Some Aggregation Operators For Bipolar-Valued Hesitant Fuzzy Information

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***Abstract*—In this article we define some aggregation operators for bipolar-valued hesitant fuzzy sets. These operations include bipolar-valued hesitant fuzzy ordered weighted averaging (BPVHFOWA) operator, bipolar-valued hesitant fuzzy ordered weighted geometric (BPVHFOWG) operator and their generalized forms. We also define hybrid aggregation operators and their generalized forms and solved a decision-making problem on these operation.**

Keywords—Bipolar-valued hesitant fuzzy sets (BPVHFSs), bipolar-valued hesitant fuzzy elements (BPVHFEs), BPVHFOWA operator, BPVHFOWG operator, BPVHFHA operator, BPVHFHG operator, score function and decision making (DM).

I. INTRODUCTION

At any level of our life decision making plays an essential role. It is a very famous research field now days. Everyone needs to take decision about the selection of best choice at any stage of his life. Using ordinary mathematical techniques, we are not able to solve DM problems

To deal with problems related to different kind of uncertainties, L. A. Zadeh [37] in 1965 initiated the concept of fuzzy sets (FSs). After his idea of FSs, researchers started to think about different extensions of FSs and some advanced forms of FSs have been established. Some of these extensions are interval-valued fuzzy set (IVFS) [5], intuitionistic fuzzy set (IFS) [1], hesitant fuzzy set (HFS) [22] and bipolar-valued fuzzy set (BVFS) [15] are some well known sets. Later on these new extensions of FSs have been extensively used in decision making [5], [16].

As different advanced forms of FSs came one after another, scientist started to merge two kinds of fuzzy information in a single set. The idea was quite useful and some very interesting extensions of FSs have been defined. These extensions include intuitionistic hesitant fuzzy sets (IHFSs), inter-valued hesitant fuzzy sets (IVHFSs) and bipolar-valued hesitant fuzzy sets (BVHFSs). The idea of merging different kind of fuzzy sets was quite useful and very shortly some new advanced forms of FSs have been established which are inter-valued intuitionistic hesitant fuzzy sets (IVIHFSs), cubic hesitant fuzzy sets (CHFSs) and bipolar-valued hesitant fuzzy sets (BPVHFSs).

Tahir M [25] introduced BPVHFSs, a new extension of FSs and a combination of HFSs and BVFSs. BPVHFSs have affiliation functions (membership function) in terms of set of some values. The positive affiliation function is a set having values in the interval [0,1] which conveys the satisfaction extent of an element belong to the given set. While the negative affiliation function is a set of some values in [-1,0] which conveys the negative or counter satisfaction degree of an element belong to given set. Tahir M [25] defines some basic operations for BPVHFSs and proved some interesting results. He also defines aggregation operators for BPVHFSs and then used these operators in DM.

In our article, we apply some order on previously defined bipolar-valued hesitant fuzzy weighted averaging and weighted geometric operators by defining bipolar-valued hesitant fuzzy ordered weighted averaging and bipolar-valued hesitant fuzzy ordered weighted geometric operators along with their generalized operators. We also defined some hybrid aggregation operators on BPVHFSs along with their generalized forms. Finally, we did solve a DM problem using these newly defined aggregation operators and get very useful results.

This article consists of 4 sections with section one as introduction. In section two we recall the definition of BPVHFSs, their properties and some aggregation operators of BPVHFSs. Section three contain BPVHFOWA operators, BPVHFOWG operators, BPVHFHA operators and BPVHFHG operators. We also solve some examples on these defined operations. In the last section, we solve a DM problem using the defined operations in section three. Finally, we finish our article by adding a conclusion to it.

II. PRELIMINARIES

This section consists of the definition of BPVHFS and some aggregation operators on BPVHFSs. We also add some properties of BPVHFSs to this section and we recall the concept of score function for BPVHFSs.

A. *Definition 1: [25]*

For any set $\mathfrak{X}$, the BPVHFS $\mathfrak{B}$ on some domain of $\mathfrak{X}$ is denoted and defined by:

$$\mathfrak{B} = \{(\kappa, (\text{Ҍ}^+(\kappa), \text{Ҍ}^-(\kappa))) : \kappa \in \mathfrak{X}\}$$

where $\text{Ҍ}^+ : \mathfrak{X} \to [0,1]$ is a finite set of few distinct values in the interval [0, 1]. It conveys the satisfaction extent of "$\varkappa$" corresponding to BPVHFS $\mathfrak{B}$ and $\text{Ҍ}^- : \mathfrak{X} \to [-1,0]$ is a finite set of few distinct values in the interval [-1, 0]. It conveys the implicit counter or negative property of "$\varkappa$" corresponding to BPVHFS $\mathfrak{B}$.

Here $\text{Ҍ} = \{\text{Ҍ}^+(\kappa), \text{Ҍ}^-(\kappa)\}$ is a BPVHFE. The set of all BPVHFEs is denoted by Φ.

Consider two BPVHFSs:

$$\mathfrak{A} = \{(\kappa, (\text{Ҍ}^+{}_{\mathfrak{A}}(\kappa), (\text{Ҍ}^-{}_{\mathfrak{A}}(\kappa))) : \kappa \in \mathfrak{X}\}$$

$$\mathfrak{B} = \{(\kappa, (\text{Ҍ}^+{}_{\mathfrak{B}}(\kappa), (\text{Ҍ}^-{}_{\mathfrak{B}}(\kappa))) : \kappa \in \mathfrak{X}\}$$

The set operations for BPVHFSs are defined as:

$\mathfrak{A} \cup \mathfrak{B} = \{ \text{ҕ} \in \text{Ҍ}_{\mathfrak{A}}(\kappa) \cup \text{Ҍ}_{\mathfrak{B}}(\kappa) : (\text{Ҍ}^+{}_{\mathfrak{A}} \cup \text{Ҍ}^+{}_{\mathfrak{B}})(\kappa), (\text{Ҍ}^-{}_{\mathfrak{A}} \cup \text{Ҍ}^-{}_{\mathfrak{B}})(\kappa)\}$

$\mathfrak{A} \cap \mathfrak{B} = \{ \text{ҕ} \in \text{Ҍ}_{\mathfrak{A}}(\kappa) \cup \text{Ҍ}_{\mathfrak{B}}(\kappa) : (\text{Ҍ}^+{}_{\mathfrak{A}} \cap \text{Ҍ}^+{}_{\mathfrak{B}})(\kappa), (\text{Ҍ}^-{}_{\mathfrak{A}} \cap \text{Ҍ}^-{}_{\mathfrak{B}})(\kappa)\}$

$$(\mathfrak{A})^c = \{(\kappa, (\text{Ҍ}^+{}_{\mathfrak{A}}(\kappa))^c, (\text{Ҍ}^-{}_{\mathfrak{A}}(\kappa))^c) : \kappa \in \mathfrak{X}\}.$$

$$(\text{Ҍ}^+{}_{\mathfrak{A}}(\kappa))^c = \{1 - \text{ҕ} : \text{ҕ} \in \text{Ҍ}^+{}_{\mathfrak{A}}(\kappa)\},$$

$$(\text{Ҍ}^-{}_{\mathfrak{A}}(\kappa))^c = \{-1 - \text{ҕ} : \text{ҕ} \in \text{Ҍ}^-{}_{\mathfrak{A}}(\kappa)\}.$$

$(\mathfrak{A} \oplus \mathfrak{B})(\kappa) = \{\text{ҕ}_1 + \text{ҕ}_2 - \text{ҕ}_1\text{ҕ}_2 : \text{ҕ}_1 \in \text{Ҍ}^+{}_{\mathfrak{A}}(\kappa), \text{ҕ}_2 \in \text{Ҍ}^+{}_{\mathfrak{B}}(\kappa), -(\text{ҕ}_1\text{ҕ}_2) : \text{ҕ}_1 \in \text{Ҍ}^-{}_{\mathfrak{A}}(\kappa), \text{ҕ}_2 \in \text{Ҍ}^-{}_{\mathfrak{B}}(\kappa)\}$

$(\mathfrak{A} \otimes \mathfrak{B})(\kappa) = \{\text{ҕ}_1\text{ҕ}_2 : \text{ҕ}_1 \in \text{Ҍ}^+{}_{\mathfrak{A}}(\kappa), \text{ҕ}_2 \in \text{Ҍ}^+{}_{\mathfrak{B}}(\kappa), -(-\text{ҕ}_1 - \text{ҕ}_2 - \text{ҕ}_1\text{ҕ}_2) : \text{ҕ}_1 \in \text{Ҍ}^-{}_{\mathfrak{A}}(\kappa), \text{ҕ}_2 \in \text{Ҍ}^-{}_{\mathfrak{B}}(\kappa)\}$

for any $\rho^{\cdot} > 0$

$\rho^{\cdot}\mathfrak{A}(\kappa) = \{1 - (1 - \text{ҕ})^{\rho^{\cdot}} : \text{ҕ} \in \text{Ҍ}^+{}_{\mathfrak{A}}(\kappa), -(-\text{ҕ}^{\rho^{\cdot}}) : \text{ҕ} \in \text{Ҍ}^-{}_{\mathfrak{A}}(\kappa)\}$

$\mathfrak{A}^{\rho^{\cdot}}(\kappa) = \{\text{ҕ}^{\rho^{\cdot}} : \text{ҕ} \in \text{Ҍ}^+{}_{\mathfrak{A}}(\kappa), -1 - \left(-(-(-1 - \text{ҕ}))^{\rho^{\cdot}}\right) : \text{ҕ} \in \text{Ҍ}^-{}_{\mathfrak{A}}(\kappa)\}$

B. *Definition 2: [25]*

Let $\text{Ҍ}_i\,(i = 1,2,3,4 \dots n)$ be a set of BPVHFEs and let $\omega^{\cdot} = (\omega^{\cdot}_1, \omega^{\cdot}_2, \omega^{\cdot}_3, \omega^{\cdot}_4, \dots \omega^{\cdot}_n)^T$ be the WV of $\text{Ҍ}_i\,(i = 1,2,3,4 \dots n)$ with $\omega^{\cdot}_i \in [0,1]$ and $\sum_{i=1}^{n} \omega^{\cdot}_i = 1$, then

1. BPVHFWA Operator is a function $\Psi^n \to \Psi$ such that

$$BPVHFWA(\text{Ҍ}_1, \text{Ҍ}_2, \dots \text{Ҍ}_n) = \bigoplus_{i=1}^{n} (\omega^{\cdot}_i \text{Ҍ}_i)$$

$$= \left\{\left\{1 - \prod_{i=1}^{n} (1 - \text{ҕ}_i)^{\omega^{\cdot}_i} : \text{ҕ}_i \in \text{Ҍ}_i^+\right\}, \left\{-\prod_{i=1}^{n} (-\text{ҕ}_i)^{\omega^{\cdot}_i} : \text{ҕ}_i \in \text{Ҍ}_i^-\right\}\right\}$$

2. BPVHFWG Operator is a function $\Psi^n \to \Psi$ such that

$$BPVHFWG(\text{Ҍ}_1, \text{Ҍ}_2, \dots \text{Ҍ}_n) = \bigotimes_{i=1}^{n} (\text{Ҍ}_i)^{\omega^{\cdot}_i}$$

$$= \left\{\left\{\prod_{i=1}^{n} (\text{ҕ}_i)^{\omega^{\cdot}_i} : \text{ҕ}_i \in \text{Ҍ}_i^+\right\}, \left\{-1 - \prod_{i=1}^{n} \left(-\left(-(-1 - \text{ҕ}_i)\right)^{\omega^{\cdot}_i}\right) : \text{ҕ}_i \in \text{Ҍ}_i^-\right\}\right\}$$

3. A GBPVHFWA Operator is a function $\Psi^n \to \Psi$ such that

$$GBPVHFWA_{\rho^{\cdot}}(\text{Ҍ}_1, \text{Ҍ}_2, \dots \text{Ҍ}_n) = \left(\bigoplus_{i=1}^{n} \left(\omega^{\cdot}_i \text{Ҍ}_i^{\rho^{\cdot}}\right)\right)^{\frac{1}{\rho^{\cdot}}}$$

$$= \left\{\left\{\left(1 - \prod_{i=1}^{n} (1 - \text{ҕ}_i^{\rho^{\cdot}})^{\omega^{\cdot}_i}\right)^{\frac{1}{\rho^{\cdot}}} : \text{ҕ}_i \in \text{Ҍ}_i^+\right\}, \left\{-1 - \left(\prod_{i=1}^{n} -\left(-\left(-1\left(-\left((-\text{ҕ}_i)^{\rho^{\cdot}}\right)^{\omega^{\cdot}_i}\right)\right)\right)\right)^{\frac{1}{\rho^{\cdot}}} : \text{ҕ}_i \in \text{Ҍ}_i^-\right\}\right\}$$

$$with\ \rho^{\cdot} > 0$$

4. A GBPVHFWG Operator is a function $\Psi^n \to \Psi$ such that

$$GBPVHFWG_{\rho^{\cdot}}(\text{Ҍ}_1, \text{Ҍ}_2, \dots \text{Ҍ}_n) = \frac{1}{\rho^{\cdot}}\left(\bigoplus_{i=1}^{n} (\rho^{\cdot}\text{Ҍ}_i)^{\omega^{\cdot}_i}\right)$$

$$= \left\{\left\{1 - \left(1 - \prod_{i=1}^{n} \left(1 - (1 - \text{ҕ}_i)^{\rho^{\cdot}}\right)^{\omega^{\cdot}_i}\right)^{\frac{1}{\rho^{\cdot}}} : \text{ҕ}_i \in \text{Ҍ}_i^+\right\}, \left\{-1 - \left(\prod_{i=1}^{n} -\left(-\left(-1 - \left(-(-\text{ҕ}_i)^{\rho^{\cdot}}\right)\right)\right)^{\omega^{\cdot}_i}\right)^{\frac{1}{\rho^{\cdot}}} : \text{ҕ}_i \in \text{Ҍ}_i^-\right\}\right\}$$

$$with\ \rho^{\cdot} > 0$$

C. *Definition 3: [25]*

Let $\text{Ҍ} = < \text{Ҍ}^+, \text{Ҍ}^- >$ be a BPVHFE, then the Score function (Accuracy Function) of Ҍ is denoted and defined by:

$$\mathcal{S}(\text{Ҍ}) = \frac{1}{\ell_{\text{Ҍ}}}(\xi_{\text{Ҍ}}^+ + \xi_{\text{Ҍ}}^-)$$

where $\xi_{\text{Ҍ}}^+$ is the sum of elements of Ҍ^+ and $\xi_{\text{Ҍ}}^-$ is the sum of elements of Ҍ^-, $\ell_{\text{Ҍ}}$ is the length of Ҍ and $\mathcal{S}(\text{Ҍ}) \in [-1, 1]$.

D. *Remark 1: [25]*

Length of Ҍ^+ and Ҍ^- are not necessarily equal.

For two BPVHFEs Ҍ_1 and Ҍ_2, if

$$\mathcal{S}(\text{Ҍ}_1) < \mathcal{S}(\text{Ҍ}_2),$$

then Ҍ_1 is said to be minor than Ҍ_2 i.e. $\text{Ҍ}_1 < \text{Ҍ}_2$.

$$\mathcal{S}(\text{Ҍ}_1) > \mathcal{S}(\text{Ҍ}_2),$$

then Ҍ_1 is said to bigger than Ҍ_2 i.e. $\text{Ҍ}_1 > \text{Ҍ}_2$.

$$\mathcal{S}(\text{Ҍ}_1) = \mathcal{S}(\text{Ҍ}_2),$$

then Ҍ_1 is indifferent (similar) to Ҍ_2 denoted by $\text{Ҍ}_1 \sim \text{Ҍ}_2$.

III. Ordered Weighted And Hybrid Operators For Bpvhfss

In this section, we define BPVHFOWA operators, BPVHFOWG operators, BPVHFHA operators and BPVHFHG operators. We also explain these operations with the help of examples.

Definition 4:

Let $\text{Ƀ}_i (i = 1,2,3,4 \dots n)$ be a set of BPVHFEs and $\text{Ƀ}_{\sigma(i)}$ the i^{th} largest among them. Let $\omega' = (\omega'_1, \omega'_2, \omega'_3, \omega'_4, \dots \omega'_n)^T$ be the aggregation associated weight vector of $\text{Ƀ}_i (i = 1,2,3,4 \dots n)$ with $\omega'_i \in [0,1]$ and $\sum_{i=1}^{n} \omega'_i = 1$. Then

1. A BPVHFOWA operator is a function $BPVHFOWA: \Psi^n \to \Psi$, such that

$$BPVHFOWA(\text{Ƀ}_1, \text{Ƀ}_2, \dots \text{Ƀ}_n) = \bigoplus_{i=1}^{n} (\omega'_i \text{Ƀ}_{\sigma(i)})$$

2. A BPVHFOWG operator is a function $BPVHFOWG: \Psi^n \to \Psi$, such that

$$BPVHFWG(\text{Ƀ}_1, \text{Ƀ}_2, \dots \text{Ƀ}_n) = \bigotimes_{i=1}^{n} (\text{Ƀ}_{\sigma(i)})^{\omega'_i}$$

1.1.1. Theorem

Let $\text{Ƀ}_i (i = 1,2,3,4 \dots n)$ be a set of BPVHFEs. Then their aggregated value determined by using BPVHFOWA operator or BPVHFOWG operator is a BPVHFE and

$$BPVHFOWA(\text{Ƀ}_1, \text{Ƀ}_2, \dots \text{Ƀ}_n)$$

$$= \left\{ \left\{ 1 - \prod_{i=1}^{n} (1 - \tau_{\sigma(i)})^{\omega'_i} : \tau_{\sigma(i)} \in \text{Ƀ}^{+}_{\sigma(i)} \right\}, \left\{ - \prod_{i=1}^{n} (-\tau_{\sigma(i)})^{\omega'_i} : \tau_{\sigma(i)} \in \text{Ƀ}^{-}_{\sigma(i)} \right\} \right\}$$

$$BPVHFWG(\text{Ƀ}_1, \text{Ƀ}_2, \dots \text{Ƀ}_n) = \left\{ \left\{ \prod_{i=1}^{n} (\tau_{\sigma(i)})^{\omega'_i} : \tau_{\sigma(i)} \in \text{Ƀ}^{+}_{\sigma(i)} \right\}, \left\{ -1 - \left(- \prod_{i=1}^{n} \left(-(-1 - \tau_{\sigma(i)}) \right)^{\omega'_i} \right) : \tau_{\sigma(i)} \in \text{Ƀ}^{-}_{\sigma(i)} \right\} \right\}$$

1.1.2. Definition

Let $\text{Ƀ}_i (i = 1,2,3,4 \dots n)$ be a set of BPVHFEs and $\text{Ƀ}_{\sigma(i)}$ the i^{th} largest among them.

Let $\omega' = (\omega'_1, \omega'_2, \omega'_3, \omega'_4, \dots \omega'_n)^T$ be the aggregation associated weight vector of $\text{Ƀ}_i (i = 1,2,3,4 \dots n)$ with $\omega'_i \in [0,1]$ and $\sum_{i=1}^{n} \omega'_i = 1$. Then

1. A GBPVHFOWA operator is a function $GBPVHFOWA: \Psi^n \to \Psi$, such that

$$GBPVHFOWA_{\rho'}(\text{Ƀ}_1, \text{Ƀ}_2, \dots \text{Ƀ}_n) = \left(\bigoplus_{i=1}^{n} \left(\omega'_i \text{Ƀ}^{\rho'}_{\sigma(i)} \right) \right)^{\frac{1}{\rho'}}$$

$$with\ \rho' > 0$$

$$= \left\{ \left\{ \left(1 - \prod_{i=1}^{n} \left(1 - \tau^{\rho'}_{\sigma(i)}\right)^{\omega'_i} \right)^{\frac{1}{\rho'}} : \tau_{\sigma(i)} \in \text{Ƀ}^{+}_{\sigma(i)} \right\}, \right.$$

$$\left. \left\{ -1 - \left(- \left(\prod_{i=1}^{n} \left(- \left(-1 - \left(- \left((-\tau_{\sigma(i)})^{\rho'} \right)^{\omega'_i} \right) \right) \right) \right)^{\frac{1}{\rho'}} \right) : \tau_{\sigma(i)} \in \text{Ƀ}^{-}_{\sigma(i)} \right\} \right\}$$

2. A GBPVHFOWG operator is a function $GBPVHFOWG: \Psi^n \to \Psi$, such that

$$GBPVHFOWG_{\rho'}(\text{Ƀ}_1, \text{Ƀ}_2, \dots \text{Ƀ}_n) = \frac{1}{\rho'} \left(\bigoplus_{i=1}^{n} \left(\rho' \text{Ƀ}_{\sigma(i)} \right)^{\omega'_i} \right)$$

$$with\ \rho' > 0$$

$$= \left\{ \left\{ 1 - \left(1 - \prod_{i=1}^{n} \left(1 - (1 - \tau_{\sigma(i)})^{\rho'} \right)^{\omega'_i} \right)^{\frac{1}{\rho'}} : \tau_{\sigma(i)} \in \text{Ƀ}^{+}_{i} \right\}, \right.$$

$$\left. \left\{ -1 - \left(- \prod_{i=1}^{n} \left(\left(- \left(-1 - \left(- (-\tau_{\sigma(i)})^{\rho} \right) \right) \right)^{\omega_i} \right)^{\frac{1}{\rho'}} \right) : \tau_{\sigma(i)} \in \text{Ƀ}^{-}_{\sigma(i)} \right\} \right\}$$

Example 1:

Let $\text{Ƀ}_1 = \{\{0.1, 0.2\}, \{-0.3, -0.2\}\}$,
$\text{Ƀ}_2 = \{\{0.5, 0.6\}, \{-0.2, -0.1\}\}$
and $\text{Ƀ}_3 = \{\{0.9, 0.8\}, \{-0.2, -0.1\}\}$ be three BPVHFEs and let $\omega = (0.3, 0.5, 0.2)^{\frac{1}{T}}$ be the aggregation- associated weight vector. Then

$$S(\text{Ƀ}_1) = \frac{1}{\ell_{\text{Ƀ}_1}} (\xi^{+}_{\text{Ƀ}_1} + \xi^{-}_{\text{Ƀ}_1}) = \frac{1}{2} (0.1 + 0.2 + (-0.3) + (-0.2)) = -0.1$$

$$S(\text{Ƀ}_2) = \frac{1}{\ell_{\text{Ƀ}_2}} (\xi^{+}_{\text{Ƀ}_2} + \xi^{-}_{\text{Ƀ}_2}) = \frac{1}{2} (0.5 + 0.6 + (-0.2) + (-0.1)) = 0.8$$

$$S(\text{Ƀ}_3) = \frac{1}{\ell_{\text{Ƀ}_3}} (\xi^{+}_{\text{Ƀ}_3} + \xi^{-}_{\text{Ƀ}_3}) = \frac{1}{2} (0.9 + 0.8 + (-0.2) + (-0.1)) = 0.7$$

Clearly as

$S(\text{Ƀ}_1) < S(\text{Ƀ}_3) < S(\text{Ƀ}_2)$

So

$\text{Ƀ}_{\sigma(1)} = \text{Ƀ}_2 = \{\{0.5, 0.6\}, \{-0.2, -0.1\}\}$,
$\text{Ƀ}_{\sigma(2)} = \text{Ƀ}_3 = \{\{0.9, 0.8\}, \{-0.2, -0.1\}\}$
and $\text{Ƀ}_{\sigma(3)} = \text{Ƀ}_1 = \{\{0.1, 0.2\}, \{-0.3, -0.2\}\}$

Now

$$GBPVHFOWA_1(\text{Ƀ}_1, \text{Ƀ}_2, \text{Ƀ}_3) = \bigoplus_{i=1}^{3} (\omega'_i \text{Ƀ}_{\sigma(i)})$$

$$= \bigcup_{\tau_1 \in \text{Ƀ}_1, \tau_2 \in \text{Ƀ}_2, \tau_3 \in \text{Ƀ}_3} \left\{ \{1 - (1 - \tau_2)^{0.3} (1 - \tau_3)^{0.5} (1 - \tau_1)^{0.2}\}, \left\{ - \left((-(\tau_2))^{0.3} (-(\tau_3))^{0.5} (-(\tau_1))^{0.2} \right) \right\} \right\}$$

$$= \{\{0.748499, 0.754354, 0.644324, 0.652605, 0.764784, 0.77026, 0.667355, 0.675099\} \{-0.21689, -0.2, -0.15337, -0.14142, -0.17617, -0.16245, -0.12457, -0.11487\}\}$$

$$GBPVHFOWA_2(\text{Ƀ}_1, \text{Ƀ}_2, \text{Ƀ}_3) = \left(\bigoplus_{i=1}^{3} \left(\omega'_i \text{Ƀ}^{2}_{\sigma(i)} \right) \right)^{\frac{1}{2}}$$

$$=\bigcup_{\hbar_1\in \mathcal{H}_1,\hbar_2\in \mathcal{H}_2,\hbar_3\in \mathcal{H}_3}\left\{\left\{1-\left((1-(\hbar_2)^2)^{0.3}(1-(\hbar_3)^2)^{0.5}(1-(\hbar_1)^2)^{0.2}\right)^{\frac{1}{2}}\right\},\left\{-1-\left(-\left(\left(-\left(-1-\left(-\left(\left((-(\hbar_2))^2\right)^{0.3}\right)\right)\right)\right)^{\frac{1}{2}}\right)\left(\left(-\left(-1-\left(-\left(\left((-(\hbar_3))^2\right)^{0.5}\right)\right)\right)\right)^{\frac{1}{2}}\right)\left(\left(-\left(-1-\left(-\left(\left((-(\hbar_1))^2\right)^{0.2}\right)\right)\right)\right)^{\frac{1}{2}}\right)\right)\right\}\right\}$$

$$=\{\{0.3683, 0.370241, 0.258863, 0.26114, 0.383151, 0.385046, 0.276287, 0.278511\},$$
$$\{-0.56486, -0.51506, -0.53846, -0.48564, -0.52151, -0.46674, -0.49248, -0.43439\}\}$$

$$GBPVHFOWG_1(\mathcal{H}_1,\mathcal{H}_2,\mathcal{H}_3)=BPVHFOWG(\mathcal{H}_1,\mathcal{H}_2,\mathcal{H}_3)=\bigotimes_{i=1}^{3}\left(\mathcal{H}_{\sigma(i)}\right)^{\omega'_i}$$

$$=\bigcup_{\hbar_1\in \mathcal{H}_1,\hbar_2\in \mathcal{H}_2,\hbar_3\in \mathcal{H}_3}\left\{\left\{(\hbar_2)^{0.3}(\hbar_3)^{0.5}(\hbar_1)^{0.2}\right\},\left\{-1-\left(-\left(-(-1-(\hbar_2))\right)^{0.3}\left(-(-1-(\hbar_3))\right)^{0.5}\left(-(-1-(\hbar_1))\right)^{0.2}\right)\right\}\right\}$$

$$=\{\{0.486197, 0.558494, 0.458391, 0.526553, 0.513531, 0.589892, 0.484162, 0.556156\}, \{-0.22108,$$
$$-0.2, -0.17383, -0.15147, -0.19307, -0.17123, -0.14412, -0.12095\}\}$$

$$GBPVHFOWG_2(\mathcal{H}_1,\mathcal{H}_2,\mathcal{H}_3)=\frac{1}{\rho}\left(\bigoplus_{i=1}^{n}\left(\rho\mathcal{H}_{\sigma(i)}\right)^{\omega'_i}\right)$$

$$=\bigcup_{\hbar_1\in \mathcal{H}_1,\hbar_2\in \mathcal{H}_2,\hbar_3\in \mathcal{H}_3}\left\{\left\{1-\left(1-(1-(1-\hbar_2)^2)^{0.3}(1-(1-\hbar_3)^2)^{0.5}(1-(1-\hbar_1)^2)^{0.2}\right)^{\frac{1}{2}}\right\},\left\{-1-\left(-\left(\left(\left(-\left(-1-\left(-\left((-(\hbar_2))^2\right)\right)\right)\right)^{0.3}\right)^{\frac{1}{2}}\right)\left(\left(\left(-\left(-1-\left(-\left((-(\hbar_3))^2\right)\right)\right)\right)^{0.5}\right)^{\frac{1}{2}}\right)\left(\left(\left(-\left(-1-\left(-\left((-(\hbar_1))^2\right)\right)\right)\right)^{0.2}\right)^{\frac{1}{2}}\right)\right)\right\}\right\}$$

$$=\{\{0.412435, 0.494075, 0.403988, 0.48297, 0.432031, 0.520179, 0.422997, 0.508084\}, \{-0.02543, -0.0202,$$
$$-0.0179, -0.01264, -0.02092, -0.01567, -0.01336, -0.00807\}\}$$

It seems that BPVHFWA, BPVHFWG, GBPVHFWA and GBPVHFWG operators concern with the weight of BPVHF argument and have no concern with their order. On the other hand BPVHFOWA, BPVHFOWG, GBPVHFOWA and GBPVHFOWG operators deal with the weight of the ordered position of each given argument and give no importance to argument itself. Therefore we need to introduce the hybrid aggregation operators for bipolar-valued hesitant fuzzy arguments. These newly defined hybrid aggregation operators give weight to any given argument and their ordered positions too.

Definition 5:

Let $\mathcal{H}_i$ $(i = 1,2,3,4 \ldots n)$ be a collection of BPVHFEs, $\omega = (\omega_1, \omega_2, \omega_3, \omega_4, \ldots \omega_n)^T$ is their weight vector with $\omega_i \in [0,1]$ and $\sum_{i=1}^{n}\omega_i = 1$, '**n**' is the balancing coefficient and $\omega' = (\omega'_1, \omega'_2, \omega'_3, \omega'_4, \ldots \omega'_n)^T$ be the aggregation associated weight vector of $\mathcal{H}_i$ $(i = 1,2,3,4 \ldots n)$ with $\omega'_i \in [0,1]$ and $\sum_{i=1}^{n}\omega'_i = 1$. Then

1. The BPVHFHA operator is a mapping $BPVHFHA: \Psi^n \to \Psi$ such that

$$\mathrm{BPVHFHA}(\mathcal{H}_1, \mathcal{H}_2, \ldots \mathcal{H}_n) = \bigoplus_{i=1}^{n}\left(\dot{\omega}_i \mathcal{H}_{\sigma(i)}\right)$$

$$=\left\{\left\{1-\prod_{i=1}^{n}\left(1-\hbar_{\sigma(i)}\right)^{\omega'_i}: \hbar_{\sigma(i)} \in \mathcal{H}^{+}_{\sigma(i)}\right\},\left\{-\prod_{i=1}^{n}\left(-\hbar_{\sigma(i)}\right)^{\omega'_i}: \hbar_{\sigma(i)} \in \mathcal{H}^{-}_{\sigma(i)}\right\}\right\}$$

where $\text{Ƀ}_{\sigma(\mathrm{j})}$ is the r^{th} largest of $\dot{\text{Ƀ}} = nw_k\text{Ƀ}_k\,(k = 1, 2, 3, \dots, n)$

2. The BPVHFHG operator is a mapping $BPVHFHA: \Psi^n \to \Psi$ such that

$$\mathrm{BPVHFHG}\left(\ddot{\text{Ƀ}}_1, \ddot{\text{Ƀ}}_2, \dots \ddot{\text{Ƀ}}_n\right) = \bigotimes_{\mathrm{j}=1}^{n}\left(\ddot{\text{Ƀ}}_{\sigma(\mathrm{j})}\right)^{\acute{\omega}_{\mathrm{j}}}$$

$$= \left\{\left\{\prod_{\mathrm{j}=1}^{n}\left(\xi_{\sigma(\mathrm{j})}\right)^{\acute{\omega}_{\mathrm{j}}}: \xi_{\sigma(\mathrm{j})} \in \ddot{\text{Ƀ}}^{+}_{\sigma(\mathrm{j})}\right\}, \left\{-1-\left(-\prod_{\mathrm{j}=1}^{n}\left(-\left(-1-\xi_{\sigma(\mathrm{j})}\right)\right)^{\acute{\omega}_{\mathrm{j}}}\right): \xi_{\sigma(\mathrm{j})} \in \ddot{\text{Ƀ}}^{-}_{\sigma(\mathrm{j})}\right\}\right\}$$

here $\ddot{\text{Ƀ}}_{\sigma(\mathrm{j})}$ is the r^{th} largest of $\dot{\text{Ƀ}} = \text{Ƀ}_k^{nw_k}\,(k = 1, 2, 3, \dots, n)$

3. A GBPVHFA operator is a function $GBPVHFHA: \Psi^n \to \Psi$, such that

$$GBPVHFHA_{\rho^{\cdot}}\left(\text{Ƀ}_1, \text{Ƀ}_2, \dots \text{Ƀ}_n\right) = \left(\bigoplus_{\mathrm{j}=1}^{n}\left(\omega^{\cdot}_{\mathrm{j}}\text{Ƀ}^{\rho^{\cdot}}_{\sigma(\mathrm{j})}\right)\right)^{\frac{1}{\rho^{\cdot}}}$$

$with\ \rho^{\cdot} > 0$

$$= \left\{\left\{\left(1-\prod_{\mathrm{j}=1}^{n}\left(1-\xi^{\rho^{\cdot}}_{\sigma(\mathrm{j})}\right)^{\acute{\omega}_{\mathrm{j}}}\right)^{\frac{1}{\rho^{\cdot}}}: \xi_{\sigma(\mathrm{j})} \in \text{Ƀ}^{+}_{\sigma(\mathrm{j})}\right\},\right.$$

$$\left.\left\{-1-\left(-\left(\prod_{\mathrm{j}=1}^{n}\left(-\left(-1-\left(-\left(\left(-\xi_{\sigma(\mathrm{j})}\right)^{\rho^{\cdot}}\right)^{\acute{\omega}_{\mathrm{j}}}\right)\right)\right)\right)^{\frac{1}{\rho^{\cdot}}}\right): \xi_{\sigma(\mathrm{j})} \in \text{Ƀ}^{-}_{\sigma(\mathrm{j})}\right\}\right\}$$

here $\text{Ƀ}_{\sigma(\mathrm{j})}$ is the m^{th} largest of $\dot{\text{Ƀ}} = nw_k\text{Ƀ}_k\,(k = 1, 2, 3, \dots, n)$

4. A GBPVHFHG operator is a function $GBPVHFHG: \Psi^n \to \Psi$, such that

$$GBPVHFHG_{\rho^{\cdot}}\left(\ddot{\text{Ƀ}}_1, \ddot{\text{Ƀ}}_2, \dots \ddot{\text{Ƀ}}_n\right) = \frac{1}{\rho^{\cdot}}\left(\bigoplus_{\mathrm{j}=1}^{n}\left(\rho^{\cdot}\ddot{\text{Ƀ}}_{\sigma(\mathrm{j})}\right)^{\acute{\omega}_{\mathrm{j}}}\right)$$

$with\ \rho^{\cdot} > 0$

$$= \left\{\left\{1-\left(1-\prod_{\mathrm{j}=1}^{n}\left(1-\left(1-\xi_{\sigma(\mathrm{j})}\right)^{\rho^{\cdot}}\right)^{\acute{\omega}_{\mathrm{j}}}\right)^{\frac{1}{\rho^{\cdot}}}: \xi_{\sigma(\mathrm{j})} \in \ddot{\text{Ƀ}}^{+}_{\mathrm{j}}\right\},\right.$$

$$\left.\left\{-1-\left(-\prod_{\mathrm{j}=1}^{n}\left(\left(-\left(-1-\left(-\left(-\xi_{\sigma(\mathrm{j})}\right)^{\rho}\right)\right)\right)^{\acute{\omega}_{\mathrm{j}}}\right)^{\frac{1}{\rho^{\cdot}}}\right): \xi_{\sigma(\mathrm{j})} \in \ddot{\text{Ƀ}}^{-}_{\sigma(\mathrm{j})}\right\}\right\}$$

here $\ddot{\text{Ƀ}}_{\sigma(\mathrm{j})}$ is the m^{th} largest of $\dot{\text{Ƀ}} = \text{Ƀ}_k^{nw_k}\,(k = 1, 2, 3, \dots, n)$

Example 2:

Let $\text{Ƀ}_1 = \{\{0.1, 0.2\}, \{-0.3, -0.2\}\}$, $\text{Ƀ}_2 = \{\{0.5, 0.6\}, \{-0.2, -0.1\}\}$ and $\text{Ƀ}_3 = \{\{0.9, 0.8\}, \{-0.2, -0.1\}\}$ be three BPVHFEs and $\omega = (0.15, 0.2, 0.65)$ be their weight vector and let $\acute{\omega} = (0.3, 0.5, 0.2)^{\frac{1}{T}}$ be the aggregation-associated vector. Then

$$\text{Ƀ}_1 = \left\{\{1-(1-0.1)^{2\times0.15}, 1-(1-0.2)^{2\times0.15}\}, \left\{-\left(-(-0.3)\right)^{2\times0.15}, -\left(-(-0.2)\right)^{2\times0.15}\right\}\right\}$$

$$\text{Ƀ}_1 = \{\{0.031114, 0.064752\}, \{-0.69685, -0.61703\}\}$$

$$\text{Ƀ}_2 = \left\{\{1-(1-0.5)^{2\times0.2}, 1-(1-0.6)^{2\times0.2}\}, \left\{-\left(-(-0.2)\right)^{2\times0.2}, -\left(-(-0.1)\right)^{2\times0.2}\right\}\right\}$$

$$\text{Ƀ}_2 = \{\{0.242142, 0.306855\}, \{-0.52531, -0.39811\}\}$$

$$\text{Ƀ}_3 = \left\{\{1-(1-0.9)^{2\times0.65}, 1-(1-0.8)^{2\times0.65}\}, \left\{-\left(-(-0.2)\right)^{2\times0.65}, -\left(-(-0.1)\right)^{2\times0.65}\right\}\right\}$$

$$\text{Ƀ}_3 = \{\{0.949881, 0.876593\}, \{-0.12341, -0.05012\}\}$$

The score values for the given sets can be calculated as follows:

$$\mathcal{S}(\text{Ƀ}_1) = \frac{1}{\ell_{\text{Ƀ}_1}}\left(\xi^{+}_{\text{Ƀ}_1} + \xi^{-}_{\text{Ƀ}_1}\right) = \frac{1}{2}\left(0.031114 + 0.064752 + (-0.69685) + (-0.61703)\right)$$

$$\mathcal{S}(\text{Ƀ}_1) = -0.60901$$

$$\mathcal{S}(\text{Ƀ}_2) = \frac{1}{\ell_{\text{Ƀ}_2}}\left(\xi^{+}_{\text{Ƀ}_2} + \xi^{-}_{\text{Ƀ}_2}\right) = \frac{1}{2}\left(0.242142 + 0.306855 + (-0.52531) + (-0.39811)\right)$$

$$\mathcal{S}(\text{Ƀ}_2) = -0.18721$$

$$\mathcal{S}(\text{Ƀ}_3) = \frac{1}{\ell_{\text{Ƀ}_3}}\left(\xi^{+}_{\text{Ƀ}_3} + \xi^{-}_{\text{Ƀ}_3}\right) = \frac{1}{2}\left(0.949881 + 0.876593 + (-0.12341) + (-0.05012)\right)$$

$$\mathcal{S}(\text{Ƀ}_3) = 0.826472$$

Now from these results it is obvious that $\mathcal{S}(\text{Ƀ}_3) > \mathcal{S}(\text{Ƀ}_2) > \mathcal{S}(\text{Ƀ}_1)$, so

$\text{Ƀ}_{\sigma(1)} = \text{Ƀ}_3 = \{\{0.949881, 0.876593\}, \{-0.12341, -0.05012\}\}$

$\text{Ƀ}_{\sigma(2)} = \text{Ƀ}_2 = \{\{0.242142, 0.306855\}, \{-0.52531, -0.39811\}\}$

$\text{Ƀ}_{\sigma(3)} = \text{Ƀ}_1 = \{\{0.031114, 0.064752\}, \{-0.69685, -0.61703\}\}$

Now

$$GBPVHFHA_1(\text{Ƀ}_1, \text{Ƀ}_2, \text{Ƀ}_3) = BPVHFHA(\text{Ƀ}_1, \text{Ƀ}_2, \text{Ƀ}_3) = \bigoplus_{\mathrm{j}=1}^{3}\left(\acute{\omega}_{\mathrm{j}}\text{Ƀ}_{\sigma(\mathrm{j})}\right)$$

$$= \bigcup_{\xi_1 \in \text{Ƀ}_1, \xi_2 \in \text{Ƀ}_2, \xi_3 \in \text{Ƀ}_3}\left\{\left\{1-\left(1-\xi_3\right)^{0.3}\left(1-\xi_2\right)^{0.5}\left(1-\xi_1\right)^{0.2}\right\}, \left\{-\left(\left(-(\xi_3)\right)^{0.3}\left(-(\xi_2)\right)^{0.5}\left(-(\xi_1)\right)^{0.2}\right)\right\}\right\}$$

$$= \{\{0.647589, 0.650071, 0.662971, 0.665344, 0.538204, 0.541456, 0.55836, 0.56147\}, \{-0.35995, -0.3513, -0.31335, -0.30582, -0.27469, -0.26808, -0.23913, -0.23338\}\}$$

$$GBPVHFHA_2(\text{Ѣ}_1, \text{Ѣ}_2, \text{Ѣ}_3) = \left(\bigoplus_{j=1}^{3}\left(\omega'_j \dot{\text{Ѣ}}^{2}_{\sigma(j)}\right)\right)^{\frac{1}{2}}$$

$$= \bigcup_{\xi_1 \in \text{Ѣ}_1, \xi_2 \in \text{Ѣ}_2, \xi_3 \in \text{Ѣ}_3} \left\{ \left\{ 1 - \left(\left(1 - (\xi_3)^2\right)^{0.3} \left(1 - (\xi_2)^2\right)^{0.5} \left(1 - (\xi_1)^2\right)^{0.2} \right)^{\frac{1}{2}} \right\}, \left\{ -1 - \left(-\left(\left(-\left(-1 - \left(-\left(\left(\left(-(\xi_3) \right)^2 \right)^{0.3} \right) \right) \right) \right)^{\frac{1}{2}} \right) \left(\left(-\left(-1 - \left(-\left(\left(\left(-(\xi_2) \right)^2 \right)^{0.5} \right) \right) \right) \right)^{\frac{1}{2}} \right) \left(\left(-\left(-1 - \left(-\left(\left(\left(-(\xi_1) \right)^2 \right)^{0.2} \right) \right) \right) \right)^{\frac{1}{2}} \right) \right) \right\} \right\}$$

$$= \{\{0.305137, 0.305361, 0.311788, 0.31201, 0.209132, 0.209387, 0.216702, 0.216955\}, \{-0.78632, -0.75585, -0.75939, -0.72507, -0.76922, -0.73631, -0.74013, -0.70307\}\}$$

Now we calculate $\ddot{\text{Ѣ}}_1, \ddot{\text{Ѣ}}_2, \ddot{\text{Ѣ}}_2$

$$\ddot{\text{Ѣ}}_1 = \left\{ \{(0.1)^{2\times 0.15}, (0.2)^{2\times 0.15}\}, \left\{ -1 - \left(-\left(-(-1 - (-0.3)) \right)^{2\times 0.15} \right), -1 - \left(-\left(-(-1 - (-0.2)) \right)^{2\times 0.15} \right) \right\} \right\}$$

$$\ddot{\text{Ѣ}}_1 = \{\{0.501187, 0.617034\}, \{-0.10148, -0.06475\}\}$$

$$\ddot{\text{Ѣ}}_2 = \left\{ \{(0.5)^{2\times 0.2}, (0.6)^{2\times 0.2}\}, \left\{ -1 - \left(-\left(-(-1 - (-0.2)) \right)^{2\times 0.2} \right), -1 - \left(-\left(-(-1 - (-0.1)) \right)^{2\times 0.2} \right) \right\} \right\}$$

$$\ddot{\text{Ѣ}}_2 = \{\{0.757858, 0.815193\}, \{-0.08539, -0.04127\}\}$$

$$\ddot{\text{Ѣ}}_3 = \left\{ \{(0.9)^{2\times 0.65}, (0.8)^{2\times 0.65}\}, \left\{ -1 - \left(-\left(-(-1 - (-0.2)) \right)^{2\times 0.65} \right), -1 - \left(-\left(-(-1 - (-0.1)) \right)^{2\times 0.65} \right) \right\} \right\}$$

$$\ddot{\text{Ѣ}}_3 = \{\{0.871998, 0.748199\}, \{-0.2518, -0.128\}\}$$

The score values for the given sets can be calculated as follows:

$$S(\ddot{\text{Ѣ}}_1) = \frac{1}{\ell_{\ddot{\text{Ѣ}}_1}}\left(\xi^{+}_{\ddot{\text{Ѣ}}_1} + \xi^{-}_{\ddot{\text{Ѣ}}_1}\right) = \frac{1}{2}\left(0.501187 + 0.617034 + (-0.10148) + (-0.06475)\right)$$

$$S(\ddot{\text{Ѣ}}_1) = 0.475996$$

$$S(\ddot{\text{Ѣ}}_2) = \frac{1}{\ell_{\ddot{\text{Ѣ}}_2}}\left(\xi^{+}_{\ddot{\text{Ѣ}}_2} + \xi^{-}_{\ddot{\text{Ѣ}}_2}\right) = \frac{1}{2}\left(0.757858 + 0.815193 + (-0.08539) + (-0.04127)\right)$$

$$S(\ddot{\text{Ѣ}}_2) = 0.723196$$

$$S(\ddot{\text{Ѣ}}_3) = \frac{1}{\ell_{\ddot{\text{Ѣ}}_3}}\left(\xi^{+}_{\ddot{\text{Ѣ}}_3} + \xi^{-}_{\ddot{\text{Ѣ}}_3}\right) = \frac{1}{2}\left(0.871998 + .748199 + (-0.2518) + (-0.128)\right)$$

$$S(\ddot{\text{Ѣ}}_3) = 0.620199$$

Now from these results it is obvious that $S(\ddot{\text{Ѣ}}_2) > S(\ddot{\text{Ѣ}}_3) > S(\ddot{\text{Ѣ}}_1)$, so

$$\ddot{\text{Ѣ}}_{\sigma(1)} = \ddot{\text{Ѣ}}_2 = \{\{0.757858, 0.815193\}, \{-0.08539, -0.04127\}\}$$

$$\ddot{\text{Ѣ}}_{\sigma(2)} = \ddot{\text{Ѣ}}_3 = \{\{0.871998, 0.748199\}, \{-0.2518, -0.128\}\}$$

$$\ddot{\text{Ѣ}}_{\sigma(3)} = \ddot{\text{Ѣ}}_1 = \{\{0.501187, 0.617034\}, \{-0.10148, -0.06475\}\}$$

$$GBPVHFHG_1(\text{Ѣ}_1, \text{Ѣ}_2, \text{Ѣ}_3) = BPVHFHG(\text{Ѣ}_1, \text{Ѣ}_2, \text{Ѣ}_3) = \bigotimes_{j=1}^{3}\left(\ddot{\text{Ѣ}}_{\sigma(j)}\right)^{\omega'_j}$$

$$= \bigcup_{\xi_1 \in \text{Ѣ}_1, \xi_2 \in \text{Ѣ}_2, \xi_3 \in \text{Ѣ}_3} \left\{ \left\{ (\ddot{\xi}_2)^{0.3} (\ddot{\xi}_3)^{0.5} (\ddot{\xi}_1)^{0.2} \right\}, \left\{ -1 - \left(-\left(-\left(-1 - (\ddot{\xi}_2) \right) \right)^{0.3} \left(-\left(-1 - (\ddot{\xi}_3) \right) \right)^{0.5} \left(-\left(-1 - (\ddot{\xi}_1) \right) \right)^{0.2} \right) \right\} \right\}$$

$$= \{\{0.7484, 0.780182, 0.693242, 0.722681, 0.764955, 0.797439, 0.708576, 0.738667\}, \{-0.1757, -0.16907, -0.11011, -0.10295, -0.16397, -0.15724, -0.09745, -0.09019\}\}$$

$$GBPVHFHG_2(\text{Ѣ}_1, \text{Ѣ}_2, \text{Ѣ}_3) = \frac{1}{\rho'}\left(\bigoplus_{j=1}^{n}\left(\rho' \ddot{\text{Ѣ}}_{\sigma(j)}\right)^{\omega'_j}\right)$$

$$= \bigcup_{\xi_1\in\hbar_1,\xi_2\in\hbar_2,\xi_3\in\hbar_3}\Bigg\{\Big\{1-\Big(1-\big(1-(1-\xi_2)^2\big)^{0.3}\big(1-(1-\xi_3)^2\big)^{0.5}\big(1-(1-\xi_1)^2\big)^{0.2}\Big)^{\frac{1}{2}}\Big\},\Big\{-1-\Big(-\Big(\Big(\Big(-\Big(-1-\Big(-\big(\big(-(\xi_2)^2\big)\big)\Big)\Big)^{0.3}\Big)^{\frac{1}{2}}\Big)\Big(\Big(\Big(-\Big(-1-\Big(-\big(\big(-(\xi_3)^2\big)\big)\Big)\Big)^{0.5}\Big)^{\frac{1}{2}}\Big)\Big(\Big(\Big(-\Big(-1-\Big(-\big(\big(-(\xi_1)^2\big)\big)\Big)\Big)^{0.2}\Big)^{\frac{1}{2}}\Big)\Big)\Big)\Big\}\Bigg\}$$

$$= \{\{0.716791, 0.762406, 0.679905, 0.718435, 0.729636, 0.778288, 0.690934, 0.731372\}, \{-0.01834, -0.01773, -0.00624, -0.00563, -0.01751, -0.01691, -0.00541, -0.00479\}\}$$

IV. Multi-Attribute Decision Making Based On Bipolar Valued Hesitant Fuzzy Sets

In this section, we describe a brief technique to deal with MADM problems using BPVHF aggregation operator in which a DM provide information in bipolar valued hesitant fuzzy decision matrix and every element is characterized by BPVHFE.

Consider that we have n substitutes $A_i\,(i = 1,2,3,4,\dots n)$ with m attributes $\kappa_j\,(j = 1,2,3,\dots m)$ and assume that $w = (w_1, w_2, \dots, w_m)$ be the weight vector such that $w_j \in [-1, 1]$, $j = 1,2,3\dots m$ and $\sum_{j=1}^{m} \omega'_j = 1$. The decision makers assigned values in the form of BPVHFEs $(\hbar_{ij})$ for the substitutes A_i under the attributes κ_j in the state of being anonymous.

The DM method is based on the following steps:

Step 1:

This is the step of decision matrix formation as each alternative A_i has assigned some values in the form of BPVHFEs $(\hbar_{ij})$ under some attributes κ_j.

Step 2:

In this step, BPVHFE $\alpha_i\,(i = 1,2,3,\dots,n)$ can be obtained for the substitutes $A_i\,(i = 1,2,3,\dots,n)$ by using bipolar-valued hesitant fuzzy hybrid operators.

Step 3:

By applying score function (accuracy function) we get the accuracy value $\mathcal{S}(\alpha_i), (i = 1,2,3,\dots,n)$ of $\alpha_i\ (i = 1,2,3,\dots,4)$.

Step 4:

To get the most suitable alternative $A_i\,(i = 1,2,3,\dots,n)$, we established an order in the score values $\mathcal{S}(\alpha_i), (i = 1,2,3,\dots,n)$.

Example 3:

A cricket board needs a head coach for their cricket team. The cricket board advertised the post in a news paper and a number of candidates applied for the post. Based on their history in the cricket field initially 4 candidates are called for an interview. The cricket board is going to appoint a coach who possesses the qualities like hard working, Creative, Committed and skillful. For the better future of cricket in the country, the cricket board needs to appoint the most suitable head coach.

Let $A = \{A_1, A_2, A_3, A_4\}$ be the set of substitutes and $X = \{\kappa_1, \kappa_2, \kappa_3, \kappa_4\}$ be the set of attributes and let $w = (0.25, 0.23, 0.35, 0.17)^T$ be the weight vector of the attributes $X_i\,(i = 1,2,3,4)$ and $\acute{\omega} = (0.3, 0.4, 0.2, 0.1)^{\frac{1}{T}}$ be the aggregation associated weight vector.

Conclusion

In our treatise, we successfully apply aggregation operators of BPVHFSs in a DM problem. The results clearly indicate that either we use BPVHF hybrid averaging operators or BPVHF hybrid geometric operators, we get the same results. Hence these two aggregation operation can be very useful in DM especially in two-sided DM. Future research will involve the generalization to bipolar-valued hesitant neutrosophic information and its aggregation operators.

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