$\rho_{cube} = \frac{M_P}{8\,L_P^3} \tag{2.2.2}$$

$$\rho_{cube} = \frac{1}{8}\sqrt{\frac{h\,c}{2\,\pi\,G}}\sqrt{\left(\frac{2\,\pi\,c^3}{h\,G}\right)^3} \tag{2.2.3}$$

$$\rho_{cube} = \frac{1}{8}\frac{2\,\pi\,c^5}{h\,G^2} = \frac{\pi\,c^5}{h\,G^2} \tag{2.2.4}$$

$$\rho_{cube} = \frac{\pi\,c^5}{4\,h\,G^2} \tag{2.2.5}$$

According to this model the average mass density of the universe at the present time is

$$\rho_{now} = \frac{\pi\,c^5}{4\,h\,G^2}\frac{T_P^2}{T_{now}^2} \tag{2.2.6}$$

$$\rho_{now} = \frac{1}{8\,G\,T_{now}^2} \tag{2.2.7}$$

The value of this density is

$$\rho_{now} = 9.843 \times 10^{-27} \frac{Kg}{m^3} \tag{2.2.8}$$

And the number of protons per cubic meter is

Agrees with the observations
$$n_{now} = \frac{\rho_{now}}{m_p} = 5.885 \, \frac{protons}{m^3} \tag{2.2.9}$$

This result agrees nicely with the latest NASA's WMAP measurements given by (1.7)

Model 3: Spherical Black Hole with Mass Equal to the Planck Mass and Radius equal to Two Times the Planck Length

The parameters of this model are: the mass, M, of a black hole and its radius, R. In this case we assume (without proof) that M is equal to the Planck mass, this is: $M = M_P$. The problem is to find the radius of this black hole knowing that the mass and the radius are related through the following general relativity's equation

Schwarzschild radius
$$R = \frac{2\, G\, M}{c^2} \tag{2.3.1}$$

We replace the mass of the black hole by the Planck mass [see eq. (1.2)]. This gives

$$R = \frac{2\, G}{c^2} \sqrt{\frac{h\, c}{2\, \pi\, G}} \tag{2.3.2}$$

Which yields

Black hole radius
$$R = 2\, L_P \tag{2.3.3}$$

Thus our assumption implies that the radius of the black hole is twice the Planck length. Because we know the mass and the radius of this black hole we can now calculate its mass density, ρ_{sphere}

$$\rho_{sphere} = \frac{M}{V} \tag{2.3.4}$$

The volume, V, of the black hole will be

$$V = \frac{4}{3} \pi \left(2\, L_P\right)^3 \tag{2.3.5}$$

Therefore then the mass density is

$$\rho_{sphere} = \frac{M_P}{\frac{4}{3} \pi \left(2\, L_P\right)^3} \tag{2.3.6}$$

$$\rho_{sphere} = \frac{3}{32\, \pi} \frac{M_P}{L_P^3} \tag{2.3.7}$$

After some algebra we find

Black hole mass density

$$\rho_{sphere} = \frac{3}{16} \frac{c^5}{hG^2} \tag{2.3.8}$$

According to this model the average mass density of the universe at the present time is

$$\rho_{now} = \rho_{sphere} \frac{T_P^2}{T_{now}^2} \tag{2.3.9}$$

Where T_P is the Planck time [see eq. (1.1)] and T_{now} is the age of the universe

$$\rho_{now} = \frac{3\,c^5}{16\,h\,G^2} \frac{T_P^2}{T_{now}^2} \tag{2.3.10}$$

Replacing the value of T_P by the second side of eq. (1.1) we get

$$\rho_{now} = \frac{3\,c^5}{16\,h\,G^2} \frac{hG}{2\,\pi\,c^5} \frac{1}{T_{now}^2} \tag{2.3.11}$$

Finally

Average mass density of the universe now

$$\rho_{now} = \frac{3}{32\,\pi} \frac{1}{G\,T_{now}^2} \tag{2.3.12}$$

The value of this density is

$$\rho_{now} = 2.3499 \times 10^{-27} \frac{Kg}{m^3} \tag{2.3.13}$$

And the number of protons per cubic meter, now, is

Does not agree with the observations

$$n_{now} = \frac{\rho_{now}}{m_p} = 1.41 \frac{protons}{m^3} \tag{2.3.14}$$

Because this result does not agree with the latest NASA's WMAP measurements given by (1.7), we conclude that this black hole model is wrong.

Model 4: Spherical Black Hole with Mass equal to the Planck Mass Divide by Two and Radius Equal to the Planck Length

The parameters of this model are: the mass, M, of a black hole and its radius, R. In this case we assume, without proof, (the proof is given in section 3) that M is equal to the Planck mass divided by 2, this is: $M = M_P/2$. The problem is to find the radius of this black hole knowing that the mass and the radius are related through the following general relativity's equation

Schwarzschild radius

$$R = \frac{2 G M}{c^2} \tag{2.4.1}$$

We replace the mass of the black hole by the Planck mass [see eq. (1.2)]. This gives

$$R = \frac{2 G}{c^2} \frac{1}{2} \sqrt{\frac{h c}{2 \pi G}} \tag{2.4.2}$$

Which yields

Black hole radius

$$R = L_P \tag{2.4.3}$$

Thus our assumption implies that the radius of the black hole is equal to the Planck length. Because we know the mass and the radius of this black hole we can now calculate its mass density, ρ_{sphere}

$$\rho_{sphere} = \frac{M}{V} \tag{2.4.4}$$

The volume, V, of the black hole will be

$$V = \frac{4}{3} \pi L_P^3 \tag{2.4.5}$$

Therefore then the mass density is

$$\rho_{sphere} = \frac{M_P}{\frac{4}{3} \pi L_P^3} \tag{2.4.6}$$

$$\rho_{sphere} = \frac{3}{4 \pi} \frac{M_P}{L_P^3} \tag{2.4.7}$$

After some algebra we find

Black hole mass density

$$\rho_{sphere} = \frac{3}{4} \frac{c^5}{h G^2} \tag{2.4.8}$$

According to this model the average mass density of the universe at the present time is

$$\rho_{now} = \rho_{sphere} \frac{T_P^2}{T_{now}^2} \tag{2.4.9}$$

Where T_P is the Planck time [see eq. (1.1)] and T_{now} is the age of the universe.

$$\rho_{now} = \frac{3\,c^5}{4\,h\,G^2}\,\frac{T_P^2}{T_{now}^2} \qquad (2.4.10)$$

Replacing the value of T_P by the second side of eq. (1.1) we get

$$\rho_{now} = \frac{3\,c^5}{16\,h\,G^2}\,\frac{h\,G}{2\,\pi\,c^5}\,\frac{1}{T_{now}^2} \qquad (2.4.11)$$

Finally

Average mass density of the universe now

$$\rho_{now} = \frac{3}{8\,\pi}\,\frac{1}{G\,T_{now}^2} \qquad (2.4.12)$$

The value of this density is

$$\rho_{now} = 9.34\times 10^{-27}\,\frac{Kg}{m^3} \qquad (2.4.13)$$

And the number of protons per cubic meter is

Agrees with the observations

$$n_{now} = \frac{\rho_{now}}{m_p} = 5.62\,\frac{protons}{m^3} \qquad (2.4.14)$$

This result is very close to the latest NASA's WMAP measurements given by (1.7). The above four models are summarized in Table 1.

(continue on next page)

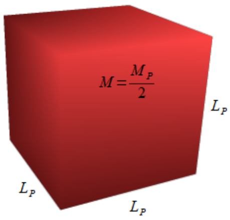
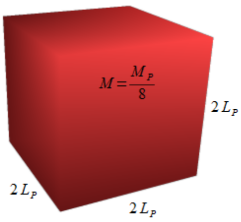
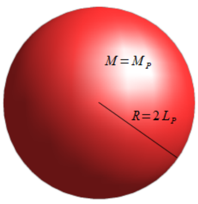
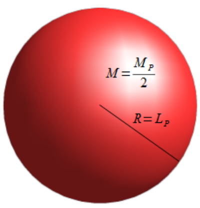
MODEL No. AND TYPE	GEOMETRY OF THE MINIMUM SIZE BLACK HOLE	MASS DENSITY OF THE MINIMUM BLACK HOLE ρ_0	MASS DENSITY NOW ρ_{now}	PROTONS / m^3 n
1 Cubic black hole		$\frac{\pi c^5}{h G^2}$	$\frac{1}{2} \frac{1}{G T_{now}^2}$	23.54 (wrong)
2 Cubic black hole		$\frac{\pi c^5}{4 h G^2}$	$\frac{1}{8} \frac{1}{G T_{now}^2}$	5.89 (right)
3 Spherical black hole		$\frac{3 c^5}{16 h G^2}$	$\frac{3}{32 \pi} \frac{1}{G T_{now}^2}$	1.41 (wrong)
4 Spherical black hole		$\frac{3 c^5}{4 h G^2}$	$\frac{3}{8 \pi} \frac{1}{G T_{now}^2}$	5.62 (right)

Table 2.1: Models of black holes of Minimum Size. *The table shows 2 cubic and 2 spherical models. Only models 2 and 4 agree with the observed average mass density of the universe we see today. Without considering the measurement errors of the average mass density, model 2 seems to be a slightly better approximation than model 4. However the problem with model 2 is that is a cubic model.*

3. Derivation of the Minimum Mass of a Black Hole from Postulate 2 and the and Schwarzschild Radius

According with Einstein's General Relativity the radius of a black hole (the Schwarzchild radius) is given by

$$R_S = \frac{2\,G\,M}{c^2} \tag{3.1}$$

Where

$c =$ speed of light in vacuum
$G =$ gravitational constant
$M =$ rest mass of the black hole
$R_S =$ Schwarzschild radius of a black hole

Let us solve this equation for M

$$M = \frac{R_S\,c^2}{2\,G} \tag{3.2}$$

The mass of the black hole will be minimum when the value of its radius to be minimum. But, according to **Postulate 2** (space quantization postulate), the minimum distance in the Universe is the Planck length, L_P . Thus, the minimum radius of any sphere in the Universe must be the Planck length (this is because the centre of the sphere must be accessible). Consequently, when R_S is equal to L_P , the mass of the black hole will be minimum, M_{min} . This fact can be expressed mathematically as follows

$$M_{min} = \frac{L_P\,c^2}{2\,G} \tag{3.3}$$

Considering the expression for the Planck length given by equation (1.2)

$$L_P = \sqrt{\frac{h\,G}{2\,\pi\,c^3}} \qquad (3.4) = (1.2)$$

we may write equation (2.3) as follows

$$M_{min} = \sqrt{\frac{h\,G}{2\,\pi\,c^3}}\,\frac{c^2}{2\,G} = \sqrt{\frac{h\,G\,c^4}{2\,\pi\,c^3\,4\,G^2}} = \sqrt{\frac{h\,c}{2\,\pi\,4\,G}} = \frac{1}{2}\sqrt{\frac{h\,c}{2\,\pi\,G}} = \frac{M_P}{2} \tag{3.5}$$

Where

$$M_P = \sqrt{\frac{h\,c}{2\,\pi\,G}} \tag{3.6}$$

Is the Planck mass. For clarity reasons we shall write the first and the last side of equation (3.5) leaving out the intermediate steps. This gives the final formula

Minimum mass of a black hole

$$M_{bh\text{-}min} = \frac{M_P}{2} \tag{3.7}$$

Thus, if space is quantized as postulated in section 1, then we have proved that the minimum mass of a black hole is one half of the Planck mass.

4. The Quantized Formula for the Mass of a Black Hole

I have adopted the spherical model 4 for the black hole of minimum size: this is a spherical black hole of mass $M_P/2$ and radius L_P. Now we shall look for general formula for the mass of the black hole. The derivation of this formula is very easy. We begin the derivation considering formula (3.2)

$$M = \frac{R_S c^2}{2G} \qquad (4.1) = (3.2)$$

This gives the mass of a black hole in terms of its radius. But we know, according to **postulate 2**, that the radius of the black hole must be quantized. Thus, the quantization relation must have the following shape

$$R_S = n L_P \tag{4.2}$$

where n is a quantum number (mathematically n is an integer greater than or equal to 1 or, in other words, a natural number). Then from equations (4.1) and (4.2) we can write

$$M = \frac{n}{2} \frac{c^2 L_P}{G} \tag{4.3}$$

But we can easily prove that the quantity $c^2 L_P / G$ is equal to the Planck mass, M_P, as follows

$$\frac{c^2 L_P}{G} = \sqrt{\frac{c^4}{G^2} \frac{hG}{2\pi c^3}} = \sqrt{\frac{hc}{2\pi G}} = M_P \tag{4.4}$$

Finally, combining equations (4.3) and (4.4) the quantized formula for the mass of a black hole turns out to be

The quantized formula for the mass of black hole

$$M_{bh}(n) = \frac{n}{2} M_P \tag{4.5}$$

n = 1, 2, 3, ...

Where we have replaced M by $M_{bh}(n)$ to indicate that the mass is the mass of a black hole. Table 4.1 shows the mass of a black hole for the first 10 values of the quantum number *n*. This is a good example of how the Planck units (in this case the Planck mass) can be used to expressed the mass of an object (In this case, the mass of a black hole).

QUANTUM NUMBER n	BLACK HOLE RADIUS R	BLACK HOLE MASS $M_{bh}(n)=\frac{n}{2}M_P$
1	L_P	$\frac{1}{2}M_P$
2	$2L_P$	M_P
3	$3L_P$	$\frac{3}{2}M_P$
4	$4L_P$	$2M_P$
5	$5L_P$	$\frac{5}{2}M_P$
6	$6L_P$	$3M_P$
7	$7L_P$	$\frac{7}{2}M_P$
8	$8L_P$	$4M_P$
9	$9L_P$	$\frac{9}{2}M_P$
10	$10L_P$	$5M_P$

Table 4.1: *This table shows the quantum number, n, the black hole radius, R, and the mass of the black hole* M_{bh} *. The minimum mass corresponds to n=1 and its value is* $0.5M_P$ *.* M_P *is the Planck mass.*

5. Conclusions

Having found in another article [2] that a black hole of minimum size coincides with the size of the universe at the beginning of time, we have proved that a black hole whose mass is equal to the Planck mass and whose radius is equal to twice the Planck length (spherical model 3) is incorrect because it can not predict the observed value of the average mass density of the universe. In other words, the prediction of model 3 is incorrect because the hypothesis is incorrect. In model 3 we assumed that the mass of the minimum black hole is equal to the Planck mass (I shall call it: the "myth" of the Planck mass) without checking the implications of this assumption. This is the reason this myth has endured so far.

The spherical model 4, on the other hand, uses the Planck mass divided by two instead of the Planck mass. Comparing the predictions of this model with the observed value of the average mass density of the universe we find that both coincide. So we came to the conclusion that

The minimum mass of a spherical black hole is $M_P/2$

The density of the spherical black hole of the model 4 turned out to be

$$\rho_{bh} = \frac{3\,c^5}{4\,h G^2} \qquad (5.1) = (2.4.8)$$

So the density depends only upon three physical constants: the Planck's constant, the speed of light and the gravitational constant and upon one mathematical constant: ¾. Without considering the errors of the measurements of the average mass density of the universe, model 2 seems to be a slightly better approximation than model 4. However, there is another problem with model 2: it is based on a cubic black hole. We do not know whether nature is capable of producing cubic black holes, even at the Planck scale. Cubic black holes appear to be counterintuitive. However, we must be cautious about it. Quantum mechanics is plagued with counterintuitive implications, such as the Copenhagen interpretation and the Feynman's path integral formulation. Because the errors provided by the WMAP spacecraft have not been taken into account we shall consider that the best model is the spherical model 4 ($M = M_P/2$ and $R = L_P$)

According to the space quantization postulate, we have also proved that the mass of a black hole must be quantized and that its value is

The quantized formula for the mass of black hole

$$M_{bh}(n) = \frac{n}{2} M_P \qquad (5.2) = (4.5)$$

$$n = 1, 2, 3, \ldots$$

We have also seen that

$$M_{bh}(1) = M_{bh\text{-}min} = \frac{M_P}{2} \qquad (5.3)$$

and therefore we have proved that equation (4.5) yields the correct mass value corresponding to the first value of the quantum number.

Appendix 1
Nomenclature

I shall use the following nomenclature for the constants and variables used in this paper.

c = speed of light in vacuum
h = Planck's constant
G = gravitational constant
L_P = Planck length

$T_P =$ Planck time (the beginning of normal time)
$M_P =$ Planck mass
$\rho = \rho(T) =$ average mass density of the universe at T
$\rho_{now} = \rho(T_{now}) =$ average mass density of the universe at the present time
$T =$ universal time
$T_{now} =$ universal time today (the age of the universe)
$m_p =$ rest mass of the proton
$\rho_P = \rho_P(T_P) =$ average mass density of the universe at the beginning of time (T_P)
(This turned out to be the reduced Planck density, $\rho_{P\text{-}red} = \rho_{P\text{-}bar}$)
$n_{now} =$ the average number of protons per cubic meter (in the observable universe, now), of any of the 4 models.
$n_{p\text{-}now} =$ Theoretical value of the average number of protons per cubic meter (in the observable universe, now), calculated with the spherical model 4 through the density-time squared cosmological equation.
$n_{p\text{-}WMAP} =$ the average number of protons per cubic meter measured by the WMAP spacecraft
$M =$ mass of any of the 4 models (rest mass of the black hole)
$R_S =$ Schwarzchild radius of the black hole
$V =$ volume of any of the 4 models
$\rho_{cube} =$ mass density of the cubic black hole models (models 1 and 2)
$\rho_{sphere} =$ mass density of the spherical black hole models (models 3 and 4)
$\rho_{bh} =$ mass density of the spherical model 4 ("the best" model)
$M_{bh} =$ mass of a quantized black hole
$M_{bh}(n) =$ mass of a quantized black hole as a function of the quantum number n
$n =$ quantum number (takes the values 1,2,3,4,5, ...)
$M_{bh\text{-}min} =$ minimum mass of a quantized black hole (which is the minimum mass a black hole can possess)

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