Construction of the Lovas-Andai Two-Qubit Function $\tilde{\chi}_2(\varepsilon) = \frac{1}{3}\varepsilon^2\left(4-\varepsilon^2\right)$ Verifies the $\frac{8}{33}$-Hilbert Schmidt Separability Probability Conjecture

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Abstract

We investigate relationships between two forms of Hilbert-Schmidt two-re[al]bit and two-qubit "separability functions"–those recently advanced by Lovas and Andai (arXiv:1610.01410), and those earlier presented by Slater (*J. Phys. A* **40** [2007] 14279). In the Lovas-Andai framework, the independent variable $\varepsilon \in [0,1]$ is the ratio $\sigma(V)$ of the singular values of the 2×2 matrix $V = D_2^{1/2} D_1^{-1/2}$ formed from the two 2×2 diagonal blocks (D_1, D_2) of a randomly generated 4×4 density matrix D. In the Slater setting, the independent variable μ is the diagonal-entry ratio $\sqrt{\frac{d_{11} d_{44}}{d_{22} d_{33}}}$–with, importantly, $\mu = \varepsilon$ or $\mu = \frac{1}{\varepsilon}$ when both D_1 and D_2 are themselves diagonal. Lovas and Andai established that their two-rebit function $\tilde{\chi}_1(\varepsilon)$ ($\approx \varepsilon$) yields the previously conjectured Hilbert-Schmidt separability probability of $\frac{29}{64}$. We are able, in the Slater framework (using cylindrical algebraic decompositions [CAD] to enforce positivity constraints), to reproduce this result. Further, we similarly obtain its new (much simpler) two-qubit counterpart, $\tilde{\chi}_2(\varepsilon) = \frac{1}{3}\varepsilon^2 \left(4 - \varepsilon^2\right)$. Verification of the companion conjecture of a Hilbert-Schmidt separability probability of $\frac{8}{33}$ immediately follows in the Lovas-Andai framework. We obtain the formulas for $\tilde{\chi}_1(\varepsilon)$ and $\tilde{\chi}_2(\varepsilon)$ by taking D_1 and D_2 to be diagonal, allowing us to proceed in lower (7 and 11), rather than the full (9 and 15) dimensions occupied by the convex sets of two-rebit and two-qubit states. The CAD's themselves involve 4 and 8 variables, in addition to $\mu = \varepsilon$. We also investigate extensions of these analyses to rebit-retrit and qubit-qutrit (6×6) settings.

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I. INTRODUCTION AND INITIAL ANALYSES

To begin our investigations, focusing on recent work of Lovas and Andai [1], we examined a certain possibility–motivated by a number of previous studies (e.g. [2–5]) and the apparent strong relevance there of the Dyson-index vantage upon random matrix theory [6]. More specifically, we ask whether the (not yet constructed by them) Lovas-Andai "separability function" $\tilde{\chi}_2(\varepsilon)$ for the standard (complex) two-qubit systems might be simply proportional (or even equal) to the square of their successfully constructed two-rebit separability function [1, eq. (9)],

$$\tilde{\chi}_1(\varepsilon) = 1 - \frac{4}{\pi^2} \int\limits_{\varepsilon}^{1} \left(s + \frac{1}{s} - \frac{1}{2}\left(s - \frac{1}{s}\right)^2 \log\left(\frac{1+s}{1-s}\right)\right) \frac{1}{s} \mathrm{d}s \tag{1}$$

$$= \frac{4}{\pi^2} \int\limits_{0}^{\varepsilon} \left(s + \frac{1}{s} - \frac{1}{2}\left(s - \frac{1}{s}\right)^2 \log\left(\frac{1+s}{1-s}\right)\right) \frac{1}{s} \mathrm{d}s.$$

Let us note that $\tilde{\chi}_1(\varepsilon)$ has a closed form,

$$\frac{2\left(\varepsilon^2\left(4\mathrm{Li}_2(\varepsilon) - \mathrm{Li}_2\left(\varepsilon^2\right)\right) + \varepsilon^4\left(-\tanh^{-1}(\varepsilon)\right) + \varepsilon^3 - \varepsilon + \tanh^{-1}(\varepsilon)\right)}{\pi^2\varepsilon^2}, \tag{2}$$

where the polylogarithmic function is defined by the infinite sum

$$\mathrm{Li}_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s},$$

for arbitrary complex s and for all complex arguments z with $|z| < 1$. Let us note also that in the proof of (1), the authors are able to formulate the problem rather concisely in terms of a "defect function" [1, App. A]

$$\Delta(\delta) = \frac{2\pi^2}{3} - \tilde{\chi}_1(e^{-\delta}) = \frac{16}{3}\int_0^{\delta} \cosh(t) - \sinh(t)^2 t \log(\frac{e^t+1}{e^t-1})\mathrm{d}t. \tag{3}$$

We will be able in sec. IV A 1 to obtain the formula (2) for $\tilde{\chi}_1(\varepsilon)$ by alternative (cylindrical algebraic decomposition [7]) means. Further, in our chief (titular) advance, in sec. IV B 1, we will apply the same basic methodology to obtain (the much simpler) formula (39) for $\tilde{\chi}_2(\varepsilon)$.

As part of their analysis, Lovas and Andai assert [1, p. 13] that

$$\mathcal{P}_{sep}(\mathbb{R}) = \frac{\int\limits_{-1}^{1}\int\limits_{-1}^{x} \tilde{\chi}_1\left(\sqrt{\frac{1-x}{1+x}} \Big/ \sqrt{\frac{1-y}{1+y}}\right)(1-x^2)(1-y^2)(x-y)\mathrm{d}y\mathrm{d}x}{\int\limits_{-1}^{1}\int\limits_{-1}^{x}(1-x^2)(1-y^2)(x-y)\mathrm{d}y\mathrm{d}x}, \tag{4}$$

with the denominator evaluting to $\frac{16}{35}$. Here, $\mathcal{P}_{sep}(\mathbb{R})$ is the Hilbert-Schmidt separability probability for the nine-dimensional convex set of two-rebit states [8]. With the indicated use of $\tilde{\chi}_1(\varepsilon)$ this probability evaluates to $\frac{29}{64}$ (the numerator of (4), they find, equalling $\frac{16}{35} - \frac{1}{4} = \frac{29}{140}$, with $\frac{29}{64} = \frac{\frac{29}{140}}{\frac{16}{35}}$), a result that had been strongly anticipated by prior analyses [9–11].

If the (Dyson-index) proportionality relationship

$$\tilde{\chi}_2(\varepsilon) \propto \tilde{\chi}_1^2(\varepsilon) \tag{5}$$

held, we would have

$$\mathcal{P}_{sep}(\mathbb{C}) \propto \frac{\int\limits_{-1}^{1}\int\limits_{-1}^{x} \tilde{\chi}_1^2\left(\sqrt{\frac{1-x}{1+x}}\Big/\sqrt{\frac{1-y}{1+y}}\right)(1-x^2)^2(1-y^2)^2(x-y)^2 \mathrm{d}y\mathrm{d}x}{\int\limits_{-1}^{1}\int\limits_{-1}^{x}(1-x^2)^2(1-y^2)^2(x-y)^2 \mathrm{d}y\mathrm{d}x}. \tag{6}$$

Here, $\mathcal{P}_{sep}(\mathbb{C})$ is–in the Lovas-Andai framework–the Hilbert-Schmidt separability probability for the fifteen-dimensional convex set of the (standard/complex) two-qubit states [12]. They expressed hope that they too would be able to demonstrate that $\mathcal{P}_{sep}(\mathbb{C}) = \frac{8}{33}$, as has been strongly indicated is, in fact, the case [9–11]. We generalized (from $\alpha = \frac{1}{2}$) the denominator of the ratio (6) to

$$\int\limits_{-1}^{1}\int\limits_{-1}^{x}(1-x^2)^{2\alpha}(1-y^2)^{2\alpha}(x-y)^{2\alpha}\mathrm{d}y\mathrm{d}x = \frac{\pi 2^{6\alpha+1}3^{-3\alpha}\alpha\Gamma(3\alpha)\Gamma(2\alpha+1)^2}{\Gamma\left(\alpha+\frac{5}{6}\right)\Gamma\left(\alpha+\frac{7}{6}\right)\Gamma(5\alpha+2)}. \tag{7}$$

Our original Dyson-index-based ansatz, then, was that

$$\frac{\int\limits_{-1}^{1}\int\limits_{-1}^{x} \tilde{\chi}_{2\alpha}\left(\sqrt{\frac{1-x}{1+x}}\Big/\sqrt{\frac{1-y}{1+y}}\right)(1-x^2)^{2\alpha}(1-y^2)^{2\alpha}(x-y)^{2\alpha} \mathrm{d}y\mathrm{d}x}{\int\limits_{-1}^{1}\int\limits_{-1}^{x}(1-x^2)^{2\alpha}(1-y^2)^{2\alpha}(x-y)^{2\alpha} \mathrm{d}y\mathrm{d}x} \tag{8}$$

would be proportional to the generalized (α-th) Hilbert-Schmidt separability probability. For $\alpha = \frac{1}{2}$, we recover the two-rebit formula (4), while for $\alpha = 1$, under the ansatz, we would obtain the two-qubit value of $\frac{8}{33}$, while for $\alpha = 2$, the two-quater[nionic]bit value of $\frac{26}{323}$ would be gotten, and similarly, for $\alpha = 4$, the (presumably) two-octo[nionic]bit value of $\frac{44482}{4091349}$ [13]. (The volume forms listed in [1, Table 1] for the self-adjoint matrices $\mathcal{M}^{sa}_{2,\mathbb{R}}$, $\mathcal{M}^{sa}_{2,\mathbb{C}}$, are $\frac{|x-y|}{\sqrt{2}}$ in the $\alpha = \frac{1}{2}$ case, and $\frac{(x-y)^2 \sin\phi}{2}$ in the $\alpha = 1$ case, respectively. Our calculations of the term $\det(1-Y^2)^d$ appearing in the several Lovas-Andai volume formulas [1, pp. 10, 12], such as this one for the volume of separable states,

$$\begin{aligned}\mathrm{Vol}\left(\mathcal{D}^s_{\{4,\mathbb{K}\}}(D)\right) = &\frac{\det(D)^{4d-\frac{d^2}{2}}}{2^{6d}} \\ &\times \int\limits_{\mathcal{E}_{2,\mathbb{K}}} \det(I-Y^2)^d \times \chi_d \circ \sigma\left(\sqrt{\frac{I-Y}{I+Y}}\right)\mathrm{d}\lambda_{d+2}(Y),\end{aligned} \tag{9}$$

[the function $\sigma(V) = \varepsilon$ being the ratio of the two singular values of the 2×2 matrix V] appear to be consistent with the use of the $(1-x^2)^{2\alpha}(1-y^2)^{2\alpha}$ terms in the ansatz (8).)

The values $\alpha = \frac{1}{2}, 1, 2, 4$ themselves correspond to the real, complex, quaternionic and octonionic division algebras. We can, further, look at the other nonnegative (non-division algebra) integral values of α. So, for $\alpha = 3$, we have the formal prediction [11, 14] of $\frac{2999}{103385}$.

In this context, let us first note that for the denominator of (6), corresponding to $\alpha = 1$, we obtain $\frac{256}{1575}$ (a result we later importantly employ (40)). Using high-precision numerical integration (http://mathematica.stackexchange.com/questions/133556/how-might-one-obtain-a-high-precision-estimate-of-the-integral-over-0-1-of-a-s) for the corresponding numerator of (6), we obtained 0.0358226206958479506059638010848. The resultant ratio (dividing by $\frac{256}{1575}$) is 0.220393076546720789860910104330, within 90% of 0.242424. However, somewhat disappointingly, it was not readily apparent as to what exact values these figures might correspond.

The analogous numerator-denominator ratio in the $\alpha = 2$ (two-quaterbit) instance was 0.0534499, while the predicted separablity probability is $\frac{26}{323} \approx 0.0804954$. It can then be seen that the required constant of proportionality ($\frac{0.0534499}{0.0804954} = 0.664013$) in the $\alpha = 2$ case is not particularly close to the square of that in the $\alpha = 1$ instance ($0.909106^2 = 0.826473$). Similarly, in the $\alpha = 4$ case, the numerator-denominator ratio is 0.00319505, while the predicted value would be $\frac{44482}{4091349} = 0.0108722$ (with the ratio of these two values being 0.293873). So, our ansatz (8) would not seem to extend to the sequence of constants of proportionality themselves conforming to the Dyson-index pattern. But the analyses so far could only address this specific issue concerning constants of proportionality.

II. EXPANDED ANALYSES

We, then, broadened the scope of the inquiry with the use of this particular formula of Lovas and Andai for the volume of separable states [1, p. 11],

$$\mathrm{Vol}(\mathcal{D}^s_{\{4,\mathbb{K}\}}) = \int\limits_{\substack{D_1, D_2 > 0 \\ \mathrm{Tr}(D_1+D_2)=1}} \det(D_1 D_2)^d f(D_2 D_1^{-1}) \mathrm{d}\lambda_{2d+3}(D_1, D_2),$$

where

$$f(D_2 D_1^{-1}) = \chi_d \circ \exp\left(-\cosh^{-1}\left(\frac{1}{2}\sqrt{\frac{\det(D_1)}{\det(D_2)}}\mathrm{Tr}\left(D_2 D_1^{-1}\right)\right)\right). \tag{10}$$

Here D_1 denotes the upper diagonal 2×2 block, and D_2, the lower diagonal 2×2 block of the 4×4 density matrix [1, p. 3],

$$D = \begin{pmatrix} D_1 & C \\ C^* & D_2 \end{pmatrix}.$$

The Lovas-Andai parameter d is defined as 1 in the two-rebit case and 2 in the standard two-qubit case (that is, in our notation, $\alpha = \frac{d}{2}$). Further, the relevant division algebra $\mathbb{K}$ is $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{Q}$, according to $d = 1, 2, 4$. The exponential term in (10) corresponds to the "singular value ratio",

$$\sigma(V) = \exp\left(-\cosh^{-1}\left(\frac{||V||_{HS}^2}{2\det(V)}\right)\right) = \exp\left(-\cosh^{-1}\left(\frac{1}{2}\sqrt{\frac{\det(D_1)}{\det(D_2)}}\mathrm{Tr}\left(D_2 D_1^{-1}\right)\right)\right), \tag{11}$$

of the matrix $V = D_2^{1/2} D_1^{-1/2}$, where the Hilbert-Schmidt norm is indicated. (In [15, sec. IV] the ratio of singular values of 2×2 "empirical polarization matrices" is investigated.)

A. Generation of random density matrices

1. *Two-rebit case*

Firstly, taking $d = 1$, we generated 687 million random (with respect to Hilbert-Schmidt measure) 4×4 density matrices situated in the 9-dimensional convex set of two-rebit states [16, App. B] [8, 17]. Of these, 311.313,185 were separable (giving a sample probability of 0.453149, close to the value of $\frac{29}{64} \approx 0.453125$, now formally established by Lovas and Andai). Additionally, we binned the two sets (separable and all) of density matrices into 200 subintervals of $[0, 1]$, based on their corresponding values of $\sigma(V)$ (Fig. 1). Fig. 2 is a plot of the estimated separability probabilities (remarkably close to linear with slope 1–as previously observed [1, Fig. 1]), while Fig. 3 shows the result of subtracting from this curve the very well-fitting (as we, of course, expected from the Lovas-Andai proof) function $\tilde{\chi}_1(\varepsilon)$, as given by ((1),(2)). (If one replaces $\tilde{\chi}_1(\varepsilon)$ by simply its close approximant ε, then the corresponding integrations would yield a "separability probability", not of $\frac{29}{64} \approx 0.453125$,

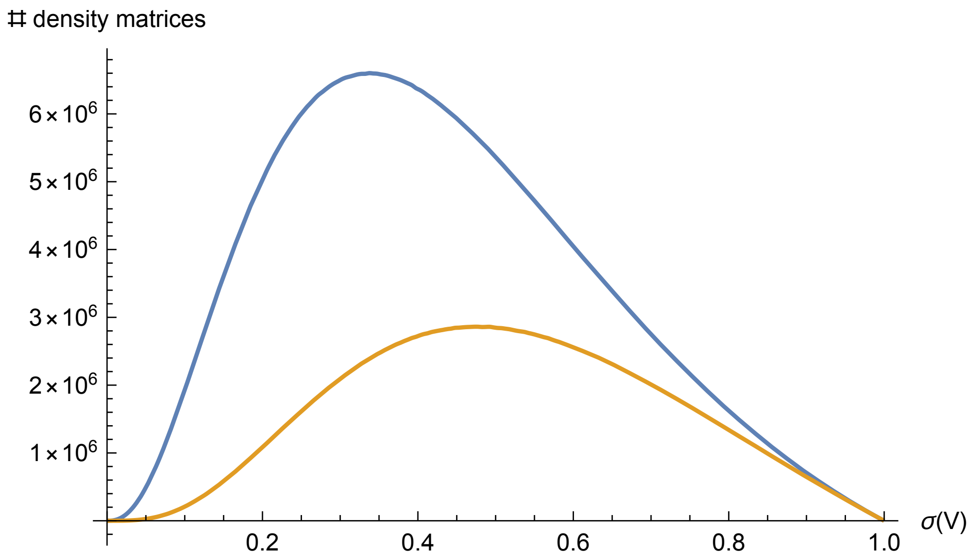

FIG. 1: Recorded counts by binned values of the singular value ratio $\sigma(V)$ of 687 million two-rebit density matrices randomly generated (with respect to Hilbert-Schmidt measure), along with the accompanying (lesser) counts of separable density matrices

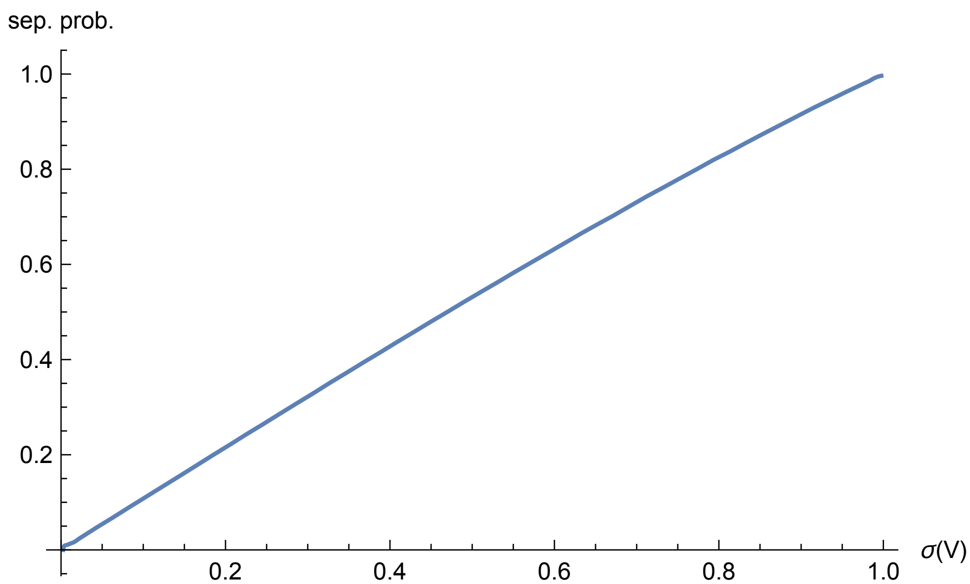

FIG. 2: Estimated two-rebit separability probabilities (close to linear with slope 1)

but of $\frac{16}{9} - \frac{35\pi^2}{256} \approx 0.428418$. If we similarly employ ε^2 in the two-qubit case, rather than the [previously undetermined] $\tilde{\chi}_2(\varepsilon)$, the corresponding integrations yield $\frac{13}{66} \approx 0.19697$, and not the presumed correct result of $\frac{8}{33} \approx 0.242424$.) Fig. 21 will serve as the two-qubit analogue of Fig. 3, further validating the formula for $\tilde{\chi}_2(\varepsilon)$ to be obtained.

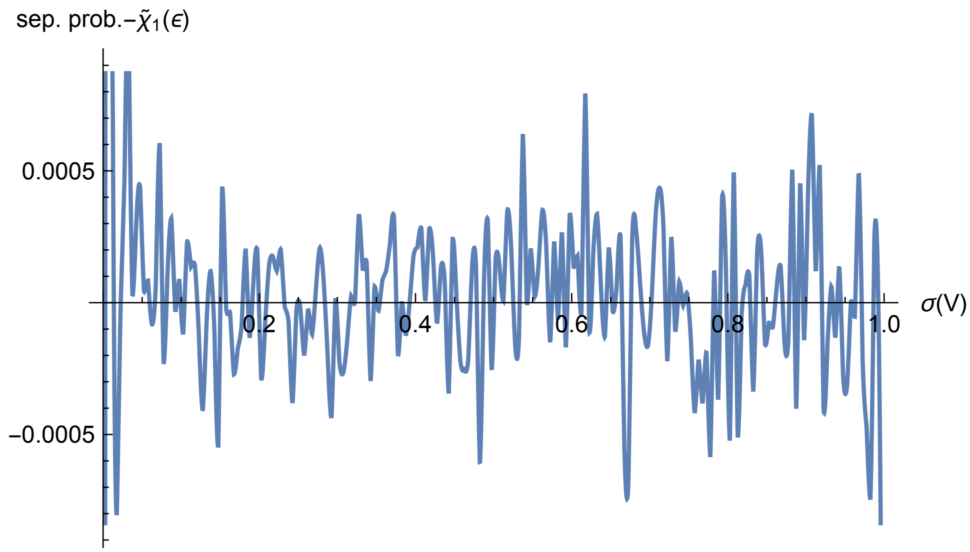

FIG. 3: Result of subtracting $\tilde{\chi}_1(\varepsilon)$ from the estimated two-rebit separability probability curve (Fig. 2). Fig. 21 will be the two-qubit analogue.

2. *Two-qubit case*

We, next, to test a Dyson-index ansatz, taking $d = 2$, generated 6,680 million random (with respect to Hilbert-Schmidt measure) 4×4 density matrices situated in the 15-dimensional convex set of (standard) two-qubit states [16, eq. (15)]. Of these, 1,619,325,156 were separable (giving a sample probability of 0.242414, close to the conjectured, well-supported [but not yet formally proven] value of $\frac{8}{33} \approx 0.242424$). We, again, binned the two sets (separable and all) of density matrices into 200 subintervals of $[0, 1]$, based on their corresponding values of $\sigma(V)$ (Fig. 4). Fig. 5 is a plot (now, clearly non-linear [cf. Fig. 2]) of the estimated separability probabilities, along with the quite closely fitting, but mainly slightly subordinate $\tilde{\chi}_1^2(\varepsilon)$ curve. Fig. 6 shows the result/residuals (of relatively small magnitude) of subtracting $\tilde{\chi}_1^2(\varepsilon)$ from the estimated separability probability curve. So, it would seem that the square of the explicitly-constructed Lovas-Andai two-rebit separability function $\tilde{\chi}_1(\varepsilon)$ provides, at least, an interesting approximation to their not then constructed two-qubit separability function $\tilde{\chi}_2(\varepsilon)$.

The Dyson-index ansatz–the focus earlier in the paper–appears to hold in some trivial/degenerate sense if we employ rather than the Lovas-Andai or Slater separability functions discussed above, the "Milz-Strunz" ones [18]. Then, rather than the singular-value ratio ε or the ratio of diagonal entries μ, one would use as the dependent/predictor variable,

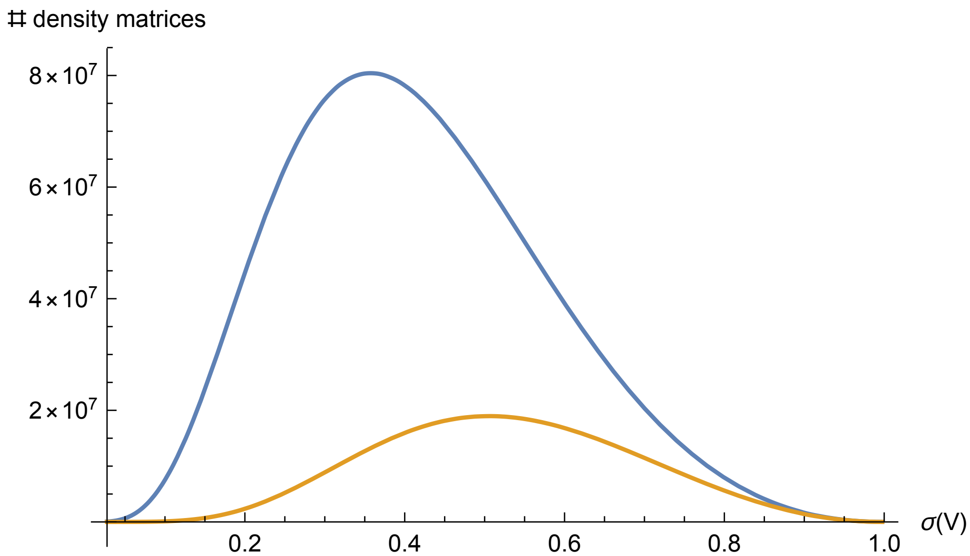

FIG. 4: Recorded counts by binned values of the singular value ratio $\sigma(V)$ of all 6,680 million two-qubit density matrices randomly generated (with respect to Hilbert-Schmidt measure), along with the accompanying (lesser) counts of separable density matrices

the Casimir invariants of the reduced systems [19]. In these cases, the separability functions become simply constant in nature. In the two-rebit and two-qubit cases, this invariant is the Bloch radius (r) of one of the two reduced systems. From the arguments of Lovas and Andai [1, Cor. 2, Thm. 2], it appears that one can assert that the Milz-Strunz form of two-rebit separability function assumes the constant value $\frac{29}{64}$ for $r \in [0, 1]$. Then, it would seem that the two-qubit counterpart would be the constant value $\frac{8}{33}$ for $r \in [0, 1]$, with the corresponding (Dyson-index ansatz) constant of proportionality being $\frac{\frac{8}{33}}{(\frac{29}{64})^2} = \frac{32768}{27753} \approx 1.1807$.

III. RELATIONS BETWEEN $\epsilon = \sigma(V)$ AND BLOORE/SLATER VARIABLE μ

Let us now note a quite interesting phenomenon, apparently relating the Lovas-Andai analyses to previous ones of Slater [2]. If we perform the indicated integration in the denominator of (4), following the integration-by-parts scheme adopted by Lovas and Andai [1, p. 12], at an intermediate stage we arrive at the univariate integrand

$$\frac{128t^3 \left(5 \left(5t^8 + 32t^6 - 32t^2 - 5\right) - 12 \left(\left(t^2 + 2\right) \left(t^4 + 14t^2 + 8\right) t^2 + 1\right) \log(t)\right)}{3 \left(t^2 - 1\right)^8}. \qquad (12)$$

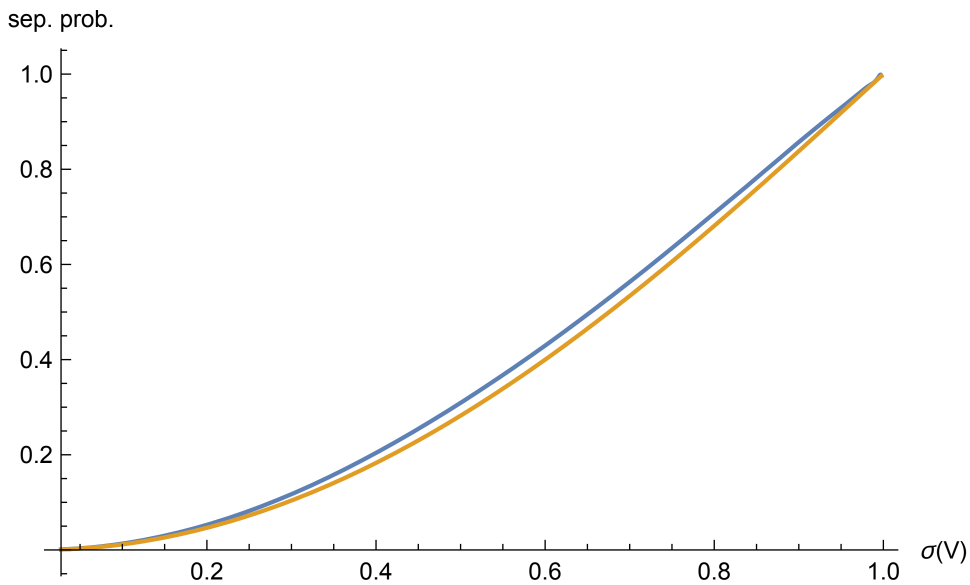

FIG. 5: Estimated two-qubit separability probabilities together with the slightly subordinate curve $\tilde{\chi}_1^2(\varepsilon)$

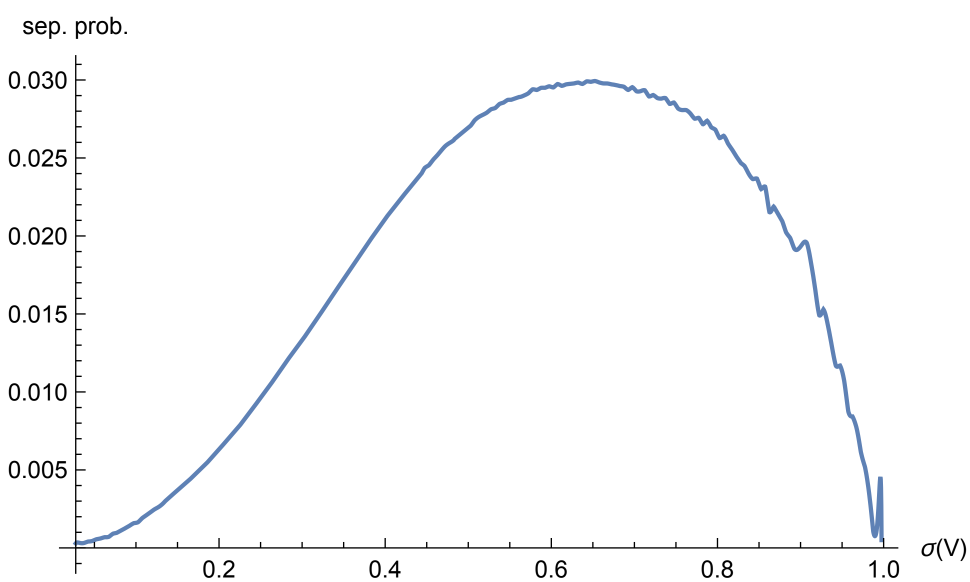

FIG. 6: Result of subtracting the (slightly subordinate) $\tilde{\chi}_1^2(\varepsilon)$ curve from the estimated two-qubit separability probability curve in Fig. 5

(Its integral over $t \in [0,1]$ equals the noted value of $\frac{16}{35}$, where $\frac{\frac{16}{35}-\frac{1}{4}}{\frac{16}{35}} = \frac{29}{64}$.) This, interestingly, bears a very close (almost identical) structural resemblance to the jacobian/volume-element

$$\mathcal{H}_{real}(\mu) = -\frac{\mu^4 \left(5 \left(5\mu^8 + 32\mu^6 - 32\mu^2 - 5\right) - 12 \left(\left(\mu^2+2\right)\left(\mu^4 + 14\mu^2 + 8\right)\mu^2 + 1\right)\log(\mu)\right)}{1890 \left(\mu^2-1\right)^9} \tag{13}$$

(integrating to $\frac{\pi^2}{2293760}$ over $\mu \in [0,1]$) reported by Slater in [2, eq. (15)] and [20, eq. (10)], also in the context of two-rebit separability functions. (We change the notation in those references from $\mathcal{J}_{real}(\nu)$ to $\mathcal{H}_{real}(\mu)$ here, since we have made the transformation $\nu \to \mu^2$, to facilitate this comparison, and the analogous one below in the two-qubit context–with the approach of Lovas and Andai. However, we will still note some results below in the original [ν] framework.) To faciltate the comparison between these two functions, we set $t = \mu = \tilde{t}$, and then divide (12) by (13), obtaining the simple ratio

$$\frac{80640 \left(1-\tilde{t}^2\right)}{\tilde{t}}. \tag{14}$$

But we note that in in [2] and [20]–motivated by work in a 3×3 density matrix context of Bloore [21]–the variable μ was taken to be the ratio $\sqrt{\frac{d_{11}d_{44}}{d_{22}d_{33}}}$ of the square root of the product of the (1,1) and (4,4) diagonal entries of the density matrix [2, eq. (1)]

$$D = \begin{pmatrix} d_{11} & z_{12}\sqrt{d_{11}d_{22}} & z_{13}\sqrt{d_{11}d_{33}} & z_{14}\sqrt{d_{11}d_{44}} \\ z_{12}\sqrt{d_{11}d_{22}} & d_{22} & z_{23}\sqrt{d_{22}d_{33}} & z_{24}\sqrt{d_{22}d_{44}} \\ z_{13}\sqrt{d_{11}d_{33}} & z_{23}\sqrt{d_{22}d_{33}} & d_{33} & z_{34}\sqrt{d_{33}d_{44}} \\ z_{14}\sqrt{d_{11}d_{44}} & z_{24}\sqrt{d_{22}d_{44}} & z_{34}\sqrt{d_{33}d_{44}} & d_{44} \end{pmatrix} \tag{15}$$

to the product of the (2,2) and (3,3) ones, while in [1], it would be the ratio $\sigma(V)$ of the singular values of the noted 2×2 matrix $D_2^{1/2} D_1^{-1/2}$. From [2, eq. (91)], we can deduce that one must multiply $\mathcal{H}_{real}(\mu)$ by $\frac{1048576}{\pi^2}$, so that its integral from 0 to 1 equal the Lovas-Andai counterpart result of $\frac{16}{35}$. (The jacobian of the transformation to the two-rebit density matrix parameterization (15) is $(d_{11}d_{22}d_{33}d_{44})^{3/2}$, and for the two-qubit counterpart, $(d_{11}d_{22}d_{33}d_{44})^3$ [20, p. 4]. These jacobians are also reported in [22].)

In [2, eq. (93)], the two-rebit separability function $S_{real}(\nu)$ was taken to be proportional to the incomplete beta function $B_\nu(\nu, \frac{1}{2}, 2) = \frac{2}{3}(3-\nu)\sqrt{\nu}$–an apparently much simpler function than the Lovas-Andai counterpart (2) above. Given the just indicated scaling by $\frac{1048576}{\pi^2}$, to achieve the $\frac{29}{140}$ separability probability numerator result of Lovas and Andai, we must take the hypothesized separability function to be, then, $\frac{3915\pi^2(3-\nu)\sqrt{\nu}}{131072}$.

A parallel phenomenon is observed in the two-qubit case, where [20, eq. (11)]

$$\mathcal{H}_{complex}(\mu) = -\frac{\mu^7 (h_1 + h_2)}{1801800(\mu^2-1)^{15}}, \tag{16}$$

with

$$h_1 = (\mu-1)(\mu+1)\left(363\mu^{12} + 10310\mu^{10} + 58673\mu^8 + 101548\mu^6 + 58673\mu^4 + 10310\mu^2 + 363\right)$$

and

$$h_2 = -140\left(\mu^2+1\right)\left(\mu^{12} + 48\mu^{10} + 393\mu^8 + 832\mu^6 + 393\mu^4 + 48\mu^2 + 1\right)\log(\mu).$$

Setting $\alpha = 1$ in the denominator formula (7), and again following the integration-by-parts scheme of Lovas and Andai, while setting $t = \mu = \tilde{t}$, the simple ratio (proportional to the square of (14)) is now

$$\frac{210862080\left(1-\tilde{t}^2\right)^2}{\tilde{t}^2}. \tag{17}$$

To achieve the $\frac{256}{1575}$ Lovas-Andai two-qubit denominator result, we must multiply $\mathcal{H}_{complex}(\nu)$ (16) by 328007680.

The two-qubit separability function $S_{complex}(\nu)$ advanced in [2] was proportional to the square of that–$B_\nu(\nu, \frac{1}{2}, 2) = \frac{2}{3}(3-\nu)\sqrt{\nu}$–employed in the two-rebit context. Now, to obtain the two-qubit numerator result of $\frac{2048}{51975}$ necessary for the $\frac{8}{33}$ separability probability outcome, we took the associated separability function to simply be $\frac{6}{71}(3-\nu)^2\nu$. We refer the reader to Figure 2 in [2] (and Figs. 13 and 14 below) to see the extraordinarily good fit of this function. (However, the two-rebit fit displayed there does not appear quite as good.)

Let us now supplement the earlier plots in [2], with some newly generated ones. (Those 2007 plots were based on quasi-Monte Carlo ["low-discrepancy" point [23]] sampling, while the ones presented here are based on more "state-of-the-art" sampling methods [16], with many more density matrices [but, of "higher-discrepancy"] generated.) In Figs. 7 and 8 we show the two-rebit separability probabilities as a function, firstly, of $\nu = \frac{d_{11}d_{44}}{d_{22}d_{33}}$ and, secondly, as a function of $\mu = \sqrt{\nu} = \sqrt{\frac{d_{11}d_{44}}{d_{22}d_{33}}}$, together with the curves $\frac{3915\pi^2(3-\nu)\sqrt{\nu}}{131072}$ and $\frac{3915\pi^2(3-\mu^2)\mu}{131072}$, respectively. In Figs. 9 and 10 we show the two-qubit separability probabilities as a function, firstly, of ν and, secondly, as a function of μ, together with the curves $\frac{6}{71}(3-\nu)^2\nu$ and $\frac{6}{71}(3-\mu^2)^2\mu^2$, respectively.

In Figs. 11, 12, 13 and 14, rather than showing the estimated separability probabilities together with the separability functions as in the previous four figures, we show the estimated

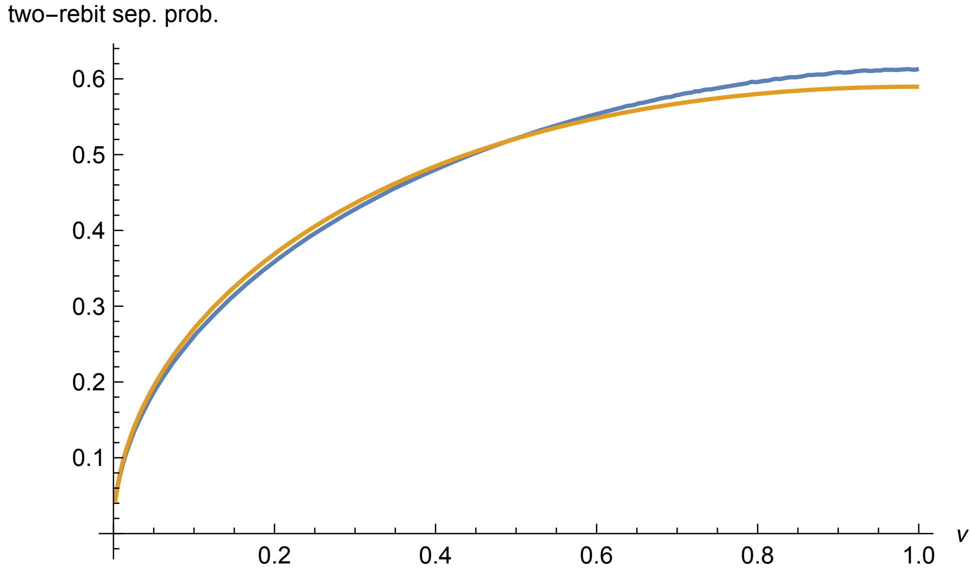

FIG. 7: Estimated two-rebit Hilbert-Schmidt separability probabilities, based on 687 million randomly-generated density matrices, together with the hypothesized (slightly subordinate) separability function $\frac{3915\pi^2(3-\nu)\sqrt{\nu}}{131072}$

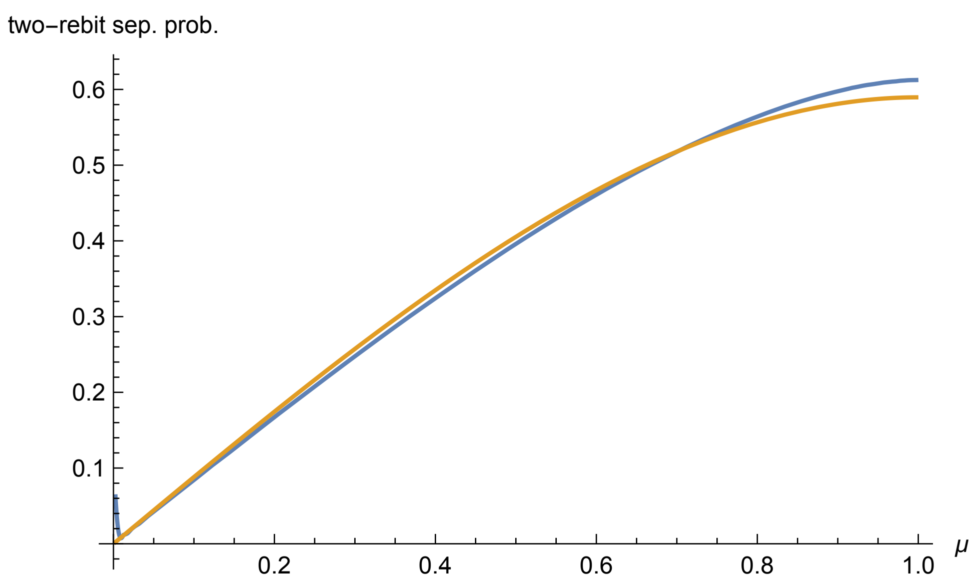

FIG. 8: Estimated two-rebit Hilbert-Schmidt separability probabilities, based on 5,077 million randomly-generated density matrices, together with the hypothesized (slightly subordinate) separability function $\frac{3915\pi^2(3-\mu^2)\mu}{131072}$

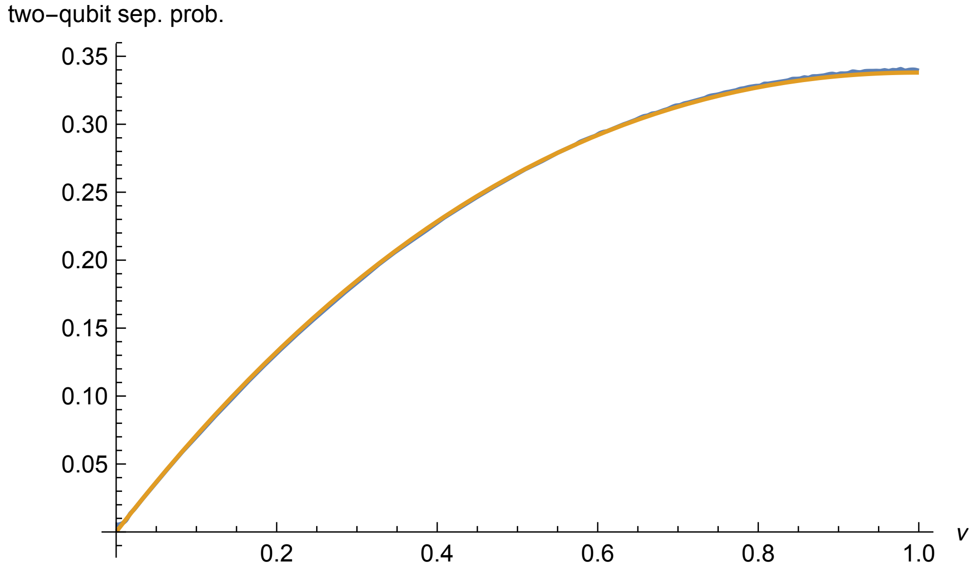

FIG. 9: Estimated two-qubit Hilbert-Schmidt separability probabilities, based on 507 million randomly-generated density matrices, together with the (indiscernibly different) separability function $\frac{6}{71}(3-\nu)^2\nu$ (cf. Fig. 13 and [2, Fig. 2] for the residuals from the fit)

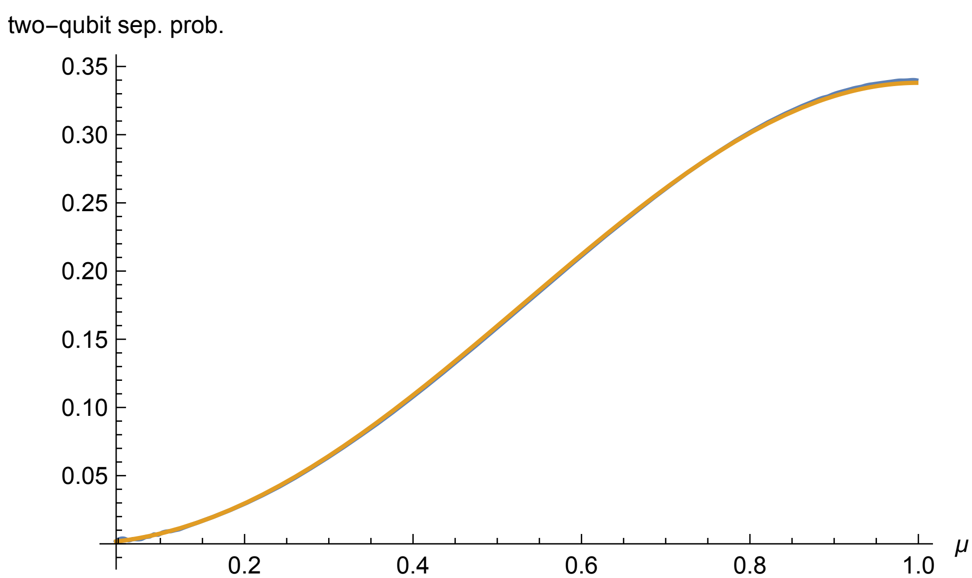

FIG. 10: Estimated two-qubit Hilbert-Schmidt separability probabilities, based on 3,715 million randomly-generated density matrices, together with the (indiscernibly different) separability function $\frac{6}{71}(3-\mu^2)^2\mu^2$

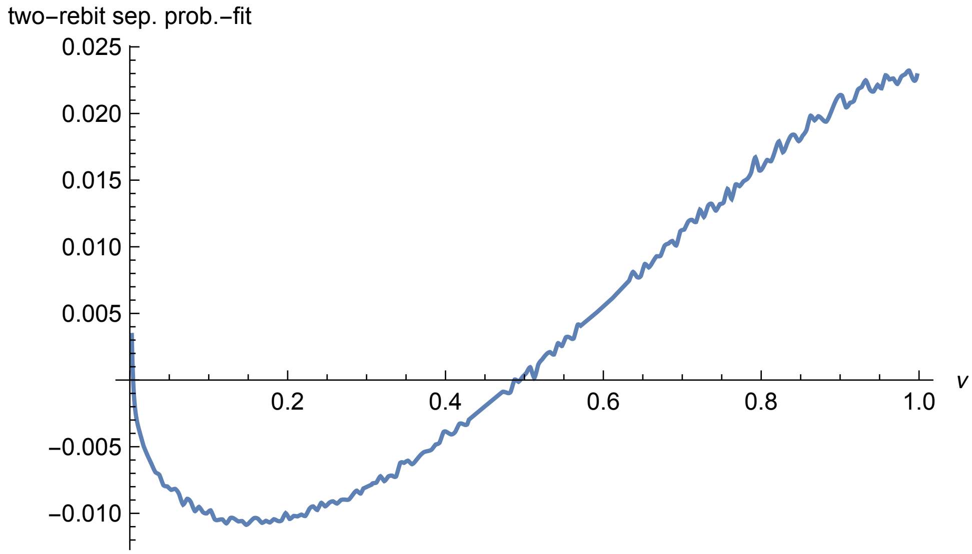

FIG. 11: Estimated two-rebit Hilbert-Schmidt separability probabilities, based on 687 million randomly-generated density matrices, *minus* the separability function $\frac{3915\pi^2(3-\nu)\sqrt{\nu}}{131072}$

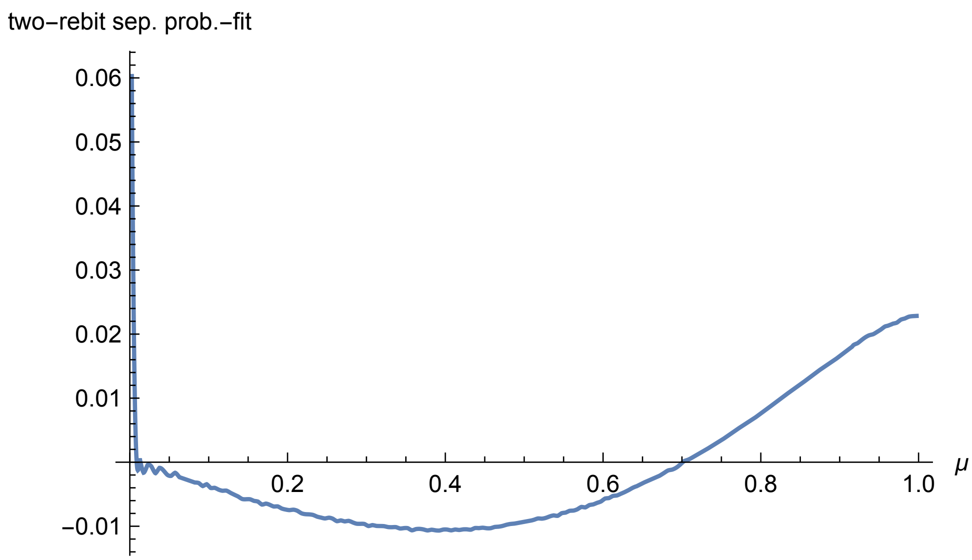

FIG. 12: Estimated two-rebit Hilbert-Schmidt separability probabilities, based on 5,077 million randomly-generated density matrices, *minus* the separability function $\frac{3915\pi^2(3-\mu^2)\mu}{131072}$

separability probabilities *minus* the separability functions, that is, the *residuals* from this fits.

So, at this stage, the evidence is certainly strong that the Dyson-index ansatz is at least of some value in approximately fitting the relationships between two-rebit and two-qubit Hilbert-Schmidt separability functions.

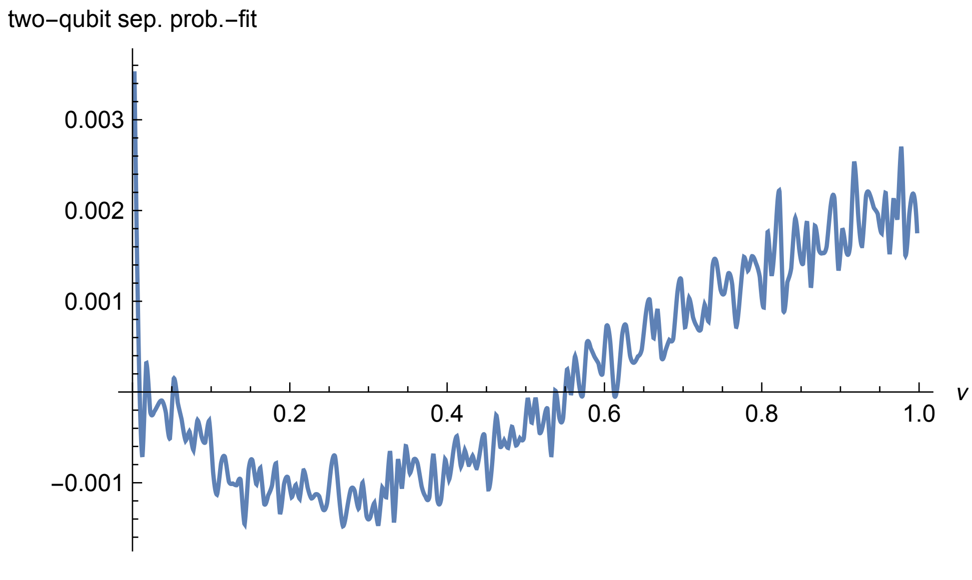

FIG. 13: Estimated two-qubit Hilbert-Schmidt separability probabilities, based on 3,715 million randomly-generated density matrices, *minus* the separability function $\frac{6}{71}(3-\nu)^2\nu$

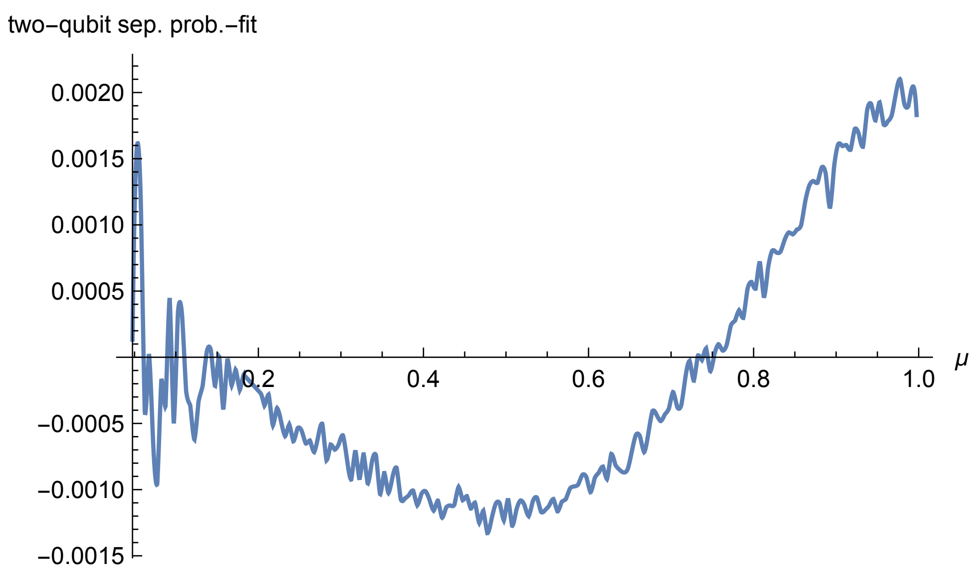

FIG. 14: Estimated two-qubit Hilbert-Schmidt separability probabilities, based on 3,715 randomly-generated density matrices, *minus* the separability function $\frac{6}{71}(3-\mu^2)^2\mu^2$

A. Formulas linking the Lovas-Andai variable ε and the Slater/Bloore variable μ

Using the two-rebit density matrix parameterization (15), then, taking the previously indicated relationship (10), which has the explicit form in this case of

$$\varepsilon = \exp\left(-\cosh^{-1}\left(\frac{-\mu^2+2\mu z_{12}z_{34}-1}{2\mu\sqrt{z_{12}^2-1}\sqrt{z_{34}^2-1}}\right)\right), \tag{18}$$

and inverting it, we find

$$\mu = \frac{1}{2}\left(\lambda-\sqrt{\lambda^2-4}\right), \tag{19}$$

where

$$\lambda = 2z_{12}z_{34}-\sqrt{z_{12}^2-1}\sqrt{z_{34}^2-1}\left(\frac{1}{\varepsilon^2}+1\right)\varepsilon.$$

For the two-qubit counterpart, we have

$$\varepsilon = \exp\left(-\cosh^{-1}\left(\frac{-\mu^2+2\mu\left(y_{12}y_{34}+z_{12}z_{34}\right)-1}{2\mu\sqrt{y_{12}^2+z_{12}^2-1}\sqrt{y_{34}^2+z_{34}^2-1}}\right)\right). \tag{20}$$

The z_{ij}'s are as in the two-rebit case (23), and the y_{ij}'s are now the corresponding imaginary parts in the natural extension of the two-rebit density matrix parameterization (15). A similar inversion yields

$$\mu = \frac{1}{2}\left(\tilde{\lambda}-\sqrt{\tilde{\lambda}^2-4}\right), \tag{21}$$

where

$$\tilde{\lambda} = -\left(\frac{1}{\varepsilon^2}+1\right)\varepsilon\sqrt{y_{12}^2+z_{12}^2-1}\sqrt{y_{34}^2+z_{34}^2-1}+2y_{12}y_{34}+2z_{12}z_{34}.$$

It appears to be a challenging problem, using these relations (10), (19) and (21), to transform the ε-parameterized volume forms and separability functions in the Lovas-Andai framework to the μ-parameterized ones in the Slater setting, and *vice versa.* (The presence of the z and y variables in the formulas, undermining any immediate one-to-one relationship between ε and μ, is a complicating factor.)

The correlation between the ε and μ variables, estimated on the basis of one million randomly-generated (with respect to Hilbert-Schmidt measure) density matrices was 0.631937 in the two-rebit instance, and 0.496949 in the two-qubit one.

Also, in these two sets of one million cases, μ was always larger than ε. This dominance effect (awaiting formal verification) is reflected in Figs. 15 and 16, being plots of the separability probabilities (again based on samples of size 5,077 and 3,715 million, respectively) as *joint* functions of ε and μ, with no results appearing in the regions $\varepsilon > \mu$.

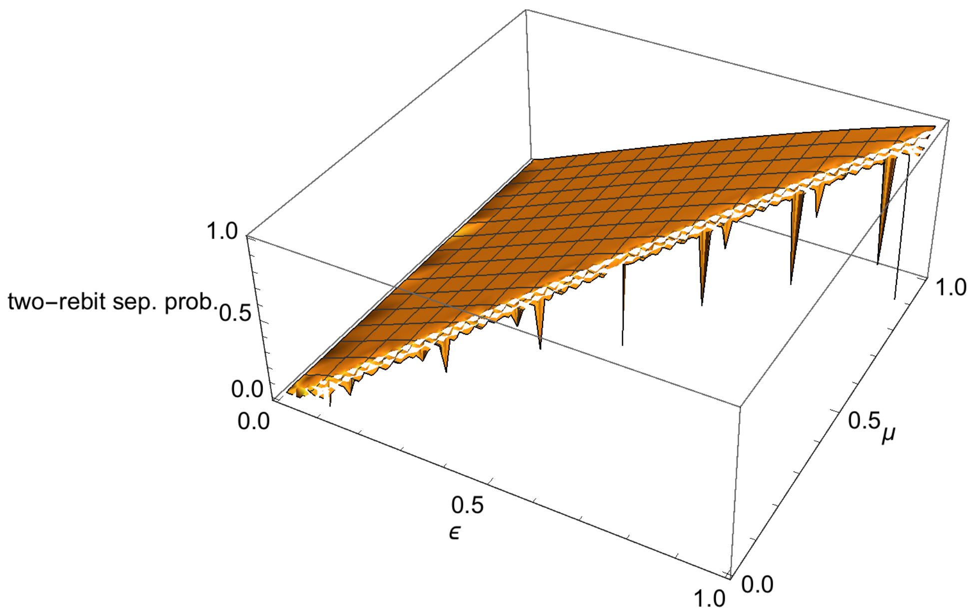

FIG. 15: Two-rebit separability probabilities as joint function of ε and μ, based on 5,077 million randomly-generated density matrices. Note the vacant region $\varepsilon > \mu$.

It has been noted (http://mathoverflow.net/questions/262943/show-that-a-certain-ratio-of-diagonal-entries-dominates-a-certain-ratio-of-singu) that for a diagonal 4×4 density matrix D that $\varepsilon = \mu$ (inverting ratios, if necessary, so that both are less than or greater than 1). This equality can also be observed by setting $z_{12} = z_{24} = 0$ (and $y_{12} = y_{24} = 0$) in the equations immediate above.

Let us now display three plots that support, but only approximately, the possible relevance of the Dyson-index ansatz for two-rebit and two-qubit separability functions. In Fig. 17, we show the ratio of the square of the two-rebit separability probabilities to the two-qubit separability probabilities, in terms of the variable employed by Slater, $\mu = \sqrt{\frac{d_{11} d_{44}}{d_{22} d_{33}}}$. In Fig. 18, we show the Lovas-Andai counterpart, that is, in terms of the ratio of singular values variable, $\varepsilon = \sigma(V)$. Further, in Fig. 19 we display the ratio of the *square* of the two-dimensional two-rebit plot (Fig. 15) to the two-dimensional two-qubit plot (Fig. 16). These three figures all manifest an upward trend in the ratios as ε and/or μ increase.

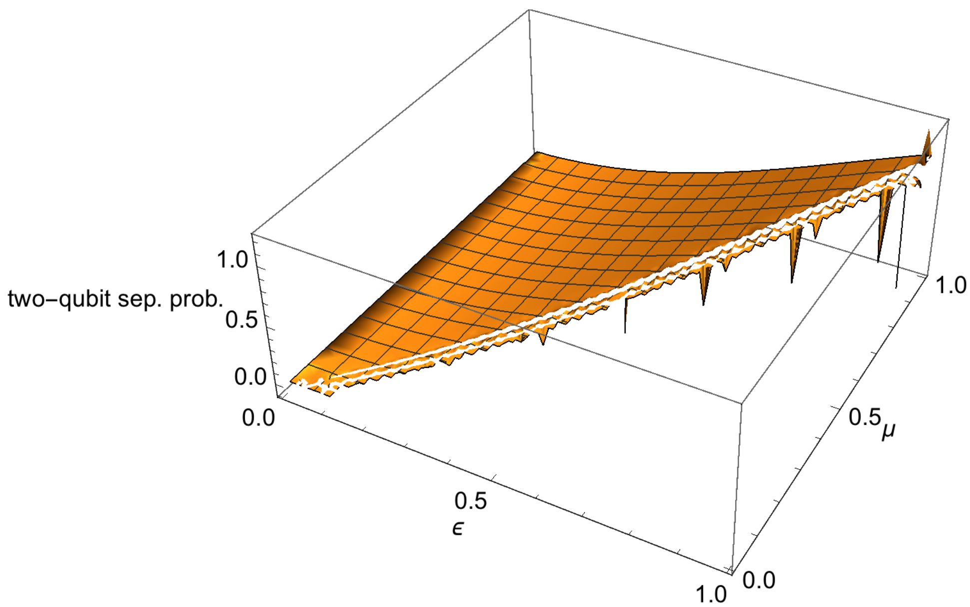

FIG. 16: Two-qubit separability probabilities as joint function of ε and μ, based on 3,715 million randomly-generated density matrices. Note the vacant region $\varepsilon > \mu$.

IV. SCENARIOS FOR WHICH $\varepsilon = \mu$ OR $\frac{1}{\mu}$

A. Seven-dimensional convex set of two-rebit states

If we set $z_{12} = z_{34} = 0$ in the relation (18), we obtain $\varepsilon = \mu$ or $\frac{1}{\mu}$. So, let us try to obtain the separability function when these null conditions are fulfilled. First, we found that the volume of the seven-dimensional convex set is equal to $\frac{1}{5040} \cdot \frac{2\pi^2}{3} = \frac{\pi^2}{7560} \approx 0.0013055$, with a jacobian for the transformation to μ equal to

$$\frac{\mu^3 \left(-11\mu^6 - 27\mu^4 + 27\mu^2 + 6\left(\mu^6 + 9\mu^4 + 9\mu^2 + 1\right)\log(\mu) + 11\right)}{210\left(\mu^2 - 1\right)^7}. \tag{22}$$

($\frac{2\pi^2}{3}$ is the normalization constant corresponding to $\chi_1(\varepsilon)$ [1, Table 2], appearing in the "defect function" (3), as well as the volume of the standard unit ball in the normed vector space of 2×2 matrices with real entries, denoted by $\mathcal{B}_1(\mathbb{R}^{2\times2})$.)

We were, further, able to impose the condition that two of the principal 3×3 minors of the partial transpose are positive. The resultant separability function (Fig. 20) was

$$\begin{cases} \frac{2\left(\sqrt{\mu^2-1}+\mu^2 \csc^{-1}(\mu)\right)}{\pi\mu^2} & \mu > 1 \\ \frac{2\sqrt{1-\mu^2}\mu+2i\log\left(\mu+i\sqrt{1-\mu^2}\right)+\pi}{\pi} & 0 < \mu < 1 \end{cases}, \tag{23}$$

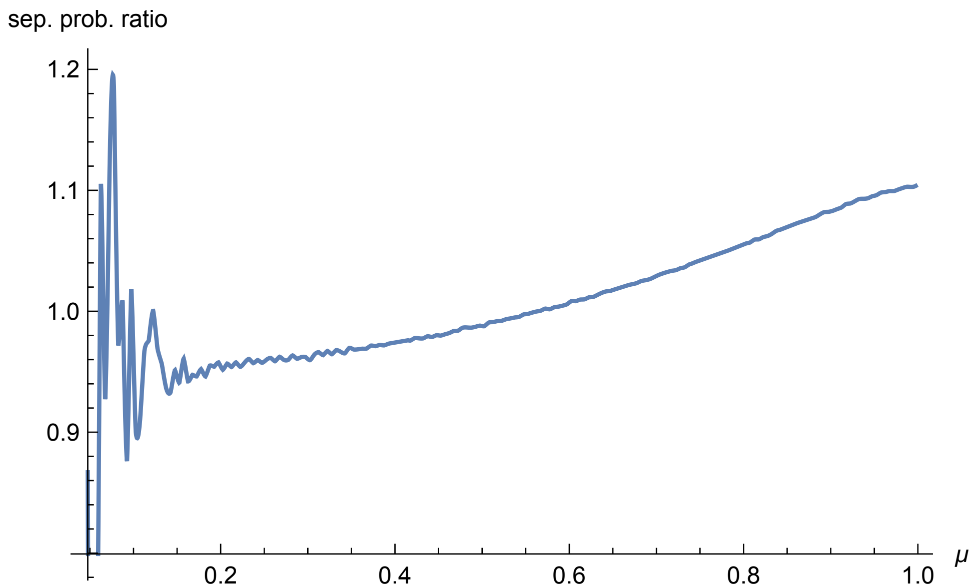

FIG. 17: Ratio of the *square* of the estimated two-rebit separability probabilities (Fig. 8) to the estimated two-qubit separability probabilities (Fig. 10), as a function of $\mu = \sqrt{\frac{d_{11} d_{44}}{d_{22} d_{33}}}$

with an associated separability probability of $\frac{71}{105} \approx 0.67619$.

We, then, sought to impose–as both necessary and sufficient for separability [24, 25]–the positivity of the partial transpose of the density matrix. First, we found that the associated separability function assumes the value 1 at $\mu = 1$. For $\mu = 2, 3$, we formulated four-dimensional constrained integration problems. Mathematica reduced them to two-dimensional integration problems, for which we were able to perform high precision calculations. Remarkably, the values obtained agreed with those for $\tilde{\chi}_1(\frac{1}{2}) = \tilde{\chi}_1(2)$ and $\tilde{\chi}_1(\frac{1}{3}) = \tilde{\chi}_1(3)$ to more than twenty decimal places. The two-dimensional integrands Mathematica yielded for $\mu = 2$ were of the form

$$\frac{3\left(\pi\sqrt{-4z_{13}^2 - z_{14}^2 + 4} - 8\sqrt{-z_{13}^2 - z_{14}^2 + 1}\sin^{-1}\left(\frac{z_{14}}{2\sqrt{1-z_{13}^2}}\right) + 2\sqrt{-4z_{13}^2 - z_{14}^2 + 4}\sin^{-1}\left(\frac{z_{14}}{\sqrt{1-z_{13}^2}}\right)\right)}{8\pi^2} \tag{24}$$

for

$$-1 < z_{13} < 1 \land z_{14} + \sqrt{1 - z_{13}^2} > 0 \land z_{14} < 0$$

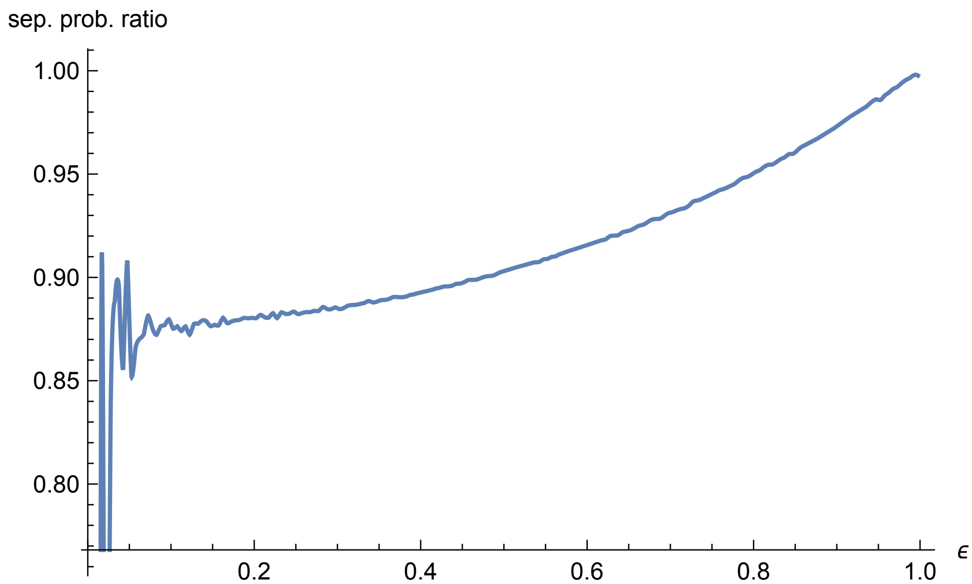

FIG. 18: Ratio of the *square* of the estimated two-rebit separability probabilities (Fig. 2) to the estimated two-qubit separability probabilities (Fig. 5), as a function of the ratio of singular values variable, $\varepsilon = \sigma(V)$

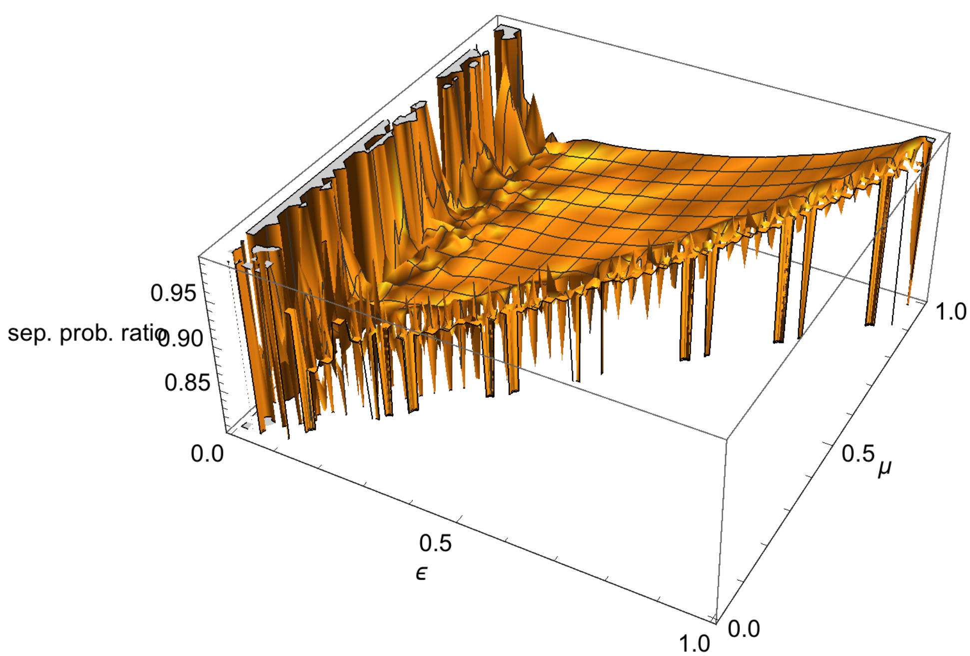

FIG. 19: The ratio of the *square* of the two-dimensional two-rebit plot (Fig. 15) to the two-dimensional two-qubit plot (Fig. 16)

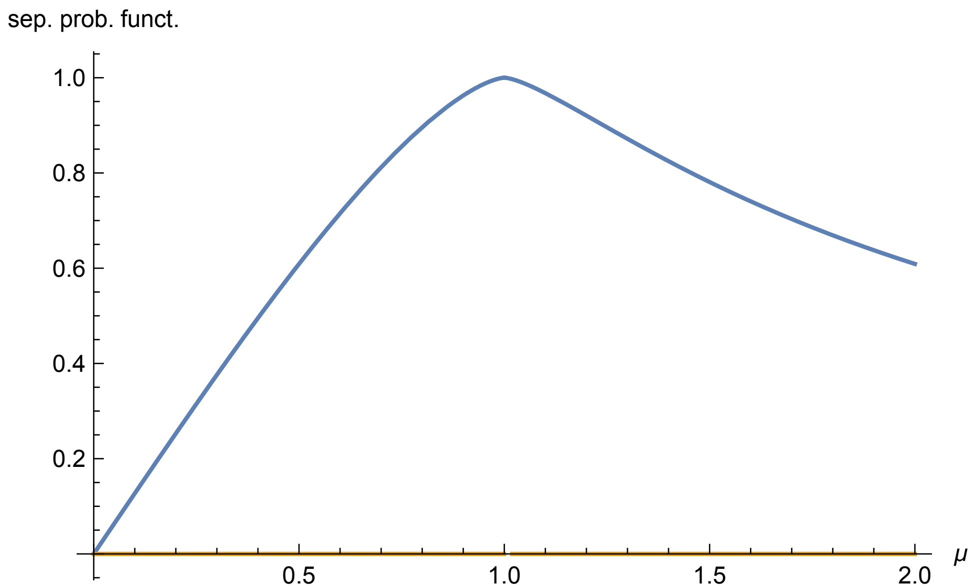

FIG. 20: Two-rebit separability probability function (23) for the seven-dimensional convex set for which $\varepsilon = \mu$ or $\frac{1}{\mu}$, based on the positivity of two principal 3×3 minors of the partial transpose

and

$$\frac{3\left(\pi\sqrt{-4z_{13}^2 - z_{14}^2 + 4} + 8\sqrt{-z_{13}^2 - z_{14}^2 + 1}\sin^{-1}\left(\frac{z_{14}}{2\sqrt{1-z_{13}^2}}\right) - 2\sqrt{-4z_{13}^2 - z_{14}^2 + 4}\sin^{-1}\left(\frac{z_{14}}{\sqrt{1-z_{13}^2}}\right)\right)}{8\pi^2} \tag{25}$$

for

$$-1 < z_{13} < 1 \wedge z_{14} > 0 \wedge \sqrt{1 - z_{13}^2} - z_{14} > 0.$$

So, in light of this evidence, we are confident in concluding that the Lovas-Andai two-rebit separability function $\tilde{\chi}_1(\varepsilon)$ serves as both the Lovas-Andai and Slater separability functions in this seven-dimensional setting.

To still more formally proceed, we were able to generalize the pair of two-dimensional integrands for the specific case $\mu = 2$ given in (24) and (25) to $\mu = 1, 2, 3, \ldots$, obtaining

$$\frac{3\left(2\mu\sqrt{-\mu^2 z_{14}^2 - z_{13}^2 + 1}\sin^{-1}\left(\frac{z_{14}}{\sqrt{1-z_{13}^2}}\right) - 2\sqrt{-z_{13}^2 - z_{14}^2 + 1}\sin^{-1}\left(\frac{\mu z_{14}}{\sqrt{1-z_{13}^2}}\right) + \pi\sqrt{-z_{13}^2 - z_{14}^2 + 1}\right)}{2\pi^2} \tag{26}$$

for

$$1 < z_{13} < 1 \wedge z_{14} > 0 \wedge \sqrt{1 - z_{13}^2} - \mu z_{14} > 0$$

and

$$\frac{3\left(-2\mu\sqrt{-\mu^2 z_{14}^2-z_{13}^2+1}\sin^{-1}\left(\frac{z_{14}}{\sqrt{1-z_{13}^2}}\right)+2\sqrt{-z_{13}^2-z_{14}^2+1}\sin^{-1}\left(\frac{\mu z_{14}}{\sqrt{1-z_{13}^2}}\right)+\pi\sqrt{-z_{13}^2-z_{14}^2+1}\right)}{2\pi^2} \tag{27}$$

for

$$-1<z_{13}<1\wedge\mu z_{14}+\sqrt{1-z_{13}^2}>0\wedge z_{14}<0.$$

1. *Reproduction of Lovas-Andai two-rebit separability function* $\tilde{\chi}_1(\varepsilon)$

Making use of these last set of relations, we were able to reproduce the Lovas-Andai two-rebit separability function $\tilde{\chi}_1(\varepsilon)$, given in (2). We accomplished this by, first, reducing the (general for integer $\mu>1$) two-dimensional integrands (26) and (27) to two piecewise one-dimensional ones of the form

$$\frac{4\left(\mu^2\sqrt{1-s^2}\sin^{-1}\left(\frac{s}{\mu}\right)+\sqrt{\mu^2-s^2}\cos^{-1}(s)\right)}{\pi^2\mu^2} \tag{28}$$

over $s\in[0,1]$ and

$$\frac{2\pi\sqrt{\mu^2-s^2}-4\mu^2\sqrt{1-s^2}\sin^{-1}\left(\frac{s}{\mu}\right)+4\sqrt{\mu^2-s^2}\sin^{-1}(s)}{\pi^2\mu^2} \tag{29}$$

over $s\in[-1,0]$.

To obtain these one-dimensional integrands, which we then were able to explicitly evaluate, we made the substitution $z_{14}\to\frac{s\sqrt{1-z_{13}^2}}{\mu}$, then integrated over $z_{13}\in[-1,1]$, with $\mu\geq 1$, so that $\varepsilon=\frac{1}{\mu}$. Let us note that in this approach, the dependent variable (μ) appears in the integrands, while in the Lovas-Andai derivation, the dependent variable (ε) appears as a limit of integration. The counterpart set of two piecewise integrands to (28) and (29) for the reciprocal case of $0<\mu\leq 1$ are

$$\frac{4\left(\mu^2\sqrt{1-s^2}\cos^{-1}\left(\frac{s}{\mu}\right)+\sqrt{(\mu-s)(\mu+s)}\sin^{-1}(s)\right)}{\pi^2\mu^2} \tag{30}$$

over $s\in[0,1]$ with $\mu>s$ and

$$\frac{2\mu^2\sqrt{1-s^2}\left(2\sin^{-1}\left(\frac{s}{\mu}\right)+\pi\right)-4\sqrt{(\mu-s)(\mu+s)}\sin^{-1}(s)}{\pi^2\mu^2} \tag{31}$$

over $s \in [-1, 0]$ with $\mu > -s$. The corresponding univariate integrations then directly yield the Lovas-Andai two-rebit separability function $\tilde{\chi}_1(\varepsilon)$, given in (2), now with $\varepsilon = \mu$, rather than $\varepsilon = \frac{1}{\mu}$.

As an interesting aside, let us note that we can obtain $\varepsilon = \mu$ in (18), in a nontrivial fashion (that is, not just by taking $z_{12} = z_{34} = 0$), by setting

$$z_{34} = \frac{z_{12}\left(-2\left(\mu^3+\mu\right)+\mu^4\left(-\sqrt{z_{12}^2-1}\right)+\sqrt{z_{12}^2-1}\right)}{\left(\mu^2-1\right)^2 z_{12}^2-\left(\mu^2+1\right)^2}, \tag{32}$$

leading to an eight-dimensional framework. However, this result did not seem readily amenable to further study/analysis.

B. Eleven-dimensional convex set of two-qubit states

Let us repeat for the 15-dimensional convex set of two-qubit states, the successful form of analysis in the preceding section, again nullifying the (1,2), (2,1), (3,4), (4,3) entries of D, so that the two diagonal 2×2 blocks D_1, D_2 are themselves diagonal. This leaves us in an 11-dimensional setting. The associated volume we computed as $\frac{1}{9979200} \cdot \frac{\pi^4}{6} = \frac{\pi^4}{59875200} \approx 1.62687 \cdot 10^{-6}$. (Here $\frac{\pi^4}{6}$ is the normalization constant corresponding to $\chi_2(1)$ [1, Table 2], as well as the volume of the standard unit ball in the normed vector space of 2×2 matrices with complex entries, denoted by $\mathcal{B}_1(\mathbb{C}^{2\times 2})$.) The associated jacobian for the transformation to the μ variable is

$$\frac{\mu^5\left(A(\mu-1)(\mu+1)-60\left(6\mu^{10}+75\mu^8+200\mu^6+150\mu^4+30\mu^2+1\right)\log(\mu)\right)}{83160\left(\mu^2-1\right)^{12}} \tag{33}$$

with

$$A = 5\mu^{10} + 647\mu^8 + 4397\mu^6 + 6397\mu^4 + 2272\mu^2 + 142.$$

The imposition of positivity for one of the 3×3 principal minors of the partial transpose yielded a separability function of $\frac{2\mu^2-1}{\mu^4}$ for $\mu > 1$, with an associated bound on the true separability probability of this set of eleven-dimensional two-qubit density matrices of $\frac{126}{181} \approx 0.696133$. (This function bears an interesting resemblance to the later reported important one (38).)

Again–as in the immediately preceding seven-dimensional two-rebit setting–imposing, as both necessary and sufficient for separability [24, 25], the positivity of the partial transpose of the density matrix, we find that the associated separability function assumes the value 1 at $\mu = 1$. For $\mu = 2$, our best estimate was 0.36848, which in line with the seven-dimensional analysis, would appear to be an approximation to the previously unknown value of $\tilde{\chi}_2(\frac{1}{2}) = \tilde{\chi}_2(2)$.

1. *Proof of the $\frac{8}{33}$-Two-Qubit Hilbert Schmidt Separability Probability Conjecture*

We applied the Mathematica command GenericCylindricalDecomposition to an eight-dimensional set (plus μ) of positivity conditions, enforcing the positive-definite nature of two-qubit (4×4) density matrices (with their (1,2), (2,1), (3,4) and (4,3) entries nullified) and of their partial transposes, for $\mu > 1$. ("GenericCylindricalDecomposition[ineqs,$x_1, x_2, ...$] finds the full-dimensional part of the decomposition of the region represented by the inequalities ineqs into cylindrical parts whose directions correspond to the successive x_i, together with any hypersurfaces containing the rest of the region.")

These density matrices had their two 2×2 diagonal blocks, themselves set diagonal in nature. The parallel two-rebit analysis (sec. IV A) succeeded in reconstructing the Lovas-Andai function $\tilde{\chi_1}(\varepsilon)$, giving us confidence in this strategy. This pair of reduction strategies rendered the corresponding sets of density matrices as 11-dimensional and 7-dimensional in nature, rather than the standard full 15- and 9-dimensions, respectively. The cylindrical algebraic decomposition (CAD)–applied to the two-qubit positivity constraints (expressible in terms of μ and four z_{ij} and four y_{ij} variables)–yielded three complementary solutions. One of these consisted of three further complementary solutions. We analyzed each of the five irreducible solutions separately, employing them to perform integrations over the same set of four (z_{23}, y_{23}, y_{24} and z_{24}) of the eight variables. Then, we summed the five results, remarkably simplifying to the four-dimensional integrand

$$\frac{12\pi^2\left(-\left(\mu^2-1\right)y_{14}^2-\left(\mu^2-1\right)z_{14}^2+y_{13}^2+z_{13}^2-1\right)\left(\mu^2\left(y_{14}^2+z_{14}^2\right)+y_{13}^2+z_{13}^2-1\right)}{2\pi^4\left(1-y_{13}^2+z_{13}^2\right)}, \quad (34)$$

subject to the constraints

$$\mu > 1 \wedge -\frac{1}{\mu} < z_{14} < \frac{1}{\mu} \wedge -\frac{\sqrt{1-\mu^2 z_{14}^2}}{\mu} < y_{14} < \frac{\sqrt{1-\mu^2 z_{14}^2}}{\mu} \quad (35)$$

$$\wedge -\sqrt{1-\mu^2\left(y_{14}^2+z_{14}^2\right)} < y_{13} < \sqrt{1-\mu^2\left(y_{14}^2+z_{14}^2\right)}$$

$$\wedge -\sqrt{\mu^2\left(-\left(y_{14}^2+z_{14}^2\right)\right)-y_{13}^2+1} < z_{13} < \sqrt{\mu^2\left(-\left(y_{14}^2+z_{14}^2\right)\right)-y_{13}^2+1}.$$

The transformation to a pair of polar coordinates

$$\{z_{13} \to r_{13}\cos(t_{13}), z_{14} \to r_{14}\cos(t_{14}), y_{13} \to r_{13}\sin(t_{13}), y_{14} \to r_{14}\sin(t_{14})\}. \tag{36}$$

gave us a somewhat simpler integrand

$$\frac{12\pi^2 r_{13} r_{14} \left(r_{14}^2\mu^2 + r_{13}^2 - 1\right)\left(-r_{14}^2\left(\mu^2-1\right) + r_{13}^2 - 1\right)}{2\pi^4\left(1-r_{13}^2\right)}. \tag{37}$$

(Note four "active" variables in the first integrand, and only two radial and no angular ones in the second.) The integration constraints (35) now simply reduced to $r_{13}^2 + r_{14}^2\mu^2 < 1$, with $\mu > 1$. The integration result

$$f(u) = \frac{4\mu^2 - 1}{3\mu^4}, \tag{38}$$

immediately followed.

Now, the function that Lovai and Andai expressed hope in employing to verify the conjecture that the Hilbert-Schmidt two-qubit separability probability is $\frac{8}{33} \approx 0.242424$, is

$$\tilde{\chi_2}(\varepsilon) = f(\frac{1}{\varepsilon}) = \frac{1}{3}\varepsilon^2\left(4-\varepsilon^2\right). \tag{39}$$

This can be seen since the denominator of the equation (6) for $\mathcal{P}_{sep}(\mathbb{C})$ evaluates, as noted earlier, to

$$\frac{256}{1575}, \tag{40}$$

while the use of the newly-constructed $\tilde{\chi_2}(\varepsilon)$ yields a numerator value of

$$\frac{2048}{51975}, \tag{41}$$

with the ratio giving the $\frac{8}{33}$ result.

It is somewhat startling to compare the quite simple nature of $\tilde{\chi_2}(\varepsilon) : [0,1] \to [0,1]$ with its two-rebit (polylogarithmic/inverse hyperbolic tangent) counterpart ((1), (2)). Let us now present (Fig. 21) the two-qubit version of Fig. 2, showing again a random distribution of residuals, serving as further validation/support for the newly-constructed $\tilde{\chi_2}(\varepsilon)$.

Let us interestingly note that it was conjectured in 2007 [2, eqs. (93), (95); sec. 9.2] that the "two-qubit separability function" (in the Slater framework) had the form

$$\frac{6}{71}(3-\mu^2)\mu^2, \tag{42}$$

somewhat similar in nature to (39) (cf. Fig. 10, 14).

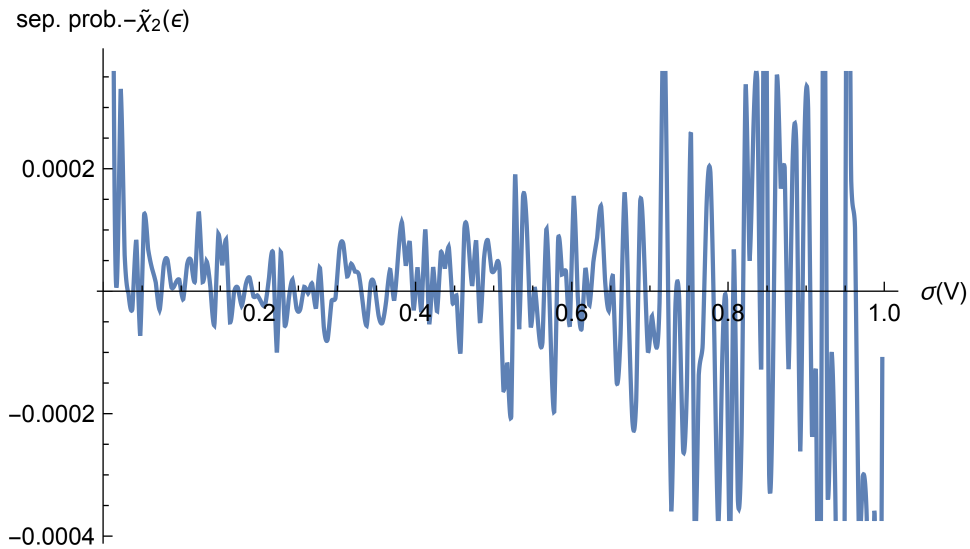

FIG. 21: Result of subtracting $\tilde{\chi}_2(\varepsilon)$ from the estimated two-qubit separability probability curve (Fig. 5). Fig. 3 is the two-rebit analogue.

V. REBIT-RETRIT AND QUBIT-QUTRIT ANALYSES

Let us now attempt to extend the two-rebit and two-qubit line of analysis above to rebit-retrit and qubit-qutrit settings–now, of course, passing from consideration of 4×4 density matrices to 6×6 ones. Lovas and Andai, in their quite recent study, had not yet addressed such issues. In [2], candidate (Slater-type) separability functions had been proposed. Two dependent variables (cf. the use of $\mu = \sqrt{\frac{d_{11}d_{44}}{d_{22}d_{33}}}$ in the lower-dimensional setting above) had been employed [2, eq. (44)]. Let us now refer to these two variables as $\tau_1 = \sqrt{\frac{d_{11}d_{55}}{d_{22}d_{44}}}$ and $\tau_2 = \sqrt{\frac{d_{22}d_{66}}{d_{33}d_{55}}}$. But, interestingly, it was argued that only a single dependent variable $\tau = \tau_1\tau_2 = \sqrt{\frac{d_{11}d_{66}}{d_{33}d_{44}}}$ sufficed for modeling the corresponding separability functions. The separability function in the rebit-retrit case was proposed to be simply proportional to τ [2, eq.(98)].

In our effort to extend the Lovas-Andai analyses [1] to this setting, we now took D_1 and D_2 to equal the upper and lower diagonal 3×3 blocks of the 6×6 density matrix in question. Then, we computed the three singular values ($s_1 \geq s_2 \geq s_3$) of $D_2^{1/2} D_1^{-1/2}$, and took the ratio variables $\varepsilon_1 = \frac{s_2}{s_1}$ and $\varepsilon_2 = \frac{s_3}{s_2}$ as the dependent ones in question. (An issue of possible concern is that, unlike the 4×4 case [26], positivity of the determinant of the partial transpose of a 6×6 density matrix is only a necessary, but not sufficient condition for separability.) Also, in the case of diagonal D, again the two variables in the μ framework are equal to those in the ε setting, or to their reciprocals.

Then, we generated 3,436 million rebit-retrit and 2,379 million qubit-qutrit density matrices, randomly with respect to Hilbert-Schmidt measure. (These sizes are much larger than those employed in 2007–for similar purposes–in [2].) We appraised the separability of the density matrices D by testing whether the partial transpose, using the four 3×3 blocks, had all its six eigenvalues positive. The separability probability estimates were $0.13180011 \pm 0.0000113109$ and $0.02785302 \pm 6.6124281 \cdot 10^{-6}$, respectively. (We can reject the qubit-qutrit conjecture of $\frac{32}{1199} \approx 0.0266889$ advanced in [2, sec. 10.2]. A possible alternative candidate is $\frac{72}{2585} = \frac{5 \cdot 11 \cdot 47}{2^3 \cdot 3^2} \approx 0.027853$, while in the rebit-retrit case, we have $\frac{298}{2261} = \frac{7 \cdot 17 \cdot 19}{2 \cdot 149} \approx 0.1318001$.) Further, our estimates of the probabilities that D had two [the most possible [27]] negative eigenvalues, and hence a positive determinant, although being entangled, were $0.0334197 \pm 0.0000409506$ in the rebit-retrit case, and $0.0103211 \pm 0.000031321$ in the qubit-qutrit instance.)

In the two-variable settings, we partition the square $[0,1]^2$ of possible separability probability results into an 80×80 grid, and in the one-variable setting, use a partitioning (as in the two-rebit and two-qubit analyses above) into 200 subintervals of [0,1]. In Fig. 22 we show the ratio of the square of the rebit-retrit separability probability to the qubit-qutrit separability probability as a function of τ, while in Fig. 23, we show a two-dimensional version. Fig. 24 is the analog of this last plot using the singular-value ratios ε_1 and ε_2. As in Figs. 17, 18 and 19, we observe a gradual increase in these Dyson-index-oriented analyses. In Figs. 25 and 26, we show the highly linear ("diagonal") rebit-retrit and qubit-qutrit separability probabilities, holding $\tau_1 = \tau_2$.

VI. CONCLUDING REMARKS

We have found (sec. IV A) that the Lovas-Andai two-rebit separability function $\tilde{\chi}_1(\varepsilon)$ also serves as the Slater separability function in a reduced (from nine to seven-dimensional) setting where the 2×2 diagonal block matrices D_1, D_2 are themselves diagonal. Additionally, we know that Lovas-Andai two-qubit separability function $\tilde{\chi}_2(\varepsilon)$ serves as the Slater separability functions in a reduced (from fifteen to eleven-dimensional) setting where the 2×2 diagonal block matrices D_1, D_2 are themselves diagonal. It remains a question of some

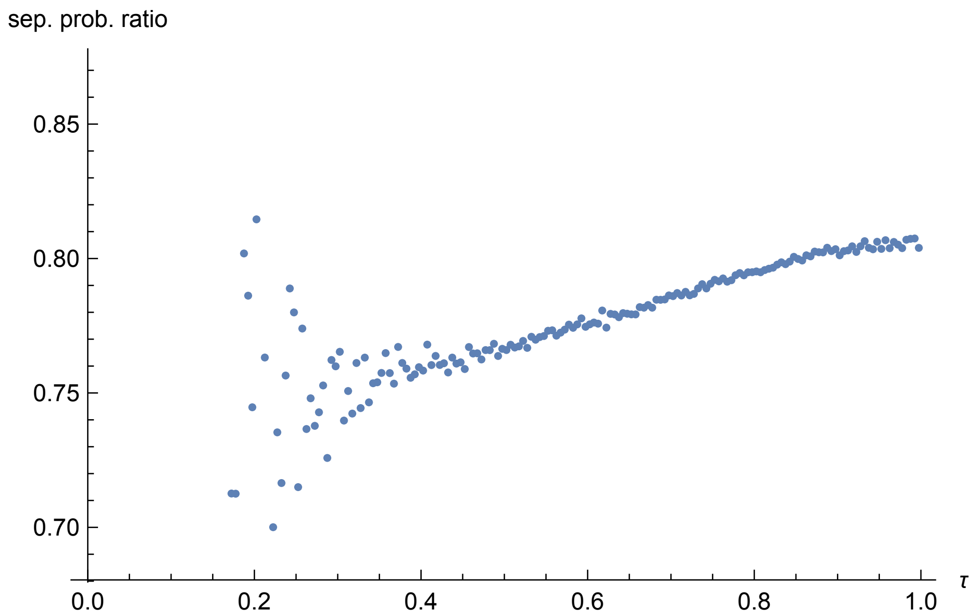

FIG. 22: The ratio of the square of the rebit-retrit separability probability to the qubit-qutrit separability probability as a function of $\tau = \tau_1 \tau_2 = \sqrt{\frac{d_{11} d_{66}}{d_{33} d_{44}}}$

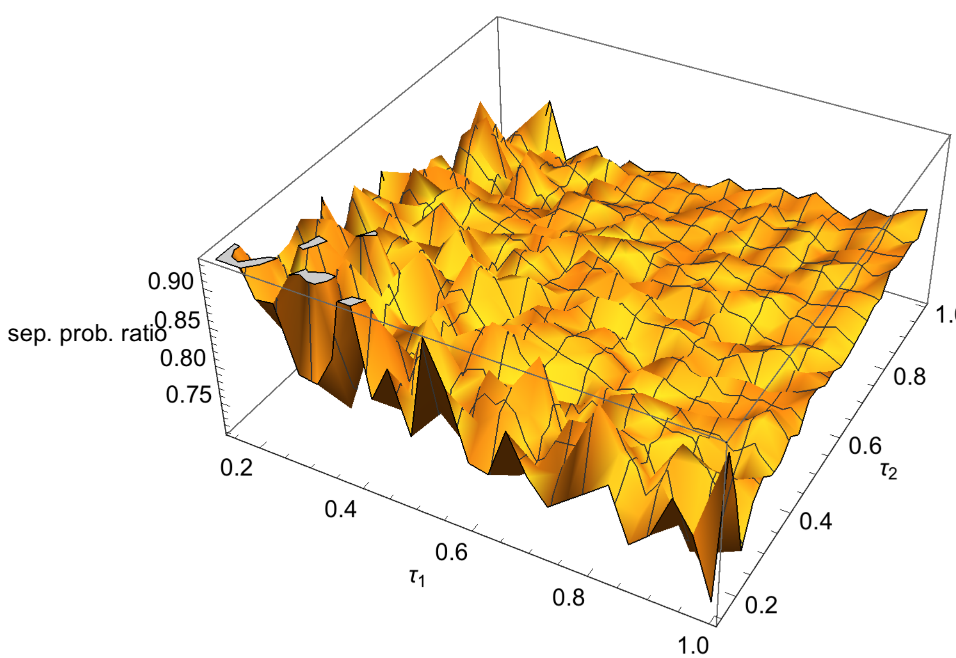

FIG. 23: The ratio of the square of the rebit-retrit separability probability to the qubit-qutrit separability probability as a function of $\tau_1 = \sqrt{\frac{d_{11} d_{55}}{d_{22} d_{44}}}$ and $\tau_2 = \sqrt{\frac{d_{22} d_{66}}{d_{33} d_{55}}}$

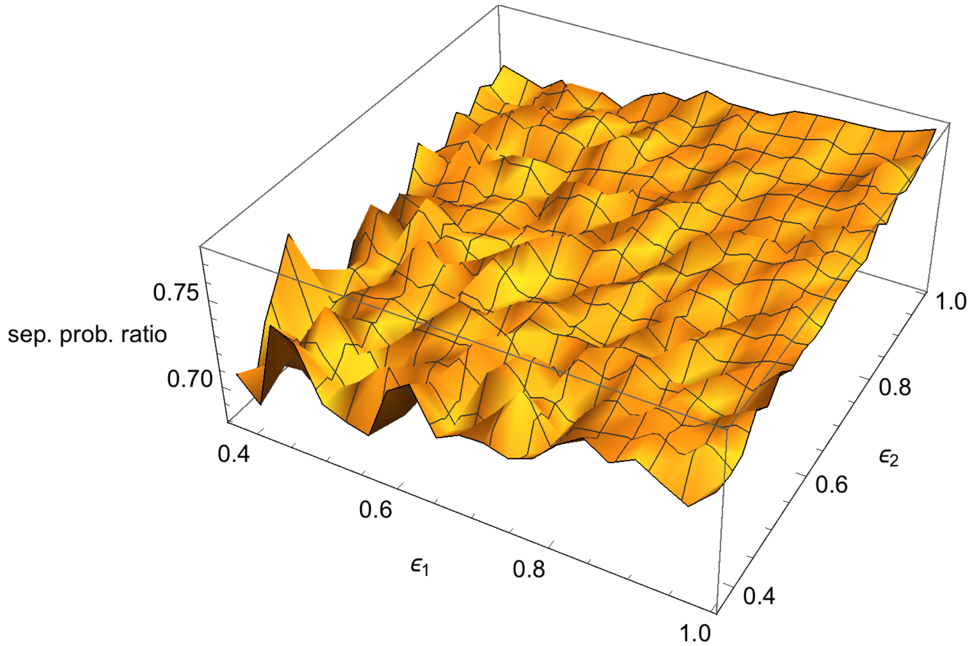

FIG. 24: The ratio of the square of the rebit-retrit separability probability to the qubit-qutrit separability probability as a function of the singular value ratios ε_1 and ε_2

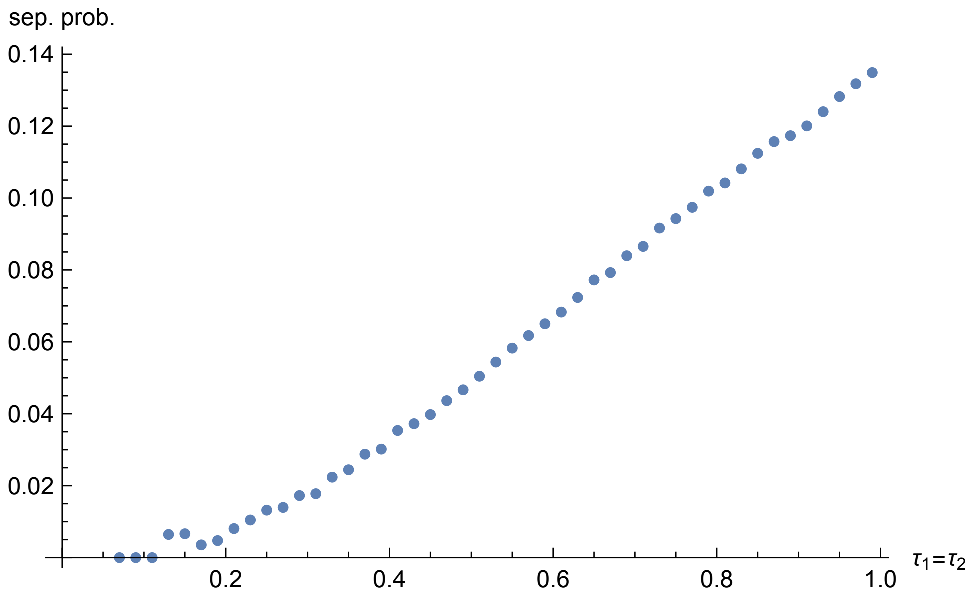

FIG. 25: Rebit-retrit separability probabilities for $\tau_1 = \tau_2$

interest as to what the Slater two-rebit and two-qubit separability functions themselves are in the full nine- and fifteen-dimensional settings, in particular, the possibility that the two-qubit separability function might be $\frac{6}{71}(3-\mu^2)\mu^2$ (cf. Figs. 10, 14). (Can the solutions in the Lovas-Andai setting be "lifted" to those in the Slater one [cf. eqs.(12)-(17)]?) Also, we

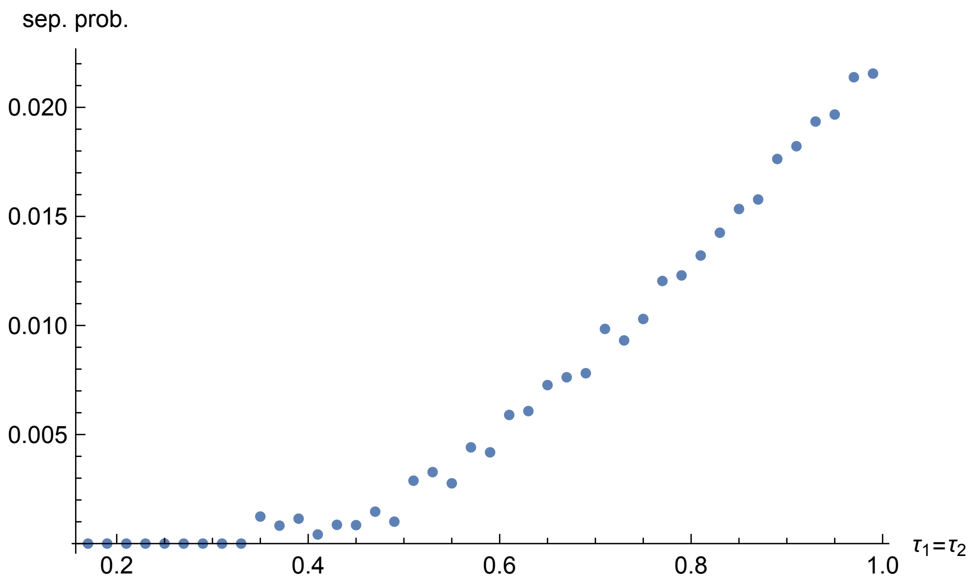

FIG. 26: Qubit-qutrit separability probabilities for $\tau_1 = \tau_2$

note that Lovas and Andai did not specifically consider D_1 and D_2 to be diagonal. So, if would be interesting to ascertain whether their same conclusions (such as the formula for $\tilde{\chi}_1(\varepsilon)$) could have been reached under such assumptions.

The counterpart rebit-retrit and qubit-qutrit 6×6 problems (sec. V) might also be productively studied when the 3×3 diagonal blocks are themselves diagonal. The problems under consideration would then be 14 and 23-dimensional in nature, as opposed to 20 and 35-dimensional, with lower-dimensional CAD's still.

A formidable challenge, to continue this line of research, is now to establish that the "two-quater[nionic]bit" Hilbert-Schmidt separability probability is $\frac{26}{323}$. This would move us, first, from the original 9-dimensional two-rebit and 15-dimensional two-qubit settings to a 27-dimensional one. But these dimensions can be reduced to 7-, 11- and 19-, using the apparently acceptable strategy–that has given us $\tilde{\chi}_1(\varepsilon)$ and $\tilde{\chi}_2(\varepsilon)$–of setting the two 2×2 diagonal blocks themselves to diagonal form. In turn, this leads to cylindrical algebraic decompositions with 4, 8 and 16 variables–with the last, quaternionic one, still seemingly computationally unfeasible. Noting that the formula (39) for $\tilde{\chi}_2(\varepsilon)$ is much simpler in form than that (2) for $\tilde{\chi}_1(\varepsilon)$, we might speculate as to the degree of complexity of the (yet unknown) quaternionic counterpart $\tilde{\chi}_4(\varepsilon)$. (Theorem 3 of [22] yields $\frac{\pi^{12}}{315071454005160652800000}$ for the [unnormalized] $\chi_4(\varepsilon)$, being as well as the volume of the standard unit ball in the normed vector space of 2×2 matrices with quaternionic entries, denoted by $\mathcal{B}_4(\mathbb{Q}^{2\times 2})$.) Our speculation/conjecture is that this degree will, as in the two-rebit case be relatively high, with the complex domain being privileged in these regards.

It should be pointed out that the manner of derivation of $\tilde{\chi_2}(\varepsilon)$ here is distinctly different from that employed by Lovas and Andai [1, App. A] in obtaining the form of $\tilde{\chi_1}(\varepsilon)$, though it has also been able to find this result here using the cylindrical algebraic decomposition approach (sec. IV A 1).

In their Conclusions, Lovas and Andai write: "The structure of the unit ball in operator norm of 2×2 matrices plays a critical role in separability probability of qubit-qubit and rebit-rebit quantum systems. It is quite surprising that the space of 2×2 real or complex matrices seems simple, but to compute the volume of the set

$$\left\{ \begin{pmatrix} a & b \\ c & e \end{pmatrix} \middle| \, a, b, c, e \in \mathbb{K}, \left\| \begin{pmatrix} a & b \\ c & e \end{pmatrix} \right\| < 1, \quad \left\| \begin{pmatrix} a & \varepsilon b \\ \frac{c}{\varepsilon} & e \end{pmatrix} \right\| < 1 \right\}$$

for a given parameter $\varepsilon \in [0, 1]$, which is the value of the function $\chi_d(\varepsilon)$, is a very challenging problem. The gist of our considerations is that the behavior of the function $\chi_d(\varepsilon)$ determines the separability probabilities with respect to the Hilbert-Schmidt measure." (The operator norm $\|\cdot\|$ is the largest singular value or Schatten-∞ norm.)

It appears that the cylindrical-algebraic-decomposition approach we have applied to 4×4 density matrices D with diagonal 2×2 diagonal blocks D_1, D_2 to obtain both $\tilde{\chi_1}(\varepsilon)$ and $\tilde{\chi_2}(\varepsilon)$ specifically answers this "very challenging" question, but in a manner quite different than Lovas and Andai applied in deriving $\tilde{\chi_1}(\varepsilon)$ [1, App. A].

VII. ADDENDUM

It certainly appears that the work of Lovas and Andai [1]–inspired by that of Milz and Strunz [18]–is highly innovative and successful in finding the two-rebit separability function $\tilde{\chi_1}(\varepsilon)$, and verifying the conjecture that the two-rebit Hilbert-Schmidt separability probability is $\frac{29}{64}$. However, in our study of the Lovas-Andai paper, we remain unconvinced by the chain of arguments on page 13 leading to the result $\frac{1}{4}$, and have posted a stack exchange question (https://mathematica.stackexchange.com/questions/144277/evaluate-a-pair-of-integrals-of-the-product-of-two-univariate-functions) in this regard.

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