A study on the time dependence of the equation-of-state parameter using Brans-Dicke theory of gravitation

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Abstract

The time dependence of the equation of state (EoS) parameter of the cosmic fluid, for a space of zero curvature, has been determined in the framework of the Brans-Dicke (BD) theory of gravity, using FRW metric. For this purpose, empirical expressions of the scale factor, scalar field and the dimensionless BD parameter have been used. The constant parameters involved in these expressions have been determined from the field equations. The dependence of the scalar field upon the scale factor and the dependence of the BD parameter upon the scalar field have been explored to determine the time dependence of the EoS parameter. Its rate of change with time has been found to depend upon a parameter that governs the time dependent behaviour of the scalar field. Time dependence of the EoS parameter has been graphically depicted.

Keywords: Brans-Dicke Theory, Time dependence of EoS parameter, Brans-Dicke parameter, Empirical scalar field, Empirical scale factor, Cosmology.

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1 Introduction

High precision astrophysical observations and their interpretations have established that the universe is expanding with acceleration [1]. This accelerated expansion is said to be caused and controlled by an entity, known as dark energy, whose true nature has not yet been determined. In theoretical calculations, this dark energy is often represented by the cosmological constant (Λ), found in General Relativity (GR). Although it accounts for the experimental observations quite well, but it has its own limitations [2]. A large number of alternative theoretical models have emerged to explain gravitational observations. The strengths as well as weaknesses of these models can be found in scientific literature [3]. Non-minimally coupled scalar field theories, particularly in the framework of Brans-Dicke (BD) theory, have been found to be highly useful in explaining the phenomenon of accelerated expansion [4]. The dimensionless parameter ω in BD theory plays a very important role in the prediction of observational results [5, 6]. In several models in BD theory, the accelerated expansion is found to be generated by a small value of ω, typically of the order of unity [4-6]. It has also been found that a Brans-Dicke scalar field alone can generate an accelerated expansion in the matter dominated era of the universe, without having to consider the presence of any quintessence matter or any interaction between the BD field and the dark matter [7]. A generalized version of BD theory by Bergman and Wagoner and a more useful form by Nordtvedt can predict this transition [8, 9, 10]. In this generalized theory, the BD parameter (ω) is regarded as a function of the scalar field (φ) and thus it becomes a function of time [10].

Cosmic microwave background radiation and observations on large scale structure indicate that the universe is highly homogeneous and isotropic on large scales [11, 12]. On the basis of observational results, the idea of an accelerating universe has emerged in the recent years [13]. To find the true nature of an exotic type of repulsive force, driving the accelerated expansion, which is said to be caused by an entity named dark energy (DE), is the aim of research in this regard. This dark energy is known to have a constant or a slightly changing energy density as the universe expands, but one does not have a clear knowledge regarding the true nature of DE [2]. The equation of state (EoS) parameter $\gamma = P/\rho$, which has conventionally characterised DE, should not be regarded as a constant. Observational results coming from SN Ia data establish that $-1.67 < \gamma < -0.62$ [14]. However, it is not at all essential to treat γas a constant. Owing to insufficient observational evidence to determine the variation of γ, the equation of state parameter has been considered in many studies to be a constant having values -1, 0, 1/3 and $+1$ for vacuum fluid, dust fluid, radiation and stiff fluid dominated universe, respectively [5]. But γ is, in general, a function of redshift or time [15]. In recent years, various models of time dependent γ have been proposed [16]. Recently Ray et al have studied variable EoS parameter for generalized dark energy model [17, 18].

In the present study, we have determined the time dependence of the equation of state parameter (γ) of the cosmic fluid, using the field equations of the Brans-Dicke theory (for flat space) and the wave equation for the scalar field (ϕ). For this formulation we have used empirical expressions of the scale factor, scalar field and the Brans-Dicke parameter (ω). Here we have a parameter (n) that controls the rate of change of the scalar field with time. This parameter is found to govern the behaviour of the EoS parameter (γ) as a function of time. Time variation of the EoS parameter has been shown graphically.

2 Theoretical Model

The field equations of generalized Brans-Dicke theory, obtained by using FRW metric, for a space of curvature k are given by,

$$3\frac{\dot{a}^2 + k}{a^2} + 3\frac{\dot{a}\dot{\phi}}{a\phi} - \frac{\omega\dot{\phi}^2}{2\phi^2} = \frac{\rho}{\phi} \tag{1}$$

$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2 + k}{a^2} + \frac{\omega\dot{\phi}^2}{2\phi^2} + 2\frac{\dot{a}\dot{\phi}}{a\phi} + \frac{\ddot{\phi}}{\phi} = -\frac{P}{\phi} \tag{2}$$

The wave equation for the scalar field (ϕ) is expressed as,

$$\ddot{\phi} + 3\frac{\dot{a}\dot{\phi}}{a} = \frac{\rho - 3P}{2\omega + 3} - \frac{\dot{\omega}\dot{\phi}}{2\omega + 3} \tag{3}$$

Combining equations (1), (2) and (3), one obtains,

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) = 0 \tag{4}$$

The equation of state of the cosmic fluid is $P = \gamma\rho$, where γ is the equation of state (EoS) parameter. Solution of equation (4), for a constant value of γ, is given by,

$$\rho = \rho_0 a^{-3(1+\gamma)} \tag{5}$$

Combining equations (2) and (3) and taking $k = 0$ (for flat space), one gets,

$$\dot{\omega}\dot{\phi} = 3\phi\left(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{\omega\dot{\phi}^2}{2\phi^2} + 2\frac{\dot{a}\dot{\phi}}{a\phi} + \frac{\ddot{\phi}}{\phi}\right) + \rho - (2\omega + 3)\,\phi\left(\frac{\ddot{\phi}}{\phi} + 3\frac{\dot{a}\dot{\phi}}{a\phi}\right) \tag{6}$$

Substituting equation (6) into (3) we have,

$$\gamma = -\frac{\phi}{\rho}\left[2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{\omega\dot{\phi}^2}{2\phi^2} + 2\frac{\dot{a}\dot{\phi}}{a\phi} + \frac{\ddot{\phi}}{\phi}\right] \tag{7}$$

Eliminating ρ from equations (1) and (7), and taking $k = 0$, one obtains,

$$\gamma = \frac{2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{\omega\dot{\phi}^2}{2\phi^2} + 2\frac{\dot{a}\dot{\phi}}{a\phi} + \frac{\ddot{\phi}}{\phi}}{3\left(\frac{\omega\dot{\phi}^2}{6\phi^2} - \frac{\dot{a}\dot{\phi}}{a\phi} - \frac{\dot{a}^2}{a^2}\right)} \tag{8}$$

Thus equation (8) is obtained as the general expression for the EoS parameter (γ) in BD theory. To determine its time dependence, following empirical expressions have been used.

$$a = a_0 Exp\left[\alpha\left(t^\beta - t_0^\beta\right)\right] \tag{9}$$

$$\phi = \phi_0\left(\frac{a}{a_0}\right)^n \tag{10}$$

$$\omega = \omega_0\left(\frac{\phi}{\phi_0}\right)^l \tag{11}$$

The scale factor (in eq. 9) has been chosen to ensure a change of sign of the deceleration parameter with time, as per many recent studies showing a transition of cosmic expansion from a phase of deceleration to acceleration [4]. Here $\alpha, \beta > 0$ to make the scale factor increasing with time in an expanding universe. Using equation (9), the Hubble parameter (H) and the deceleration parameter (q) are obtained as,

$$H = \alpha\beta t^{\beta-1} \tag{12}$$

$$q = -1 + \frac{1-\beta}{\alpha\beta}t^{-\beta} \tag{13}$$

For $0 < \beta < 1$ and $\alpha > 0$, we get $q > 0$ at $t = 0$ and, for $t \to \propto$, we have $q \to -1$.
Taking $H = H_0$ and $q = q_0$, at $t = t_0$, one gets,

$$\alpha = \frac{H_0}{1 - H_0 t_0 (1+q_0)} t_0^{\{H_0 t_0 (1+q_0)\}} \tag{14}$$

$$\beta = 1 - H_0 t_0 (1+q_0) \tag{15}$$

The scalar field (in equation 10) has been chosen on the basis of some studies on Brans-Dicke theory [10].
The empirical expression of BD parameter (in eq. 11) has been chosen according to the generalized Brans-Dicke theory, where ω is regarded as a function of the scalar field (ϕ) [4]. The values of ω_0 and l have to be determined from the field equations.
Using equation (10) along with the relation $G = 1/\phi$ one obtains,

$$n = -\frac{1}{H_0}\left(\frac{\dot{G}}{G}\right)_{t=t_0} \tag{16}$$

With the help of experimental observations regarding H_0 and $\left(\frac{\dot{G}}{G}\right)_{t=t_0}$, n can be determined from equation (16). Experimental observations regarding $\left(\frac{\dot{G}}{G}\right)_{t=t_0}$, obtained from many researchers, are found to be both positive and negative [19]. According to S. Weinberg, we have $\left|\left(\frac{\dot{G}}{G}\right)_{t=t_0}\right| \leq 4 \times 10^{-10} Yr^{-1}$[20]. Thus, from equation (16), one may express this requirement as,

$$|n| \leq \frac{4 \times 10^{-10} Yr^{-1}}{H_0} \tag{17}$$

From equation (17) one gets, $|n| \leq 5.44$ taking $H_0 = 7.348 \times 10^{-11} Yr^{-1}$
An expression of ω_0, determined from equation (1), taking $k = 0$, is given by,

$$\omega_0 = \frac{6}{n^2}\left\{1 + n - \frac{\rho_0}{3\phi_0 H_0^2}\right\} \tag{18}$$

Using equation (10) in (8) and writing all parameter values for $t = t_0$, one gets,

$$\gamma_0 = \frac{2\left(1 + n + n^2\right) + \omega_0 n^2 - 2q_0(n+2)}{\omega_0 n^2 - 6(n+1)} \tag{19}$$

The value of ω_0 in equation (19) has to be taken from equation (18).
Using equation (11) in (3) and writing all parameter values at $t = t_0$, one gets,

$$l = \frac{\rho_0\left(1 - 3\gamma_0\right)}{\phi_0 \omega_0 n^2 H_0^2} + \frac{\left(2\omega_0 + 3\right)\left(q_0 - n - 2\right)}{n\omega_0} \tag{20}$$

Substituting for γ_0 in equation (20) from equation (19), one gets,

$$l = \frac{\rho_0/\phi_0 + q_0 H_0^2\left(2n\omega_0 - 6\right) + 3H_0^2(1-n) - 0.5\omega_0 n^2 H_0^2 - 4n\omega_0 H_0^2}{\omega_0 n^2 H_0^2} \tag{21}$$

The value of ω_0 in equation (21) should be taken from equation (18).
Combining the equations (9), (10) and (11), the time dependence of the BD parameter is obtained as,

$$\omega = \omega_0 Exp\left[ln\alpha\left(t^\beta - t_0^\beta\right)\right] \tag{22}$$

The values of α, β, ω_0 in equation (22) can be obtained from the equations of (14), (15) and (18) respectively. Time dependence of γ can be studied from the following equation (eqn. 23) which is obtained by using equations (10) and (11) in (8).

$$\gamma = \frac{\omega n^2 + 2\left(1 + n + n^2\right) - 2q(n+2)}{\omega n^2 - 6(n+1)} \tag{23}$$

The values of q, n, ω in equation (23) can be obtained from equations (13), (16) and (22) respectively.
The values of different cosmological parameters used for the present study are given below.

$H_0 = 72 Km/s/Mpc = 2.33 \times 10^{-18} sec^{-1}$, $q_0 = -0.55$, $\rho_0 = 2.83 \times 10^{-27} Kgm^{-3}$, $\phi_0 = 1/G_0 = 1.498 \times 10^{10} Kg^2 m^{-2} N^{-1}$, $t_0 = 4.36 \times 10^{17} sec$

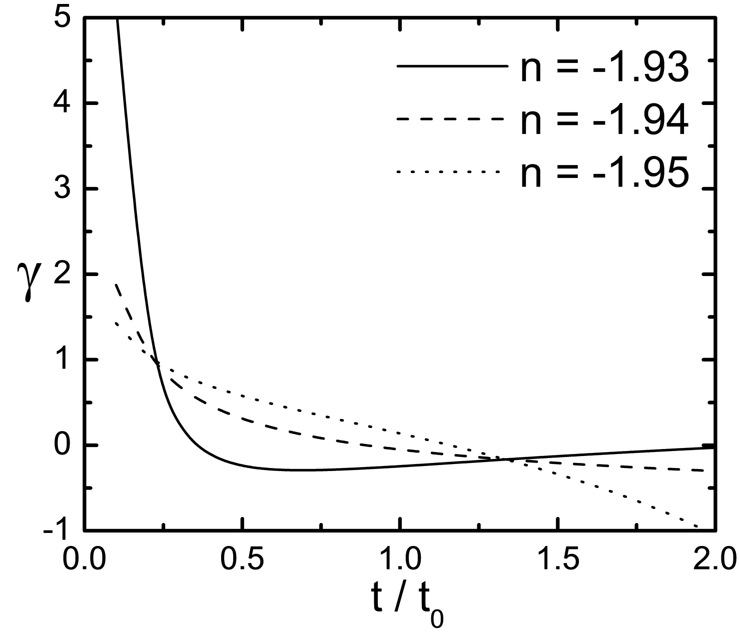

Figure 1: Plots of γ vs. time with γ_0 values close to zero.

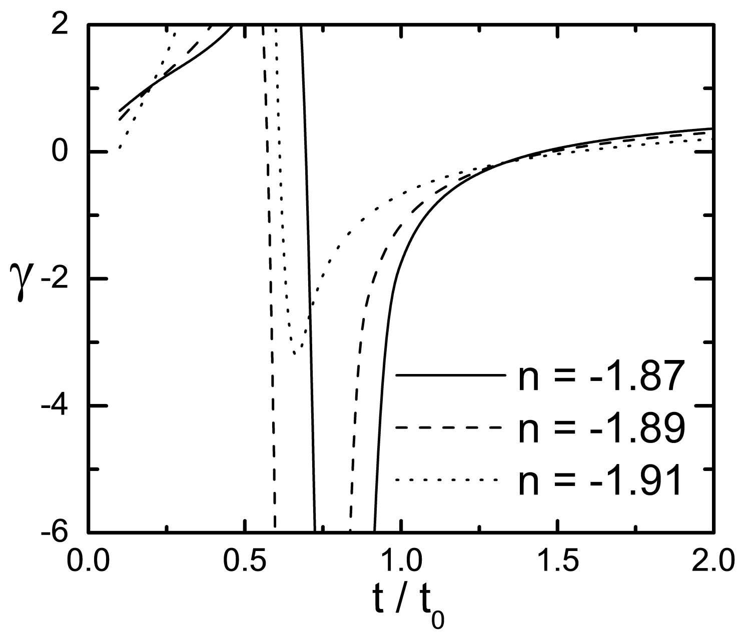

Figure 2: Plots of γ vs. time with γ_0 values close to -1.

3 Results

Figure 1 shows three plots of γ vs. time, for the values of n for which γ_0 is close to zero. For the largest value of n here, γ initially decreases very fast upto the time of $t = 0.5t_0$ and increases at a very slow rate thereafter. It has a positive slope at the present epoch ($t = t_0$). For the more negative values of n, we find a negative slope for γ everywhere. The time dependence of γ, in this figure, is similar to that shown in figure 2 of an article by A Pradhan, based on Einstein's theory of gravity [18]. Figure 2 shows three plots of γ vs. time, for the values of n for which γ_0 is close to -1. After a very sharp initial fall, each curve rises, at a gradually smaller rate, to a value close to zero. This behaviour is similar to that shown in figure 6 of Pradhan's study on dark energy models with anisotropic fluid [19]. The time dependence of γ in figure 2 is also similar to that obtained in a study of dark energy models by S Ray [17]. It is found from equation (19) that the values of γ_0, for positive values of n, are large negative numbers, contrary to observations according to which γ_0 should have a small negative value close to -1 [17]. So, one must choose only those negative values of n for which γ_0 is negative. For $n = -1.9426735$, γ_0 is very close to zero, which was generally used as the value of γ for the matter dominated universe [17]. For this value of n, γ decreases with time at a gradually decreasing rate.

4 Conclutions

The present study shows that the parameter n determines the behavious of γ as a funtion of time, since the parameters ω_0, γ_0 and l are all explicitly dependent upon the parameter n. The parameter n, by its definition in equation 10, determines the rate at which the scalar field changes with time. Thus, one is likely to draw a conclusion that any change of the scalar field ($\phi \equiv 1/G$) with time is strongly connected to a change in the value of the equation of state parameter (γ) of the cosmic fluid during that time. According to a study by Banerjee and Pavon, the range of variation of ω_0 is $-3/2 < \omega_0 < 0$ [7]. It is found from equation (18) that ω_0 is always positive for positive values of n. Therefore, only the negative values of n can be expected to predict the correct behaviour of cosmic expansion. It is found from equation (19) that $\gamma_0 < 0$ for $n \geq -1.9426735$. Here γ_0 is found to be zero for a value of n between -1.9426736 and -1.9426735. From equation (19) we find that, $\gamma_0 \leq -1$ for $n \geq -1.8935$ and $\gamma_0 > -1$ for $n < -1.8935$. Observational results, coming from SN Ia data, establish that the range of variation of γ is $-1.67 < \gamma < -0.62$ [14]. The values of γ_0, shown in figure 2, correspond to this range and these have been obtained here by taking n close to -1.89. Equation (23) shows that, for $n = -1.8935$, the universe makes a transition from a phase of $\gamma < -1$ (phantom fluid dominated universe) to a phase of $\gamma > -1$ (quintessence), passing through a stage of $\gamma = -1$ (vacuum fluid dominated universe) which is also evident from the curves of figure 2. Here γ, as a function of time, gradually approaches a stauration value, at an increasingly smaller rate. An improvement over this theoretical model can be achieved if one uses a scale factor which has been obtained as a solution of the field equations after incorporating the empirical scaler field into them. For this study, one may also choose a different empirical expression for the scale factor satisfying the condition that the deceleration parameter (q) shows a signature flip. One may work with empirical expressions of the scalar field (ϕ) and the BD parameter (ω), which are different from those used in the present study (equations 10, 11). To obtain more tunable parameters which can be varied to predict better values of γ_0, consistent with astrophysical observations, one may use weight factors while combining the equations (2) and (3) to get equation (6). One may have different weightages of the equations (2) and (3) in the study by taking different combinations of these weight factors. The importance of the present study lies in the fact that, unlike other recent studies on variable EoS parameter, the time dependence of EoS parameter has been determined from Brans-Dicke field equations without considering or incorporating any role of dark energy in governing its behaviour and using the simple FRW space-time, assuming isotropy and homogeneity of space.

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