The asymptotic behavior of defocusing nonlinear Schrödinger equations

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Abstract

This article is concerned with the scattering problem for the defocusing nonlinear Schrödinger equations (NLS) with a power nonlinear $|u|^p u$ where $2/n < p < 4/n$. We show that for any initial data in $H_x^{0,1}$ the solution will eventually scatter, i.e. $U(-t)u(t)$ tends to some function u_+ as t tends to infinity.

We consider the defocusing nonlinear Schrödinger equations (NLS)

$$iu_t + \frac{1}{2}\triangle u = |u|^p u, \quad u(0) = u_0, \tag{1}$$

where u is a complex value function $u : \mathbb{R} \times \mathbb{R}^n \to \mathbb{C}$, $u_0 \in H_x^{0,1}$, and $\frac{2}{n} < p < \frac{4}{n}$.

There are many papers on the scattering theory for the NLS. For both focusing or defocusing problems, it is well known that for $p \leq \frac{2}{n}$ there will be no scattering[1]. For $p > \frac{2}{n}$, it is known that $U(-t)u(t)$ converges weakly in H_x^1 for any finite energy solution of NLS[7], if we assume additionally that $u_0 \in H_x^{1,1}$, then it is know that $U(-t)u(t)$ converges strongly in L_x^2[11]. For the asymptotic completeness problem, when $n \geq 3$, for any free solution in L_x^2 or H_x^1 there exists a solution of NLS which appoaches the free solution in the same space as t tends to infinity[6]. In the defocusing case, if $p > 8/\left(\sqrt{(n+2)^2 + 8n} + n - 2\right)$, then we have the asymptotic completeness in $H^{1,1}$[4, 10, 8]. In present paper we combining methods used in [11, 5], which gives similar result for a wider class of solutions. When $u_0 \in H_x^{0,1}$, we have $U(-t)u(t)$ converges strongly in L_x^2 and converging rate $t^{\frac{1}{2}-\frac{np}{4}}$ which was implicitly indicate in [11]. Our main result follows:

Theorem 1 : *Consider the equation* (1) *with* $u_0 \in H_x^{0,1}$, *then there exists a unique global solution* u *with regularity* $U(-t)u(t) \in C(\mathbb{R}; H_x^{0,1})$, *and a function* $u_+ \in L_x^2(\mathbb{R}^n)$ *satisfying*

$$\lim_{t\to\infty} \|U(-t)u(t) - u_+\|_{L_x^2} \lesssim \lim_{t\to\infty} t^{\frac{1}{2}-\frac{np}{4}} = 0. \tag{2}$$

Notation:

Let $\mathcal{F}\varphi$ and $\hat{\varphi}$ be the Fourier transform of φ defined by

$$\mathcal{F}\varphi(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi}\varphi(x)dx.$$

Let $U(t)$ be the free Schrödinger group defined by

$$U(t)\phi = (2\pi i t)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i|x-y|^2/2t}\varphi(y)dy.$$

Note that $U(t) = M(t)D(t)\mathcal{F}M(t)$, where $D(t)$ is the dilation operator $D(t)f(x) = i^{-\frac{n}{2}}t^{-\frac{n}{2}}f\left(\frac{x}{t}\right)$, and $M(t) = e^{\frac{i|x|^2}{2t}}$. Hence $U(-t) = M(-t)\mathcal{F}^{-1}D^{-1}(t)M(-t)$.

Let $P_{\leq N}\phi$, $P_{\geq N}\phi$ be the Littlewood-Paley projections:

$$P_{\leq N}\phi = \mathcal{F}^{-1}\mathcal{X}\left(\frac{\xi}{N}\right)\hat{\phi}(\xi), \quad P_{\geq N} = \phi - P_{\leq N}\phi$$

where $\mathcal{X}$ is a Schwartz radial symmetry bump function.

Let $H^{m,k}$ be the norm define by

$$\|\varphi\|^2_{H^{m,k}} = \left\|(1-\triangle)^{\frac{m}{2}}\varphi\right\|^2_{L^2} + \left\|\left(1+|x|^2\right)^{\frac{k}{2}}\varphi\right\|^2_{L^2}, \quad m,k \geq 0.$$

1 Well-posedness and energy estimate.

The equation (1) is locally L^2_x well-posed with $u_0 \in L^2_x$ by Strichartz estimate for the linear inhomogeneous problem

$$\left(i\partial_t + \frac{1}{2}\triangle\right)u = f, \qquad u(0) = u_0,$$

which gives us

$$\|u\|_{L^\infty_t L^2_x} + \|u\|_{L^a_t L^{2+p}_x} \lesssim \|u_0\|_{L^2_x} + \|f\|_{L^{a'}_t L^{(2+p)'}_x},$$

where $a = \frac{4(2+p)}{np}$ satisfying the equation $\frac{2}{a} + \frac{n}{2+p} = \frac{n}{2}$. Appying Hölder inequality to the inhomogeneous term we obtain the unique local solution via the contraction principle in the space $L^\infty_t\left(0,T;L^2_x\right)\bigcap L^a_t\left(0,T;L^{2+p}_x\right)$ provided that T is small enough. The global well-posedness of u is due to the conservation of the mass $\|u(t)\|_{L^2_x} = \|u_0\|_{L^2_x}$.

Denoting $L_x u$ be the vector field $L = x + it\nabla$, which is the conjugate of x with respect to the linear flow, $L_x = U(t)xU(-t)$. Naturally we have

$$\left[i\partial_t + \frac{1}{2}\triangle, L_x\right] = 0$$

and the equation of $L_x u$ has the form

$$\left(i\partial_t + \frac{1}{2}\triangle\right)L_x u = \left(1+\frac{p}{2}\right)|u|^p L_x u - \frac{p}{2}u^2|u|^{p-2}\overline{L_x u},$$

which is the linearization of (1). The well-posedness of the $L_x u$ equation is also obtained by the same Strichartz estimate and conservation of the mass. This shows the globally well-posed for initial data in $H^{0.1}$. See [3, 4]. Denoting

$$w(t,v) = t^{\frac{n}{2}} e^{-it|v|^2/2} u(t,tv), \tag{3}$$

we have $it^{\frac{n}{2}} e^{-it|v|^2/2} \left(L_x u\right)(t,tv) = \partial_v w(t,v)$, hence $w \in C\left(\mathbb{R}\backslash\{0\}; H_v^1\right)$ and also globally well-posed.. It can also be written as $w(t,v) = i^{-\frac{n}{2}} D^{-1}(t) M(-t) u$ and gives the differential equation

$$iw_t + \frac{1}{2t^2} \triangle w = t^{-\frac{np}{2}} |w|^p w \tag{4}$$

for $t \in \mathbb{R}\backslash\{0\}$. Multiplying (4) with $\overline{w}_t$ and takes the real part, this leads us to the following equation, the formal calculation of which can be justified by the regularizing technique of Ginibre and Velo [3]

$$\partial_t \left(\frac{1}{4} \|\nabla w\|_{L_v^2}^2 + \frac{1}{2+p} t^{2-\frac{np}{2}} \|w\|_{L_v^{2+p}}^{2+p} \right) = \frac{4-np}{4+2p} t^{1-\frac{np}{2}} \|w\|_{L_v^{2+p}}^{2+p} \tag{5}$$

and use the relation $\nabla w = -it^{\frac{n}{2}} L_x u(t,tv)$ to rewrite (5) into the form

$$\frac{1}{4} \|L_x u(t)\|_{L_x^2}^2 + \frac{1}{2+p} t^2 \|u(t)\|_{L_x^{2+p}}^{2+p} = \frac{1}{4} \|xu_0\|_{L_x^2}^2 + \int_0^t \frac{4-np}{4+2p} s \|u(s)\|_{L_x^{2+p}}^{2+p} ds. \tag{6}$$

Hence by Gronwall's inequality we get the growth

$$\|L_x u\|_{L_x^2} = \|\nabla w\|_{L_v^2} \lesssim_{\|xu_0\|_{L_x^2}} t^{1-\frac{np}{4}}, \tag{7}$$

and

$$t^{\frac{np}{2}} \|u\|_{L_x^{2+p}}^{2+p} = \|w\|_{L_v^{2+p}}^{2+p} \lesssim_{\|xu_0\|_{L_x^2}} 1. \tag{8}$$

Note that $0 < 1 - \frac{np}{4} < \frac{1}{2}$.

2 Wave packets and the asymptotic equation.

To study the global decay properties of solutions we use the method of testing by wave packets developing by Ifrim and Tataru [5]. A wave packet is an approximate solution localized in both space and frequency on the scale of the uncertainty principle. We define a wave packet Ψ_v adapted to the ray $\Gamma_v := \{x = vt\}$ and measure u along Γ_v by considering

$$\gamma(t,v) = \int u(t,x) \overline{\Psi_v}(t,x) dx.$$

The test function Ψ_v is of the form

$$\Psi_v(t,x) = \mathcal{X}\left(\frac{x-vt}{\sqrt{t}}\right) e^{i\phi}$$

where the phase function $\phi = \frac{|x|^2}{2t}$. Here for the computation purpose, rewrite γ as

$$\gamma = P_{\leq\sqrt{t}} w,$$

which is the same definition as the original one.

A direct computation yields

$$i\gamma_t = \mathcal{F}\left[D\mathcal{X}\left(\frac{\xi}{\sqrt{t}}\right)\cdot\frac{\xi}{2t^{\frac{3}{2}}} + \frac{|\xi|^2}{2t^2}\mathcal{X}\left(\frac{\xi}{\sqrt{t}}\right)\right]\hat{w} + t^{-\frac{np}{2}}P_{\leq\sqrt{t}}|w|^p w := I_1 + I_2. \tag{9}$$

We apply the similar arguement of Tsutsumi and Yajima [11] by computing the decaying rate of $\|\gamma(t)-\gamma(s)\|_{L_v^2}^2$ when t,s goes to infinity to prove that γ converges to some function. Since

$$I_1 = \mathcal{F}\left[D\mathcal{X}\left(\frac{\xi}{\sqrt{t}}\right)\cdot\frac{\xi}{2t^{\frac{3}{2}}} + \frac{|\xi|^2}{2t^2}\mathcal{X}\left(\frac{\xi}{\sqrt{t}}\right)\right]\hat{w},$$

and $\mathcal{X}$ is a Schwartz function, we get

$$\|I_1(t)\|_{L_x^2} \lesssim t^{-\frac{3}{2}}\||\xi|\,\hat{w}\|_{L_\xi^2} = t^{-\frac{3}{2}}\|L_x u\|_{L_x^2}. \tag{10}$$

For the nonlinear part

$$I_2 = t^{-\frac{np}{2}}P_{\leq\sqrt{t}}|w|^p w,$$

by using (8) and Hölder's inequality, we have for any $s \geq r \geq 1$ and any $T \geq 1$

$$\begin{aligned}\left|\left\langle\int_s^r I_2(\sigma)d\sigma, \gamma(T)\right\rangle\right| &= \left|\int_s^r \sigma^{-\frac{np}{2}}\left\langle P_{\leq\sqrt{\sigma}}|w|^p w(\sigma), \gamma(T)\right\rangle d\sigma\right| \\ &\lesssim \int_s^r \sigma^{-\frac{np}{2}}\|w(\sigma)\|_{L_v^{2+p}}^{1+p}\|\gamma(T)\|_{L_v^{2+p}}d\sigma \\ &\lesssim \int_s^r \sigma^{-\frac{np}{2}}\|w(\sigma)\|_{L_v^{2+p}}^{1+p}\|w(T)\|_{L_v^{2+p}}d\sigma \\ &\lesssim_{\|xu_0\|_{L_x^2}} s^{1-\frac{np}{2}} - r^{1-\frac{np}{2}}.\end{aligned} \tag{11}$$

By the relation $\gamma(T) = \gamma(1) - i\int_1^T I_1(\sigma)d\sigma - i\int_1^T I_2(\sigma)d\sigma$ which directly gives

$$\gamma(r) - \gamma(s) = -i\int_s^r I_1(\sigma)d\sigma - i\int_s^r I_2(\sigma)d\sigma. \tag{12}$$

Since $\|\gamma(T)\|_{L_v^2} \leq \|u(T)\|_{L_x^2} = \|u_0\|_{L_x^2}$, and by (7), (10) display

$$\int_s^r \|I_1(\sigma)\|_{L_v^2}d\sigma \lesssim_{\|xu_0\|_{L_x^2}} \int_s^r \sigma^{-\frac{1}{2}-\frac{np}{4}}d\sigma \lesssim_{\|xu_0\|_{L_x^2}} s^{\frac{1}{2}-\frac{np}{4}} - r^{\frac{1}{2}-\frac{np}{4}},$$

and (11) which gives us

$$\langle\gamma(r)-\gamma(s), \gamma(r)-\gamma(s)\rangle \lesssim_{\|xu_0\|_{L_x^2}} \|\gamma(r)-\gamma(s)\|_{L_v^2}\left(s^{\frac{1}{2}-\frac{np}{4}} - r^{\frac{1}{2}-\frac{np}{4}}\right) + s^{1-\frac{np}{2}} - r^{1-\frac{np}{2}}. \tag{13}$$

From above equations there $\exists g \in L_v^2$ such that $\lim_{t\to\infty}\|\gamma(t)-g\|_{L_v^2} = 0$, moreover we have $\|\gamma(t)-g\|_{L_v^2}^2 \lesssim_{\|xu_0\|_{L_x^2}} t^{\frac{1}{2}-\frac{np}{4}}\|\gamma(t)-g\|_{L_v^2} + t^{1-\frac{np}{2}}$ which gives us

$$\lim_{t\to\infty}\|\gamma(t)-g\|_{L_v^2} \lesssim_{\|xu_0\|_{L_x^2}} \lim_{t\to\infty} t^{\frac{1}{2}-\frac{np}{4}} = 0. \tag{14}$$

At last, if we take $u_+ = i^{\frac{n}{2}}\mathcal{F}^{-1}g$, then there is the estimation

$$\begin{aligned} &\|U(-t)u(t) - u_+\|_{L^2_x} = \left\|i^{\frac{n}{2}}M(-t)\mathcal{F}^{-1}w(t) - i^{\frac{n}{2}}\mathcal{F}^{-1}g\right\|_{L^2_\xi} \\ \lesssim &\left\|M(-t)\mathcal{F}^{-1}\left(w(t)-\gamma(t)\right)\right\|_{L^2_\xi} + \left\|M(-t)\mathcal{F}^{-1}\gamma(t) - \mathcal{F}^{-1}\gamma\right\|_{L^2_\xi} \\ &\quad + \left\|\mathcal{F}^{-1}\gamma(t) - \mathcal{F}^{-1}g\right\|_{L^2_\xi} \\ :=& R_1 + R_2 + R_3. \end{aligned} \tag{15}$$

It's obvious that $R_3(t) = \|\gamma(t) - g\|_{L^2_v}$. For R_1, by direct computation which yields

$$R_1(t) = \quad \|w(t) - \gamma(t)\|_{L^2_v} = \left\|P_{\geq\sqrt{t}}w(t)\right\|_{L^2_v} \lesssim t^{-\frac{1}{2}}\|\nabla w\|_{L^2_v} \lesssim_{\|xu_0\|_{L^2_x}} t^{\frac{1}{2}-\frac{np}{4}}. \tag{16}$$

Use the Taylor expansion of e^{ix}we have that

$$\begin{aligned} R_2(t) = &\left\|\left(e^{-\frac{i|\xi|^2}{2t}} - 1\right)\mathcal{F}^{-1}\gamma(t)\right\|_{L^2_\xi} \lesssim \left\|\frac{|\xi|}{\sqrt{t}}\mathcal{X}\left(\frac{\xi}{\sqrt{t}}\right)\hat{w}(t)\right\|_{L^2_\xi} \\ &= t^{-\frac{1}{2}}\|\nabla w(t)\|_{L^2_v} \lesssim_{\|xu_0\|_{L^2_x}} t^{\frac{1}{2}-\frac{np}{4}} \end{aligned} \tag{17}$$

Together by (14), (15), (16), and (17)

$$\lim_{t\to\infty}\|U(-t)u(t) - u_+\|_{L^2_x} \lesssim_{\|xu_0\|_{L^2_x}} \lim_{t\to\infty} t^{\frac{1}{2}-\frac{np}{4}} = 0. \tag{18}$$

By the time symmetry property of NLS, we have the same result when $t \to -\infty$.

From coservation of mass we have $\|u_+\|_{L^2_x} = \|g\|_{L^2_v} = \|u_0\|_{L^2_x}$and (14), (18) display $\|\nabla\gamma\|_{L^2_v} \lesssim \|L_x u\|_{L^2_x} \lesssim_{u_0} t^{1-\frac{np}{4}}$, $0 < \alpha \leq \frac{np}{2} - 1$

$$\|\langle x\rangle^\alpha u_+\|_{L^2_x} = \lim_{t\to\infty}\|\langle\nabla\rangle^\alpha\gamma\|_{L^2_v} \lesssim_{u_0} 1. \tag{19}$$

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