Fractals and Pi

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Abstract

This note presents a collection of fractals related with constant pi

1.

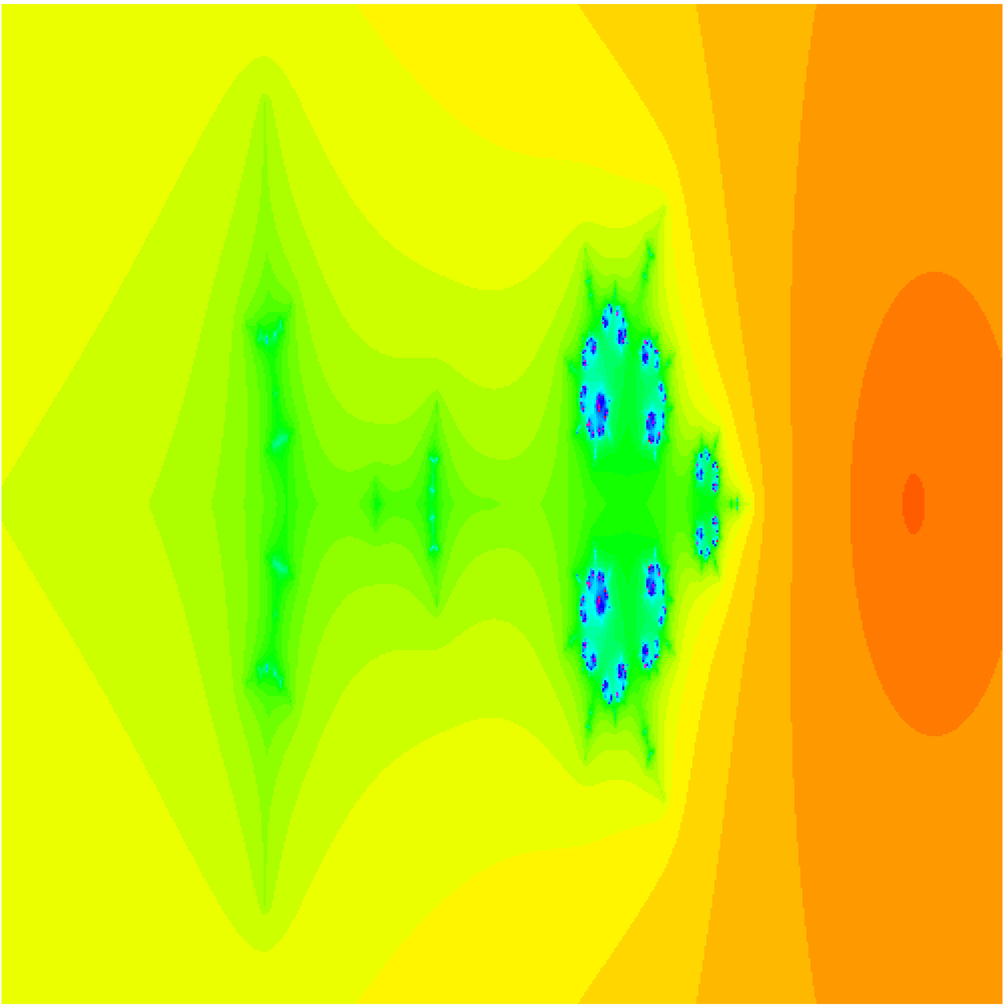

Figure 1. $f(z) = z \ln z - z - 1$

$(bl, ur) = (-11 - 3i, 5 + 3i)$

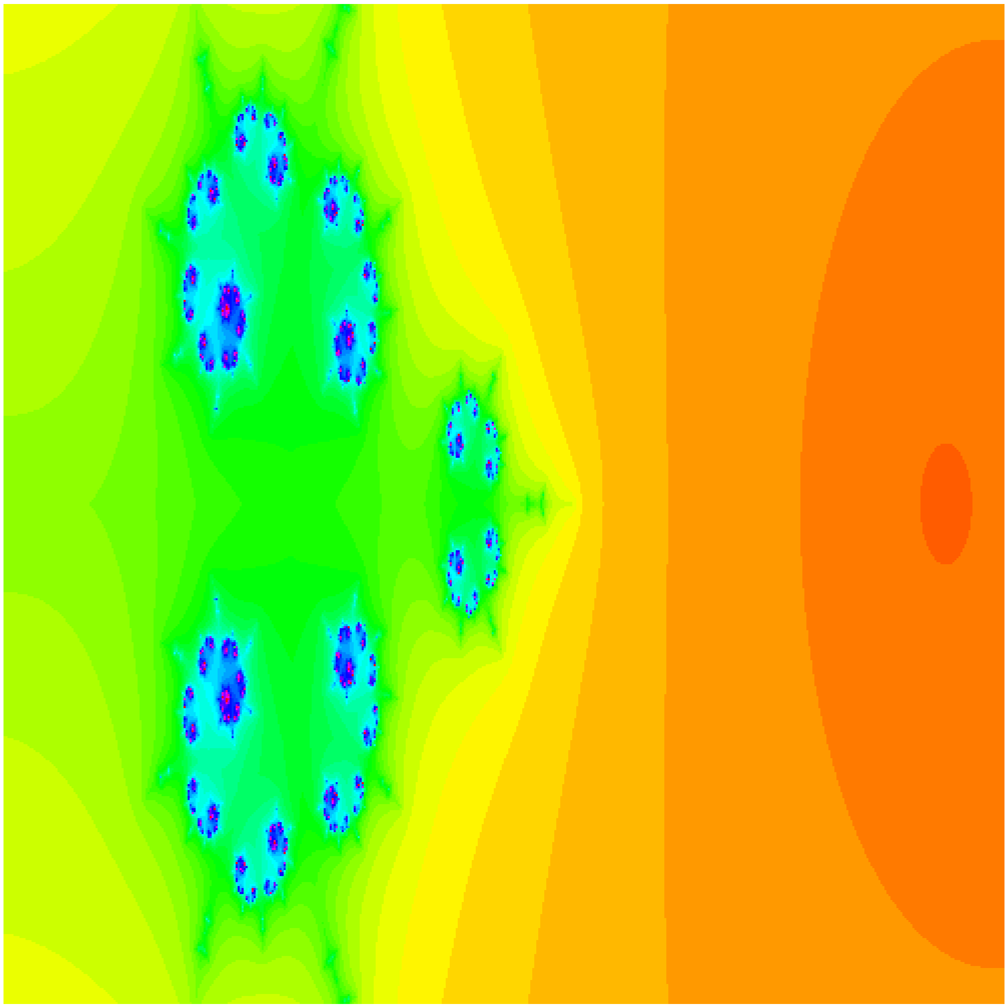

Figure 2. $f(z) = z \ln z - z - 1$

$(bl, ur) = (-3 - 1.5i, 4 + 1.5i)$

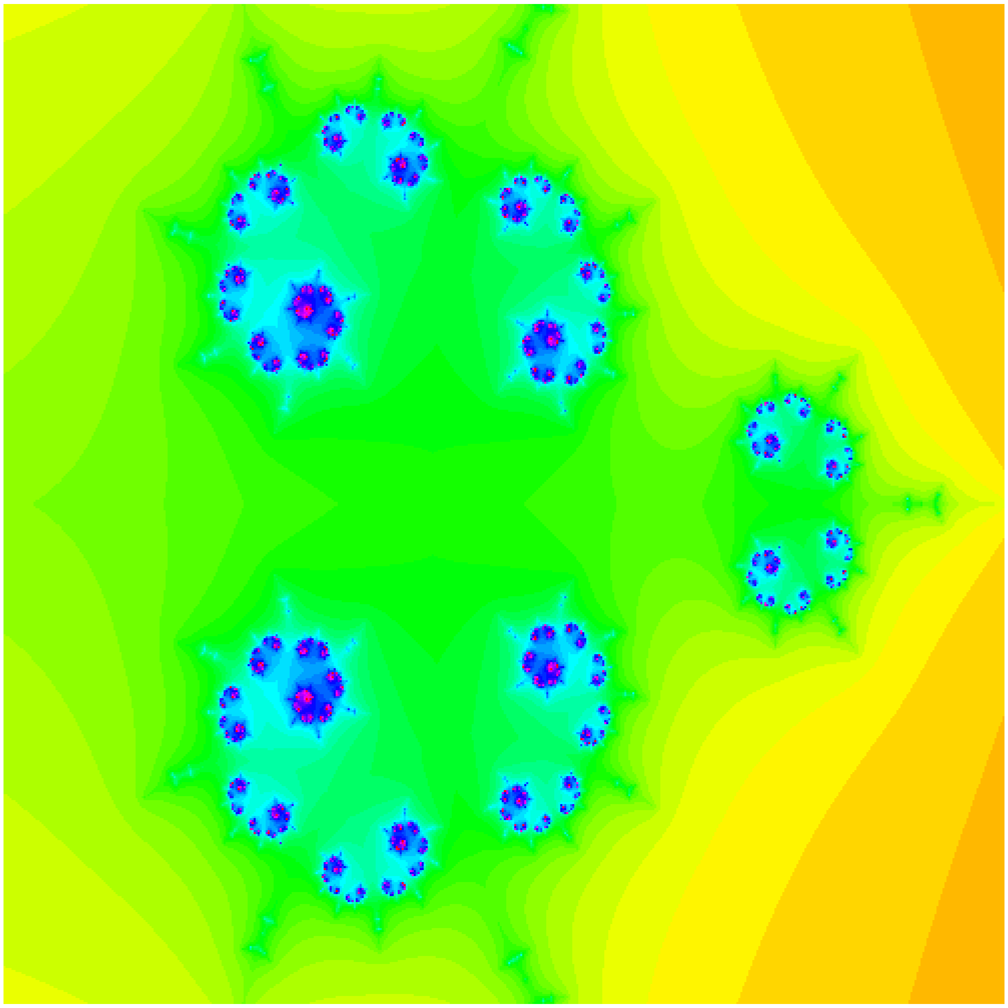

Figure 3. $f(z) = z\ln z - z - 1$, $(bl, ur) = (-2.5 - 1.5i, 1 + 1.5i)$

$$\pi = \int_0^\infty \ln x \ln\left(1 + \frac{\alpha^2}{x^2}\right) dx \tag{1}$$

$$\alpha = e^{1+e^{-1-e^{-1-e^{-1-\dots}}}} = 3.591121\dots \tag{2}$$

$$\alpha \ln\alpha - \alpha - 1 = 0 \tag{3}$$

2.

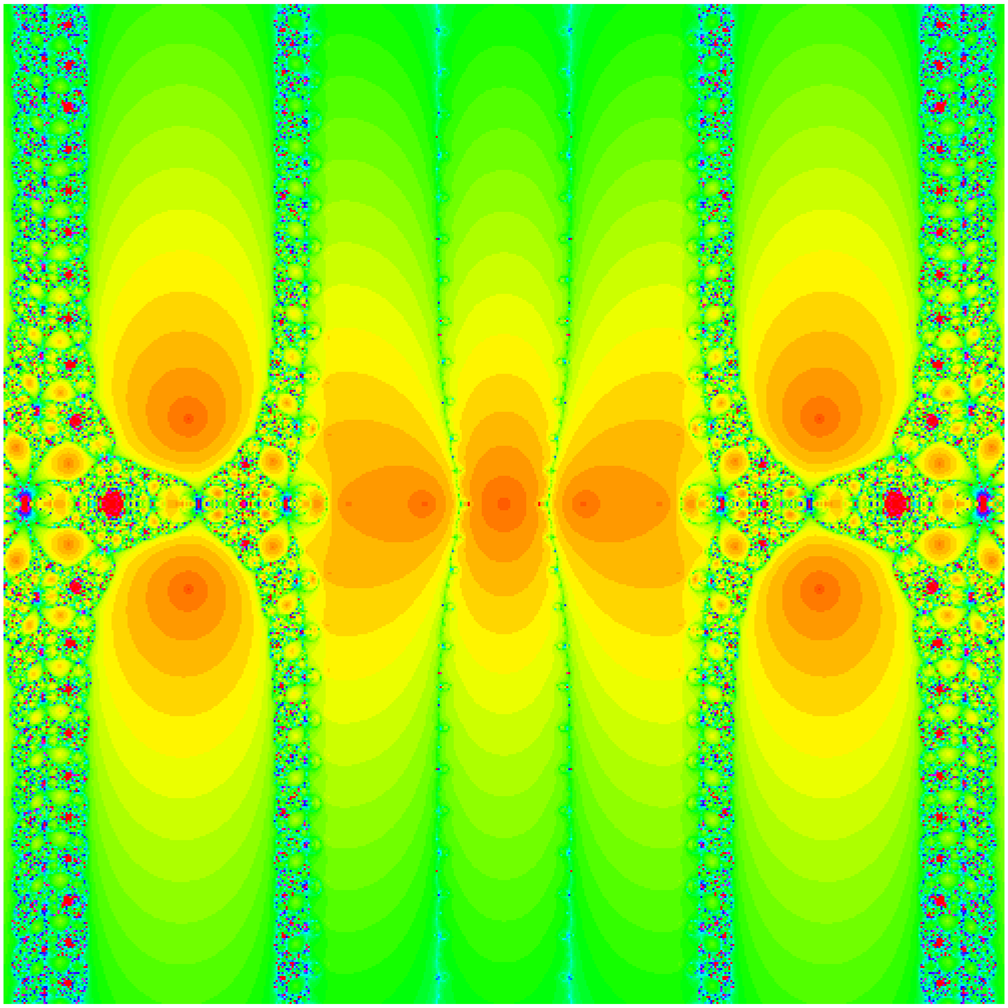

Figure 4. $f(z) = \sin(2z) - z$, $(bl, ur) = (-6-6i, 6+6i)$

$$\pi = \int_0^\infty \sin(\alpha x) \ln\left|\frac{x+2}{x-2}\right| dx \quad (4)$$

$$\sin(2\alpha) - \alpha = 0 \, , \, \alpha = 0.947747... \quad (5)$$

$$\alpha = \sin\left(2\sin\left(2\sin(2...)\right)\right) \quad (6)$$

3.

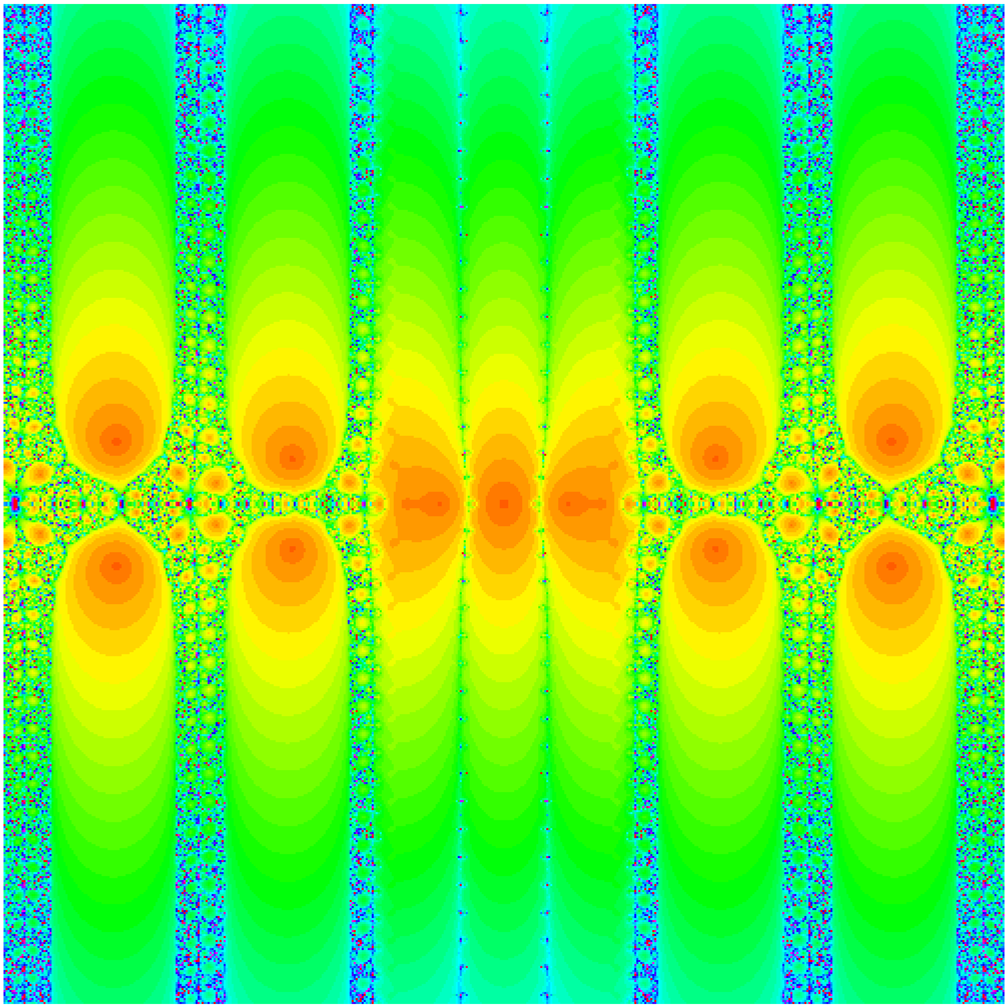

Figure 5. $f(z)=\sin(3z)-z$, $(bl,ur)=(-6-6i,6+6i)$

$$\pi=\int_0^\infty \sin(\alpha x)\ln\left|\frac{x+3}{x-3}\right|dx \tag{7}$$

$$\sin(3\alpha)-\alpha=0\,,\ \alpha=0.759620... \tag{8}$$

4.

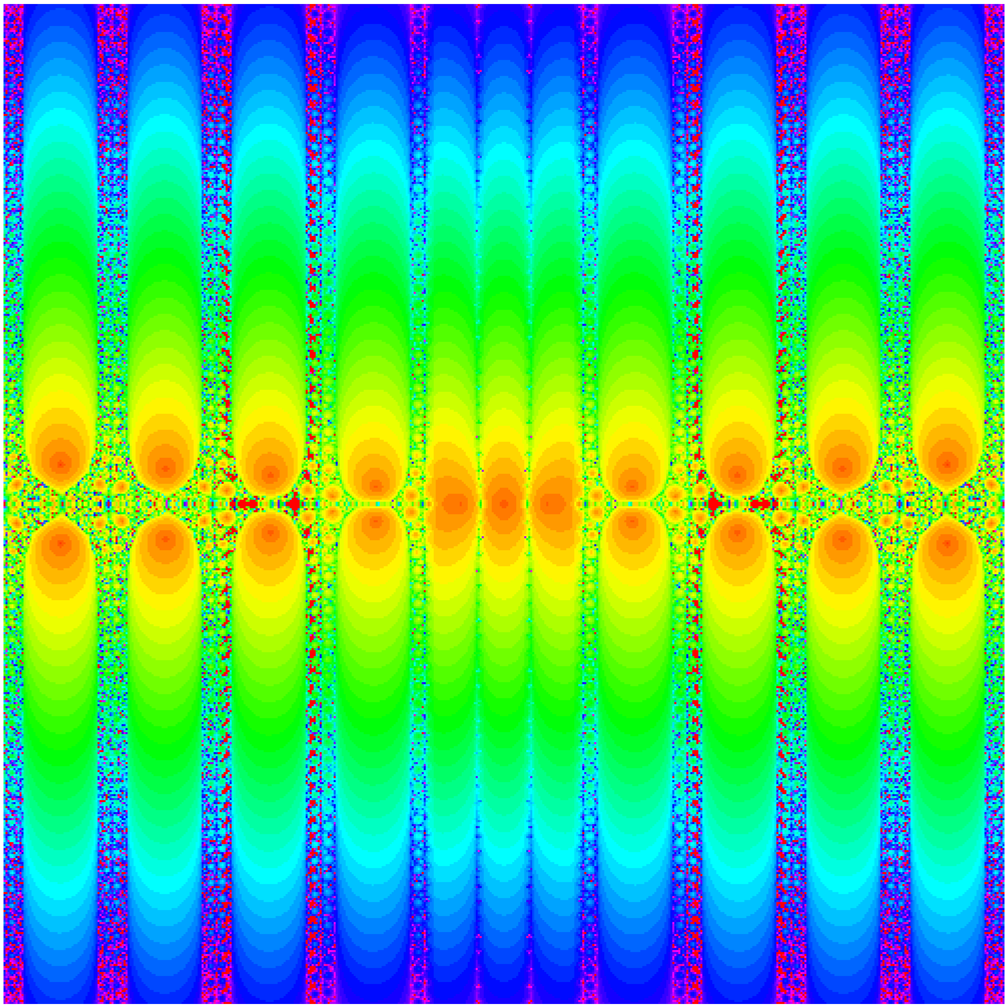

Figure 6. $f(z)=\sin(5z)-z$, $(bl,ur)=(-6-6i,6+6i)$

$$\pi=\int_0^\infty \sin(\alpha x)\ln\left|\frac{x+5}{x-5}\right|dx \qquad (9)$$

$$\sin(5\alpha)-\alpha=0\,,\ \alpha=0.519147... \qquad (10)$$

5.

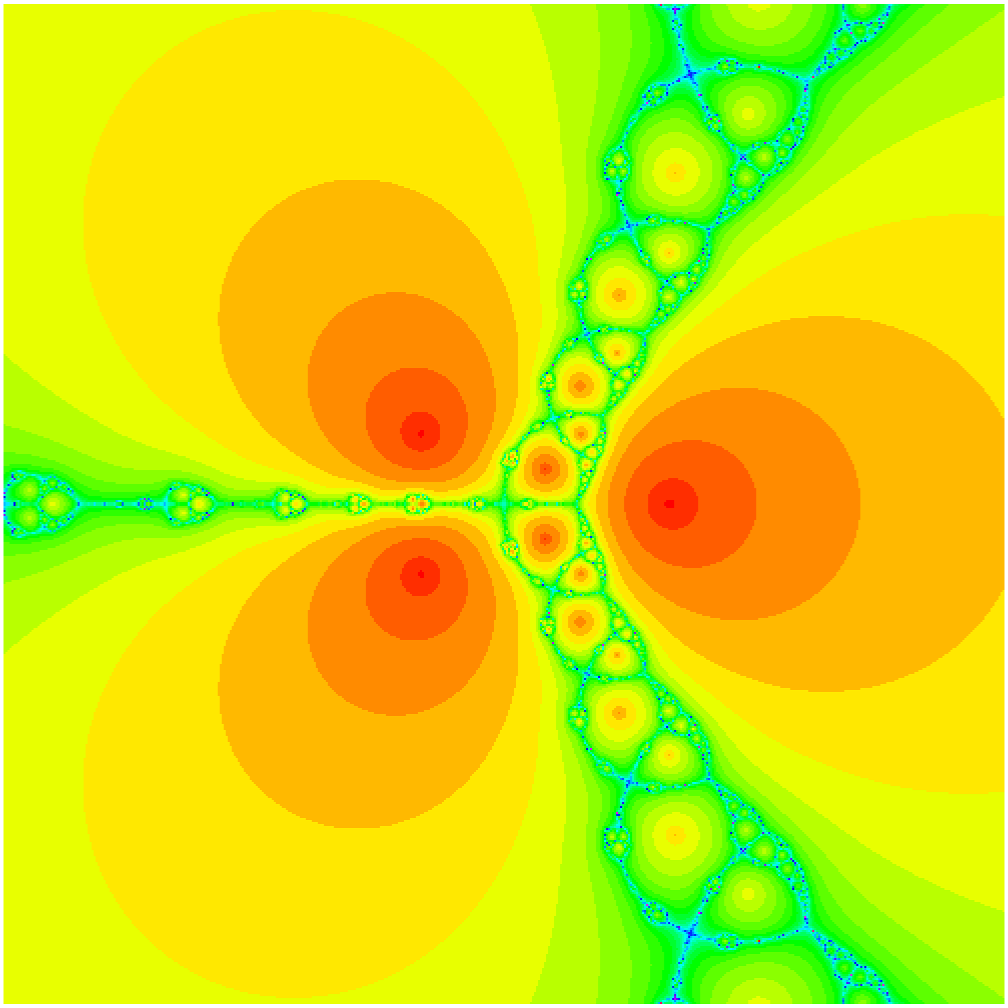

Figure 7. $f(z) = z^3 - z - 1$, $(bl, ur) = (-4-4i, 4+4i)$

$$\pi = \alpha^2 \int_{-1}^{1} \frac{4x^3 - 3x}{\left(1+\alpha^2 - 2\alpha x\right)\sqrt{1-x^2}}\,dx \quad (11)$$

$$\alpha^3 - \alpha - 1 = 0\,,\ \alpha = 1.324717... \quad (12)$$

$$\alpha = \sqrt[3]{1+\sqrt[3]{1+\sqrt[3]{1+...}}} \quad (13)$$

6.

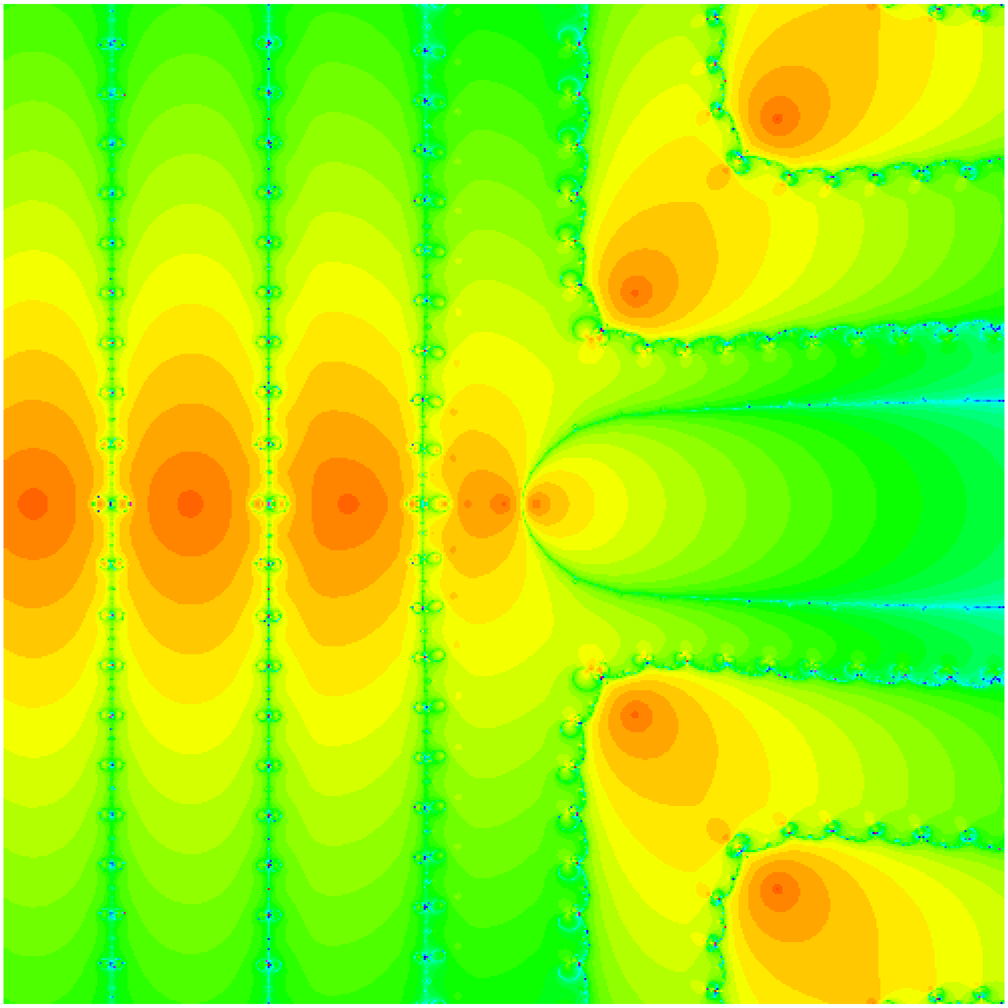

Figure 8. $f(z) = ze^{z} - 2\sin z$, $(bl, ur) = (-10-10i, 10+10i)$

$$\pi = \int_0^{\infty} \sin(\alpha x) \ln\left(\frac{1+(x+1)^2}{1+(x-1)^2}\right) dx \tag{14}$$

$$\alpha e^{\alpha} - 2\sin\alpha = 0, \ \alpha = 0.626790... \tag{15}$$

7.

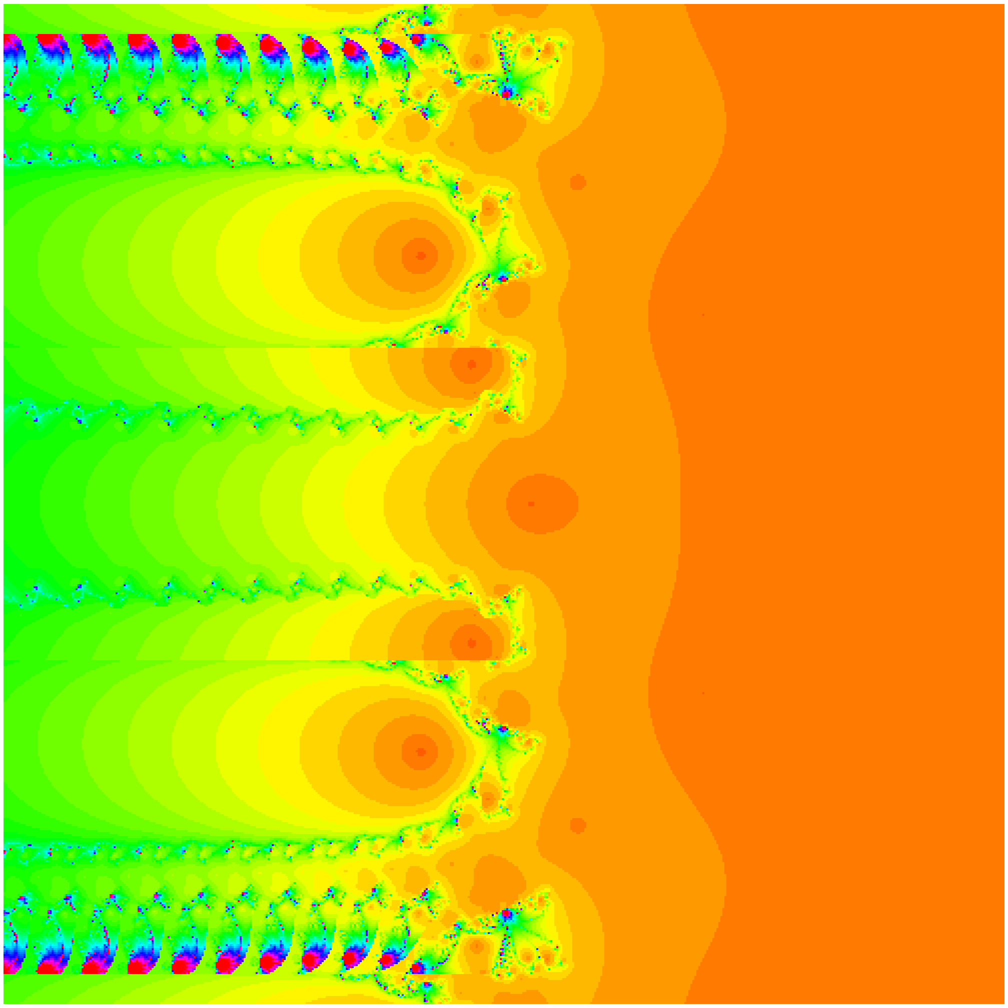

Figure 9. $f(z) = z - \cosh(z)\ln\left(1 + e^{-z}\right), (bl, ur) = (-10 - 10i, 10 + 10i)$

$$\pi = \int_0^\infty \frac{\cos x}{\alpha^2 + x^2} \ln(2 + 2\cos x)\,dx \tag{16}$$

$$\alpha - \cosh(\alpha)\ln\left(1 + e^{-\alpha}\right) = 0,\ \alpha = 0.529486... \tag{17}$$

8.

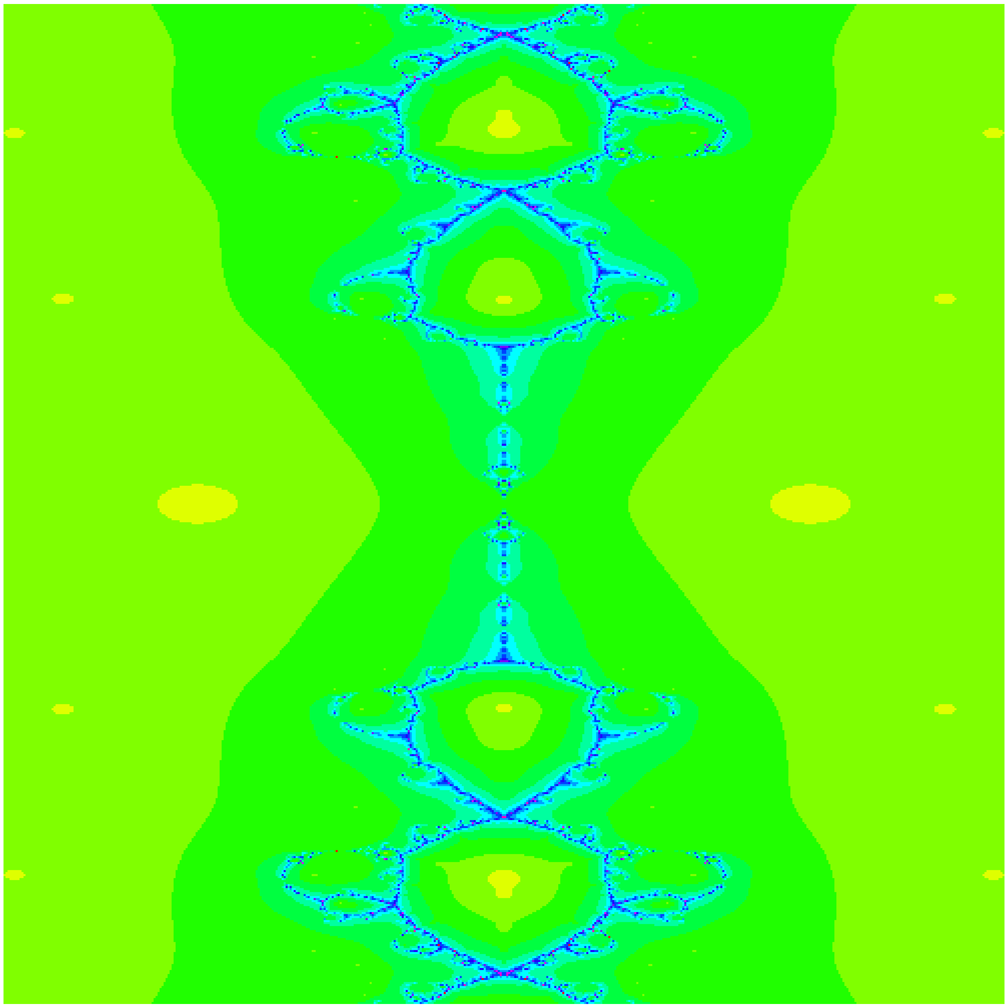

Figure 10. $f(z) = z\left(1 + \frac{2}{e^{2z}-1}\right) - 3$, $(bl, ur) = (-5-10i, 5+10i)$

$$\frac{1}{\pi} = \int_0^\infty \left(\frac{\sin(\alpha x)}{\sinh(\pi x)} \right)^2 dx \qquad (18)$$

$$\alpha\left(1 + \frac{2}{e^{2\alpha}-1}\right) - 3 = 0, \ \alpha = 2.984704... \qquad (19)$$

9.

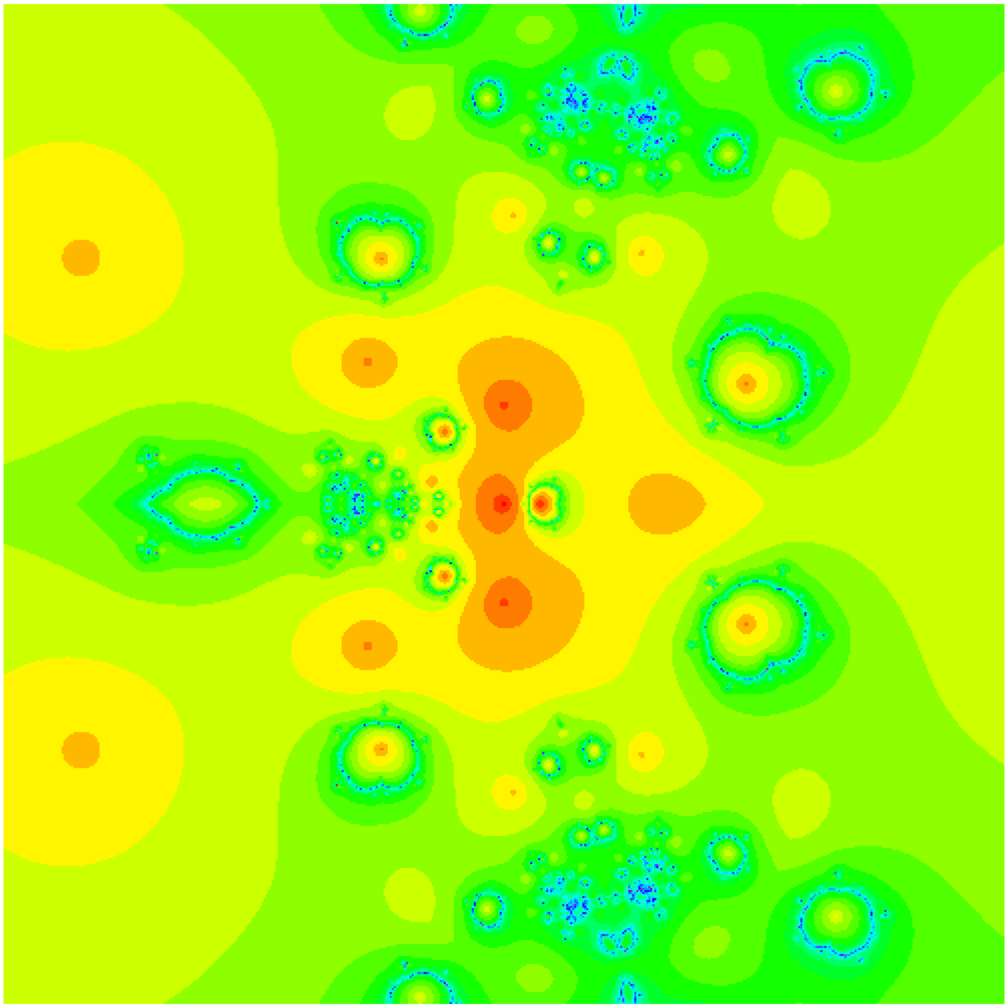

Figure 11. $f(z) = z + \ln(1 - z^2), (bl, ur) = (-10 - 10i, 10 + 10i)$

$$\frac{1}{\pi} = \int_0^\infty J_1(\alpha x) N_0(x) dx \quad (20)$$

$$\alpha + \ln(1 - \alpha^2) = 0, \alpha = 0.714556... \quad (21)$$

$$\alpha = \sqrt{1 - e^{-\sqrt{1 - e^{-\sqrt{1 - e^{-\cdots}}}}}} \quad (22)$$

10.

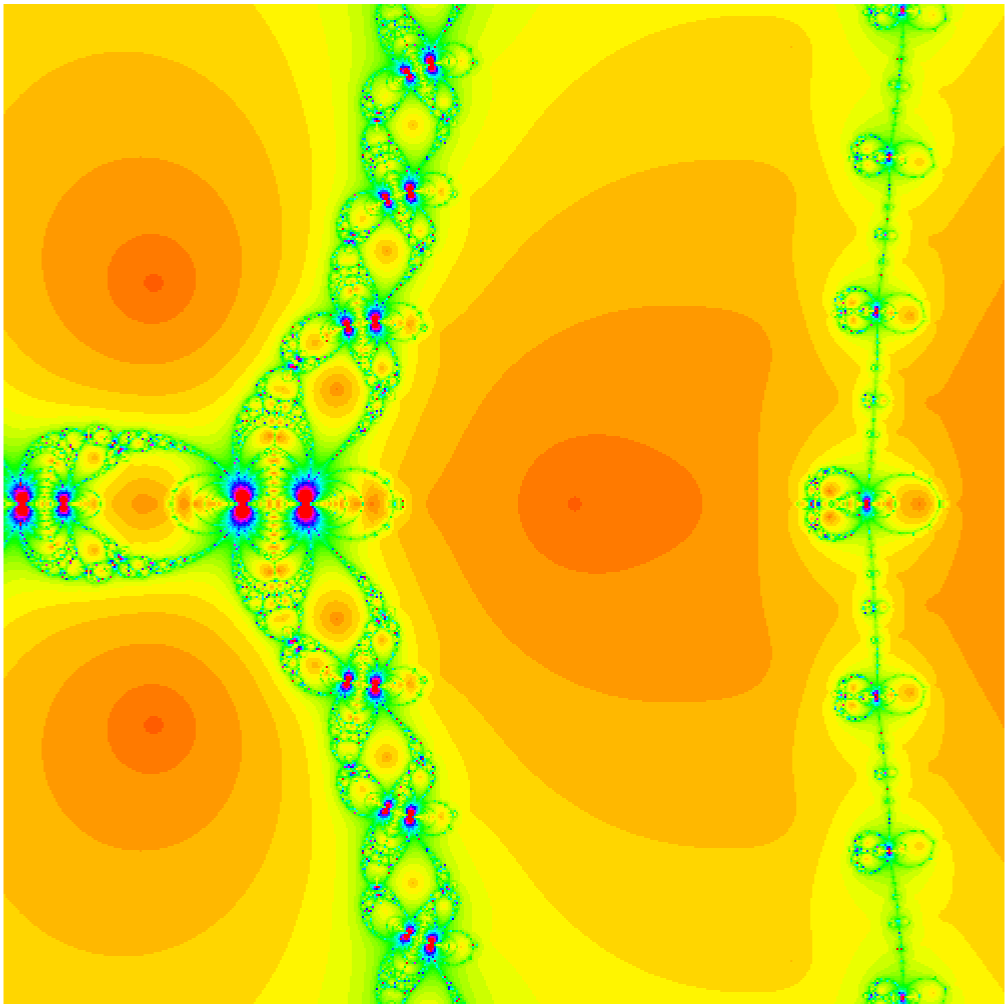

Figure 12. $f(z) = z + \sin z + z\sin z - 1, (bl, ur) = (-3-3i, 3+3i)$

$$\pi = 4\tan^{-1}(\alpha) + 4\tan^{-1}(\sin\alpha) \tag{23}$$

$$f(\alpha) = 0 \quad , \alpha = 0.420362... \tag{24}$$

11.

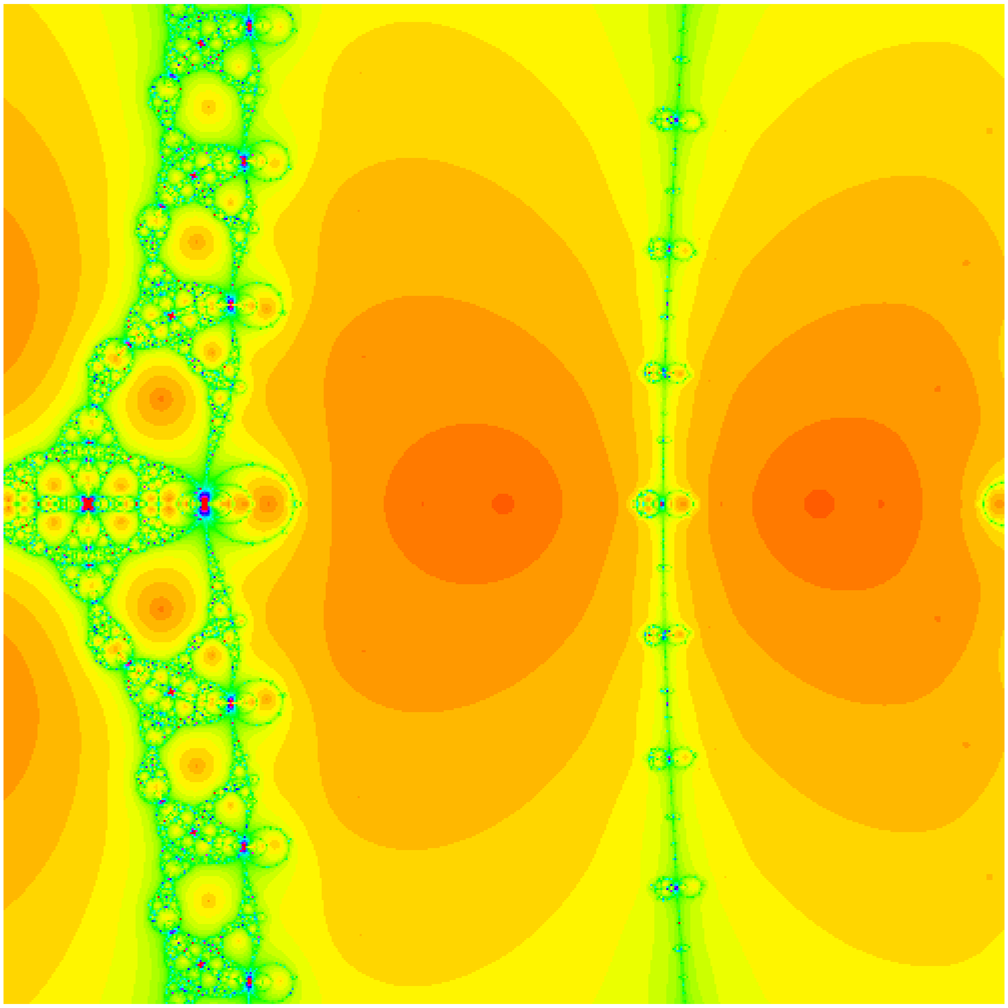

Figure 13. $f(z) = z + \cos z + z\cos z - 1, (bl, ur) = (-3-3i, 3+3i)$

$$\pi = 4\tan^{-1}(\alpha) + 4\tan^{-1}(\cos\alpha) \tag{25}$$

$$f(\alpha) = 0, \alpha = 1.881764... \tag{26}$$

12.

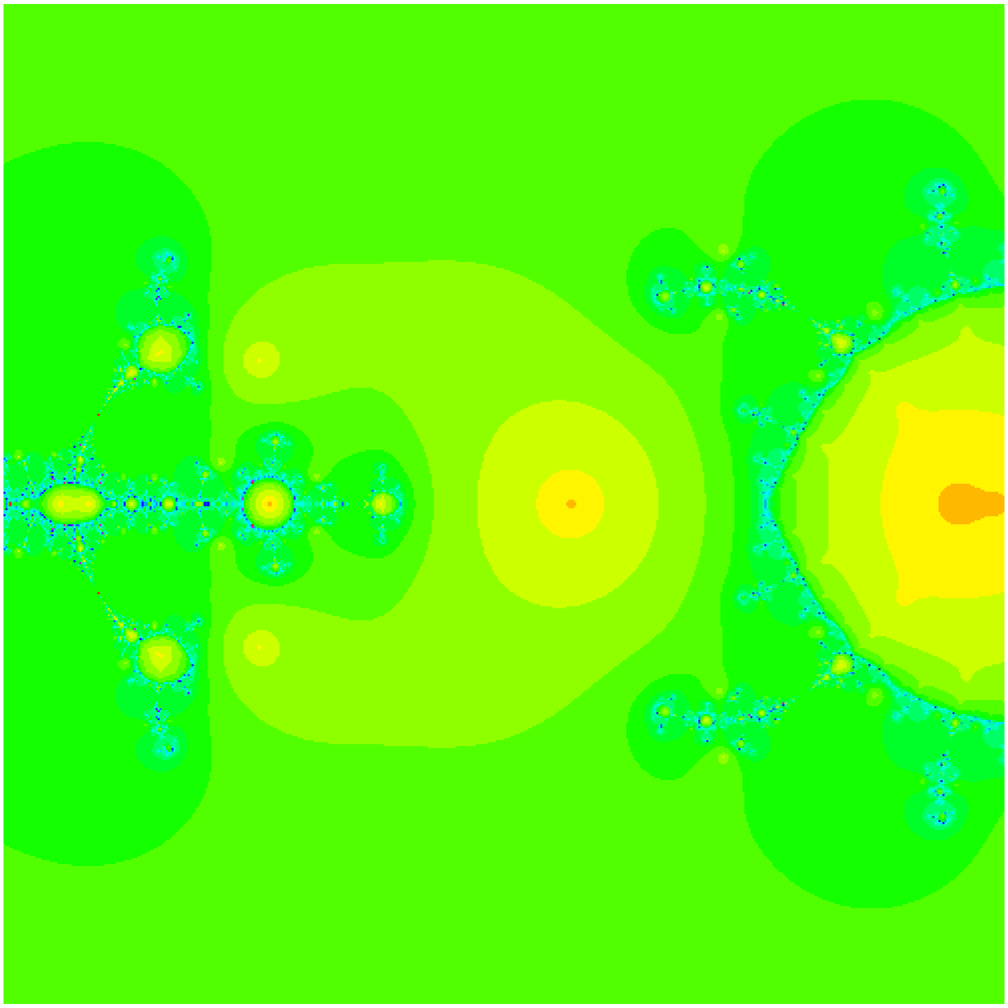

Figure 14. $f(z) = z + \tan z + z\tan z - 1$, $(bl, ur) = (-3-3i, 3+3i)$

$$\pi = 4\alpha + 4\tan^{-1}(\alpha) \tag{27}$$

$$f(\alpha) = 0\,,\ \alpha = 0.402628... \tag{28}$$

13.

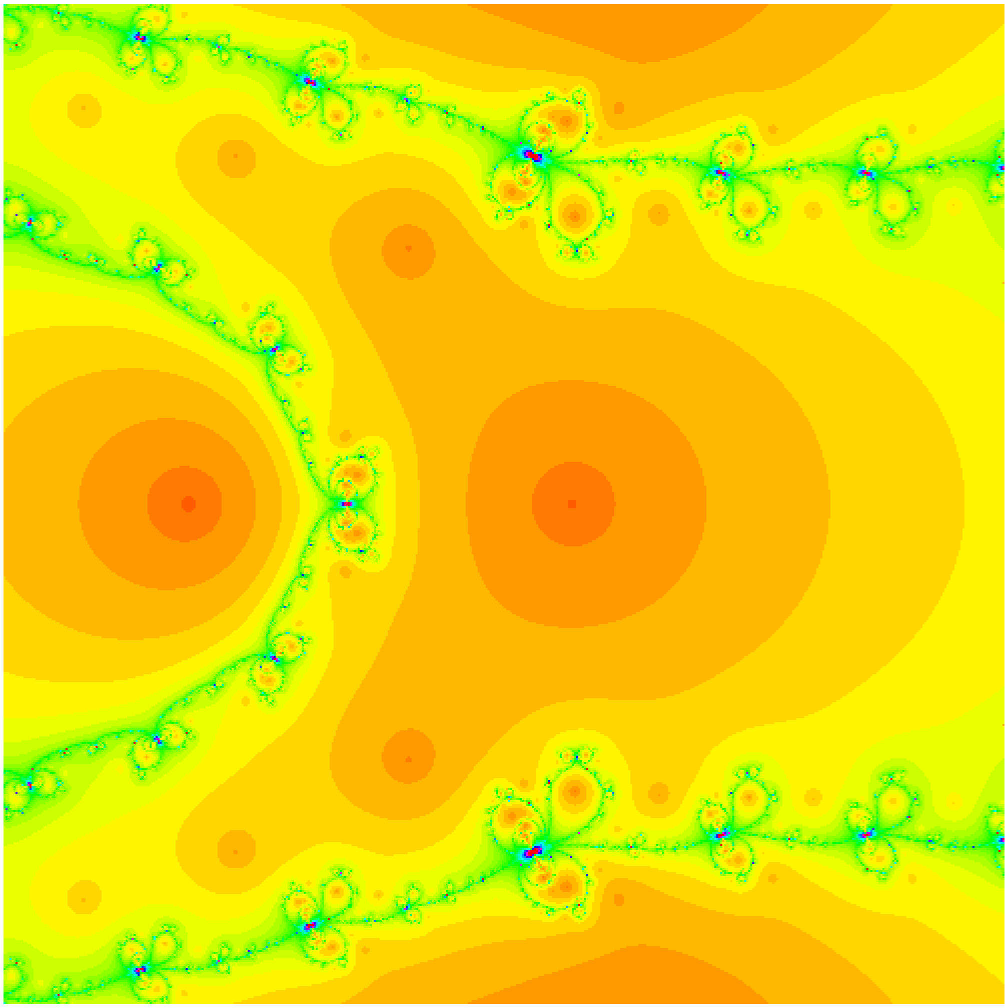

Figure 15. $f\left(z\right)=z+\sinh z+z\sinh z-1$, $\left(bl,ur\right)=\left(-3-3i,3+3i\right)$

$$\pi=4\tan^{-1}\left(\alpha\right)+4\tan^{-1}\left(\sinh\alpha\right) \tag{29}$$

$$f\left(\alpha\right)=0\,,\ \alpha=0.408497... \tag{30}$$

14.

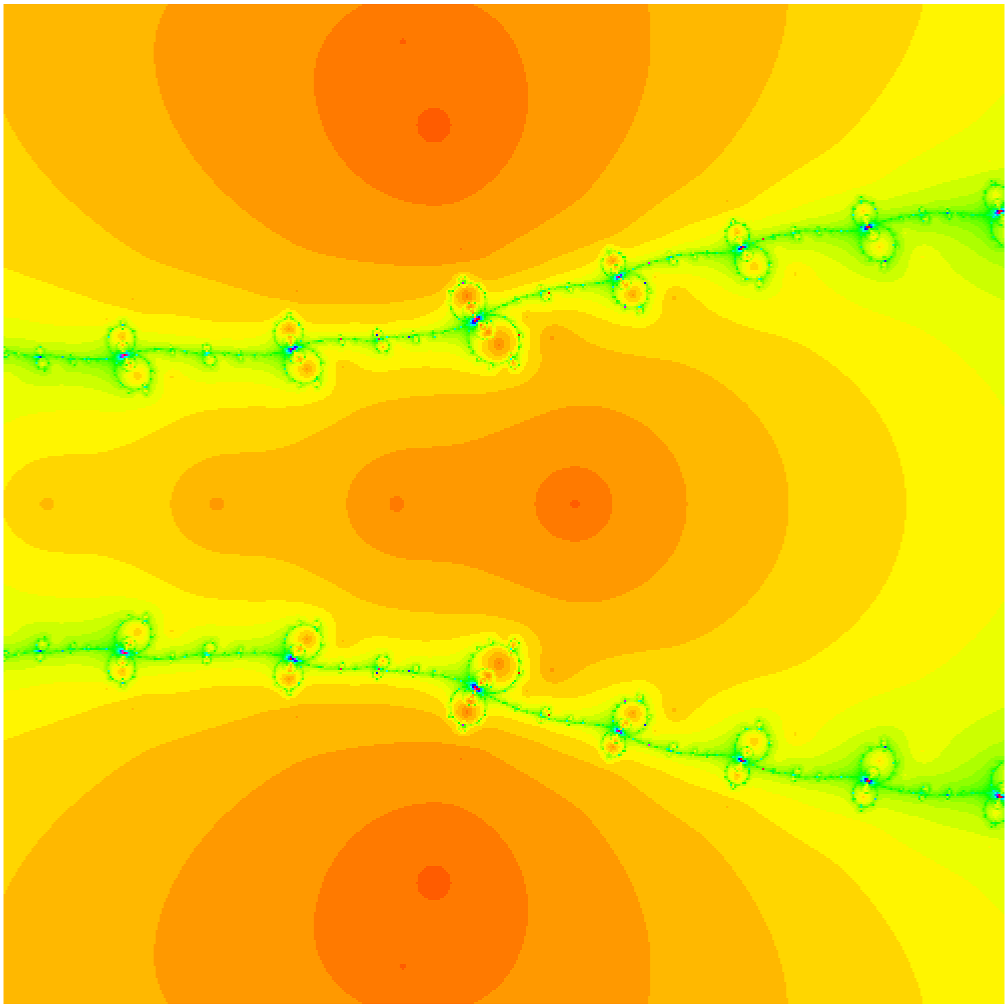

Figure 16. $f(z) = ze^{z} - e^{-z}$, $(bl, ur) = (-3-3i, 3+3i)$

$$\pi = 4\tan^{-1}(\alpha) + 4\tan^{-1}(\tanh\alpha) \quad (31)$$

$$f(\alpha) = 0\,,\ \alpha = 0.426302... \quad (32)$$

$$\alpha = e^{-2e^{-2e^{-2...}}} \quad (33)$$

15.

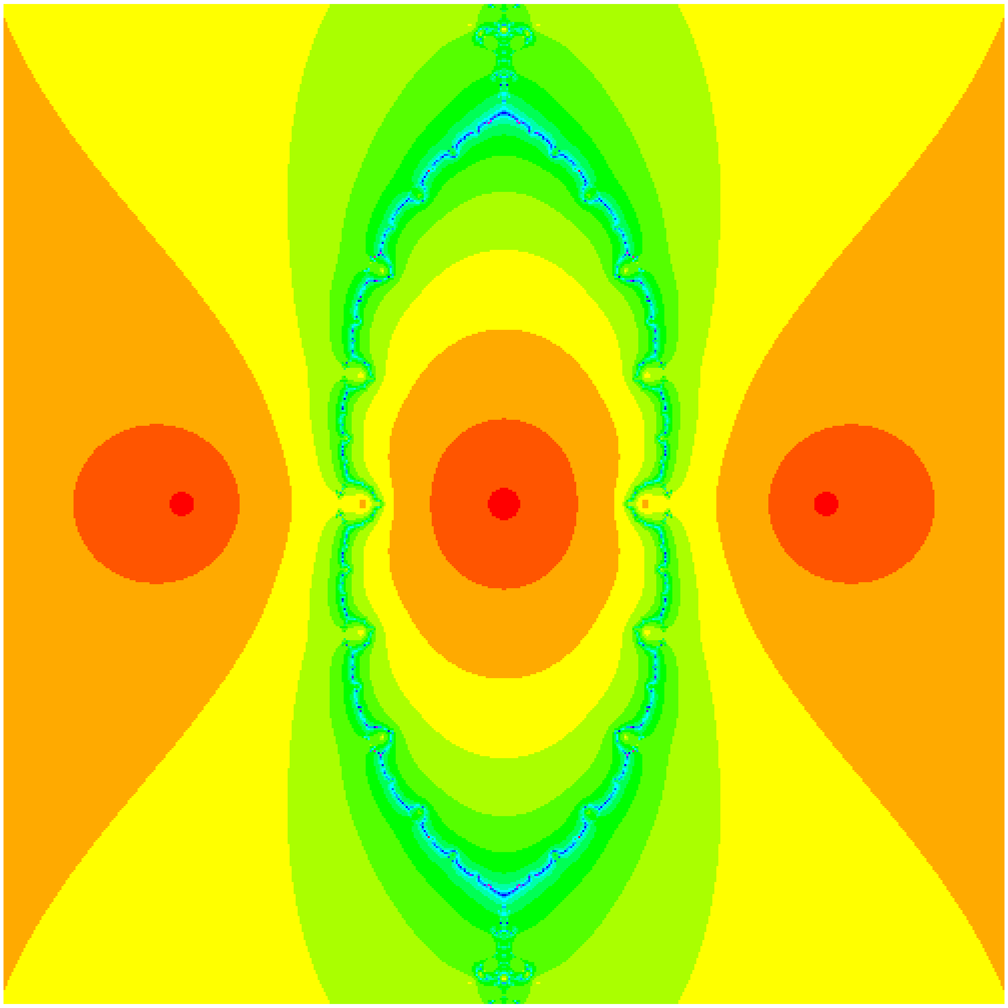

Figure 17. $f(z) = z - 3\tanh\left(\frac{z}{2}\right), (bl, ur) = (-4-4i, 4+4i)$

$$\pi = 4\tan^{-1}\left(\frac{\alpha}{3}\right) + 4\tan^{-1}\left(e^{-\alpha}\right) \qquad (34)$$

$$f(\alpha) = 0\,,\ \alpha = 2.575678... \qquad (35)$$

16.

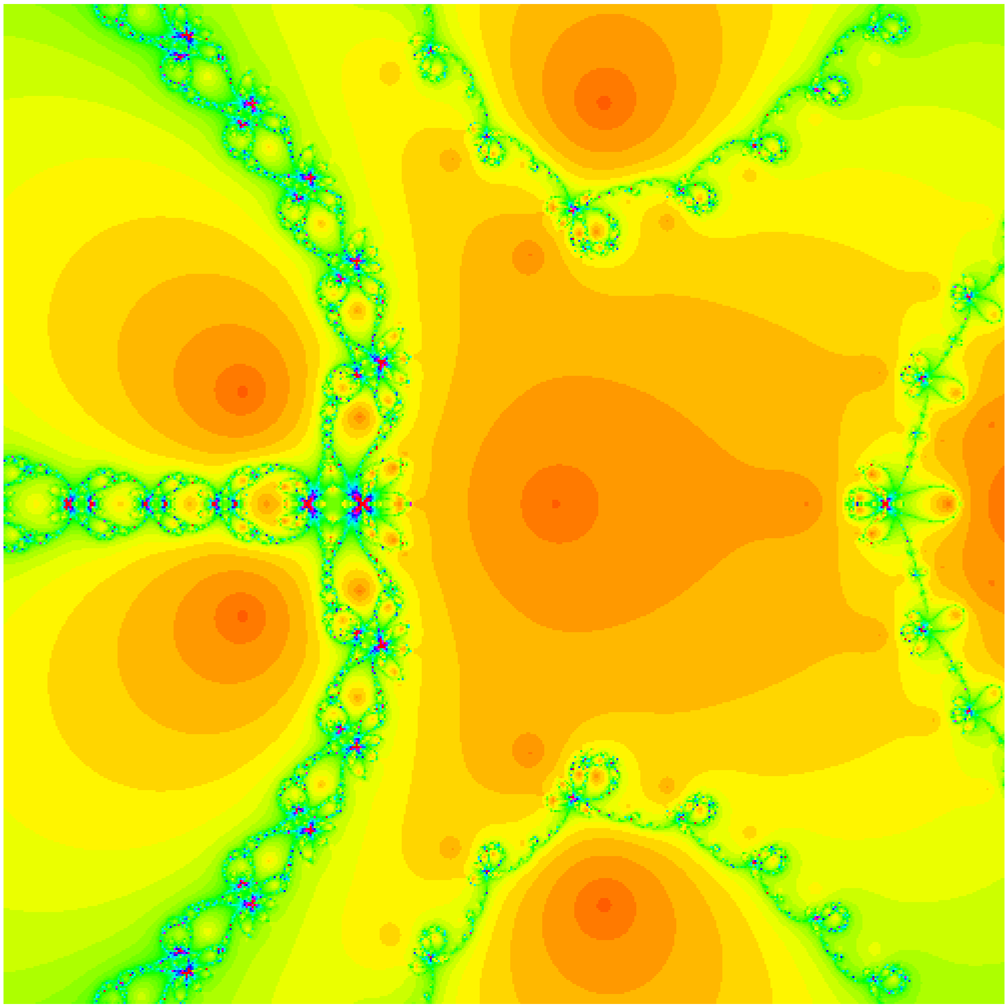

Figure 18. $f(z)=\sin z+\sinh z+\sin z\sinh z-1$, $(bl,ur)=(-4-4i,4+4i)$

$$\pi=4\tan^{-1}(\sin\alpha)+4\tan^{-1}(\sinh\alpha) \tag{36}$$

$$f(\alpha)=0\,,\ \alpha=0.414161... \tag{37}$$

17.

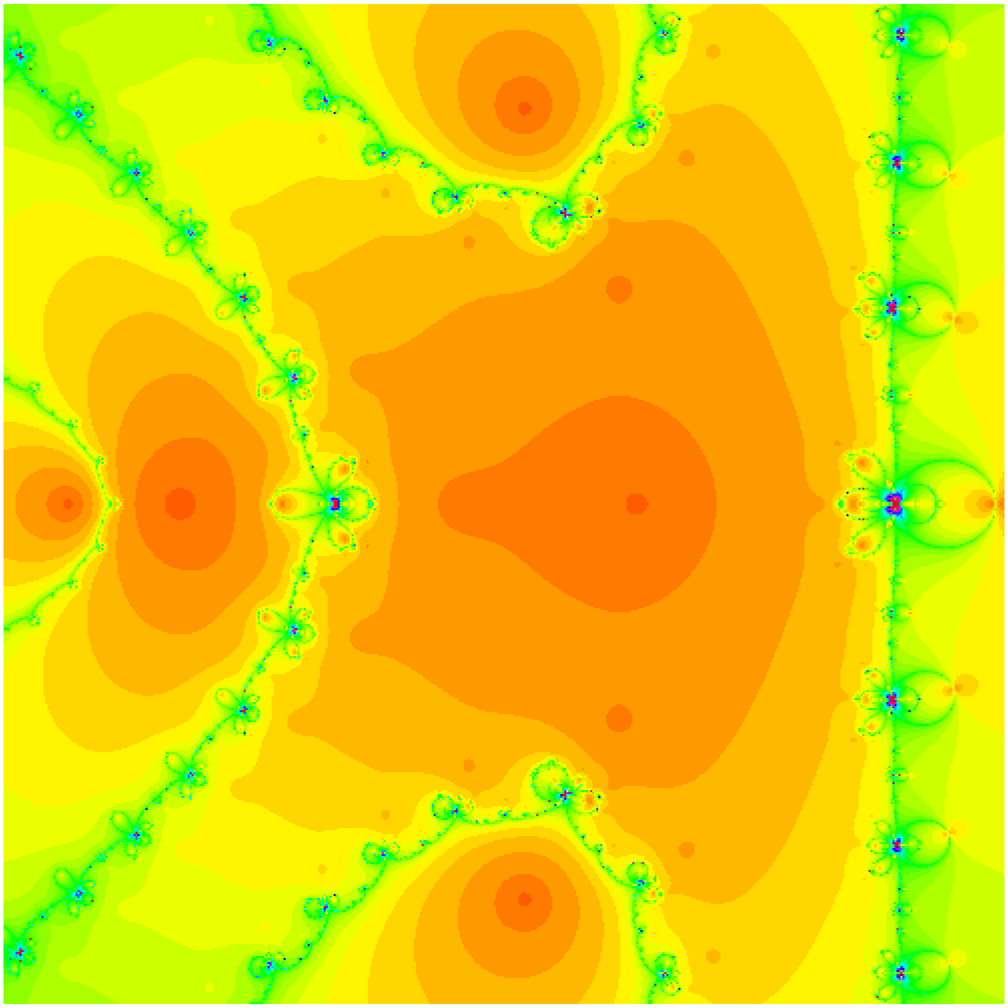

Figure 19. $f\left(z\right)=e^{-z}+\cos z+e^{-z}\cos z-1,\left(bl,ur\right)=\left(-4-4i,4+4i\right)$

$$\pi=4\tan^{-1}\left(e^{-\alpha}\right)+4\tan^{-1}\left(\cos\alpha\right) \tag{38}$$

$$f\left(\alpha\right)=0\,,\,\alpha=1.062781... \tag{39}$$

18.

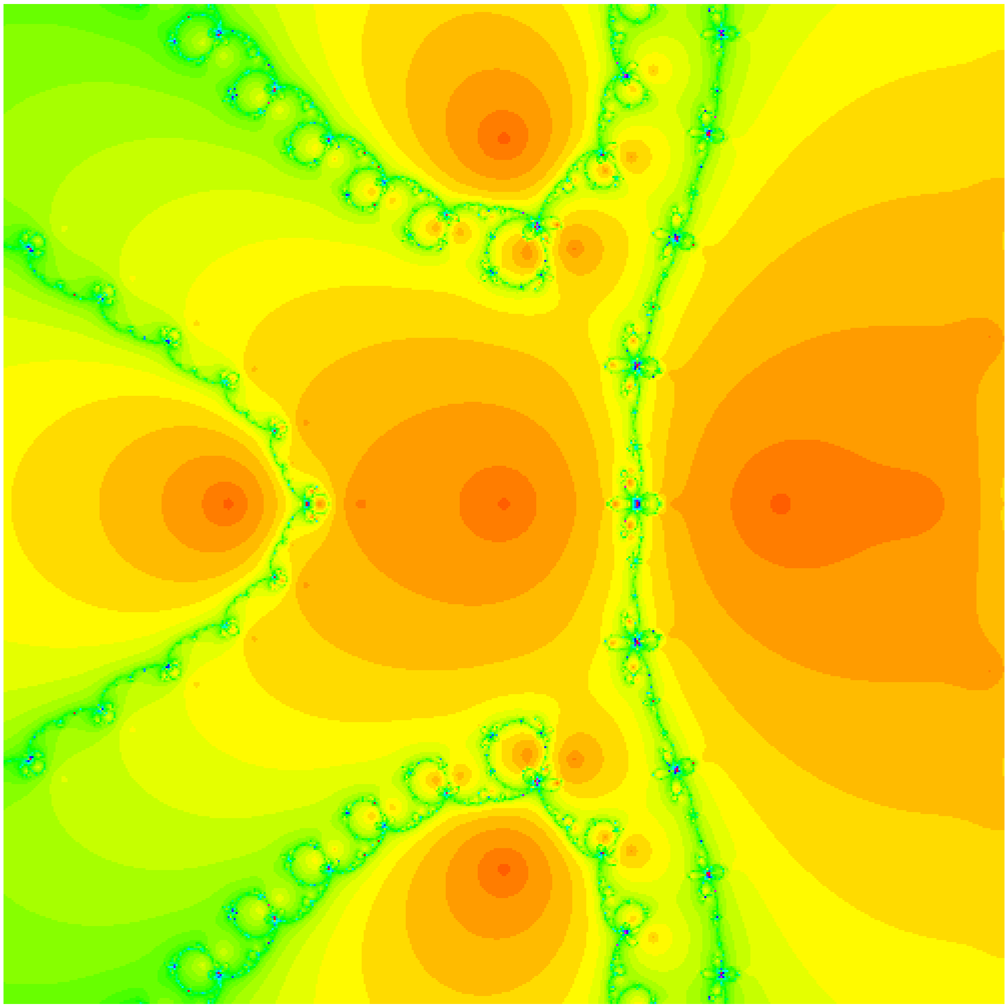

Figure 20. $f(z)=e^{-z}+\sin z+e^{-z}\sin z-1,\ (bl,ur)=(-4-4i,4+4i)$

$$\pi=4\tan^{-1}\left(e^{-\alpha}\right)+4\tan^{-1}\left(\sin\alpha\right) \tag{40}$$

$$f(\alpha)=0\,,\ \alpha=2.210365... \tag{41}$$

19.

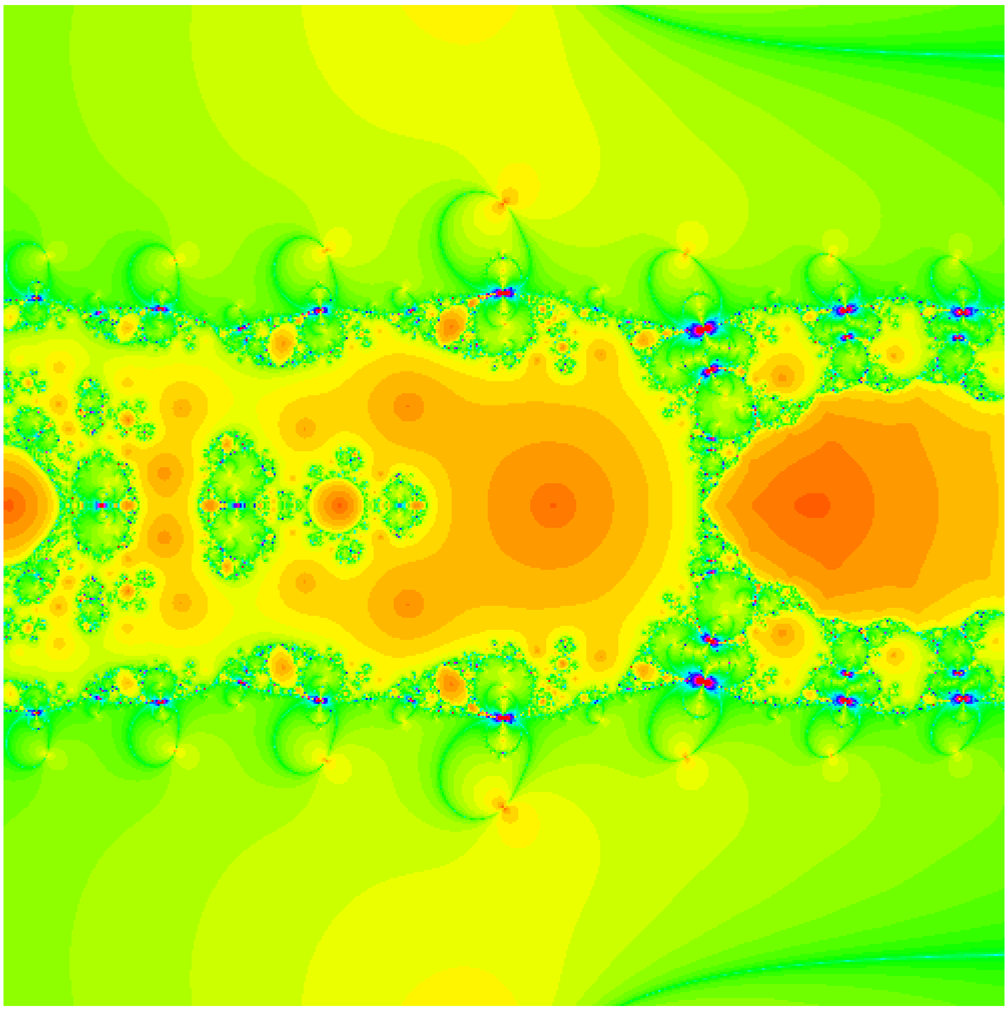

Figure 21. $f(z) = \tan z + \sinh z + \tan z \sinh z - 1$, $(bl, ur) = (-4-4i, 4+4i)$

$$\pi = 4\alpha + 4\tan^{-1}(\sinh\alpha) \quad (42)$$

$$f(\alpha) = 0 \,,\; \alpha = 0.397744... \quad (43)$$

20.

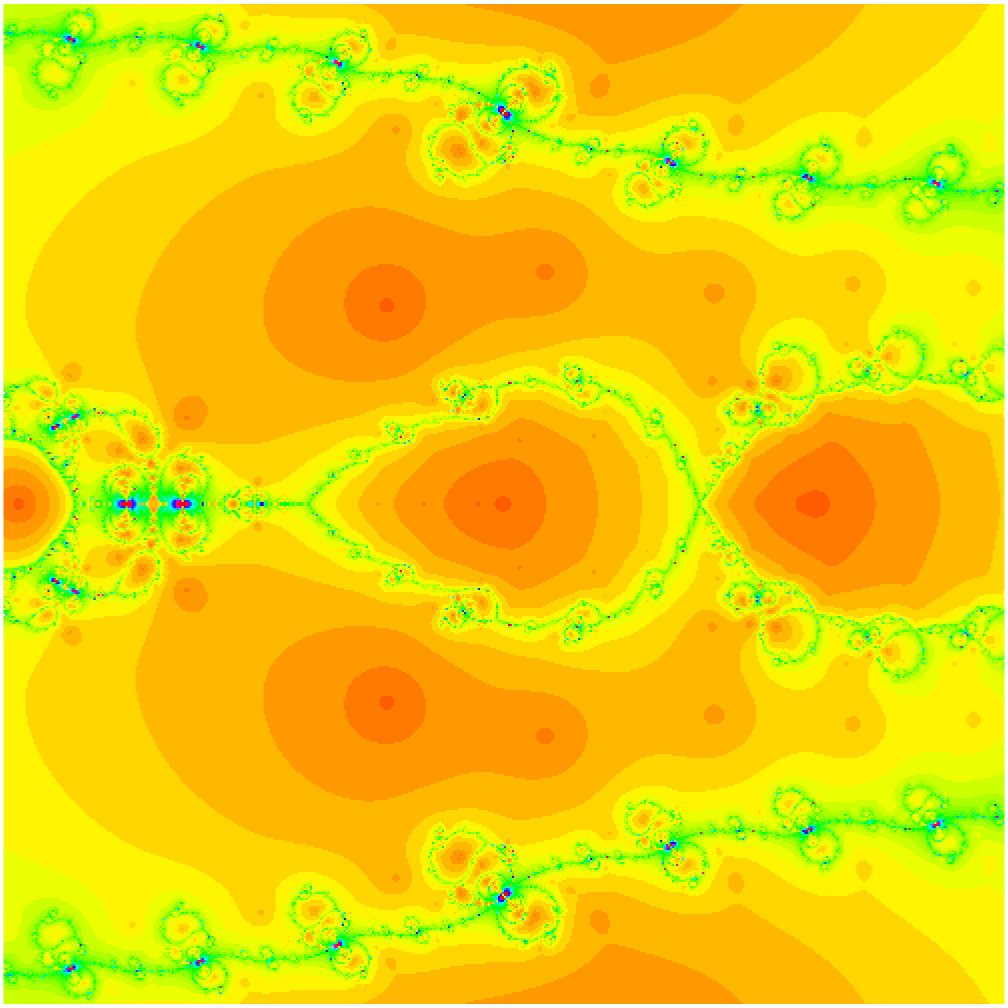

Figure 22. $f(z)=\tan z+\cosh z+\tan z\cosh z-1,\ (bl,ur)=(-4-4i,4+4i)$

$$\pi=\frac{4}{5}\alpha+\frac{4}{5}\tan^{-1}(\cosh\alpha) \tag{44}$$

$$f(\alpha)=0\ ,\ \alpha=2.515440... \tag{45}$$

21.

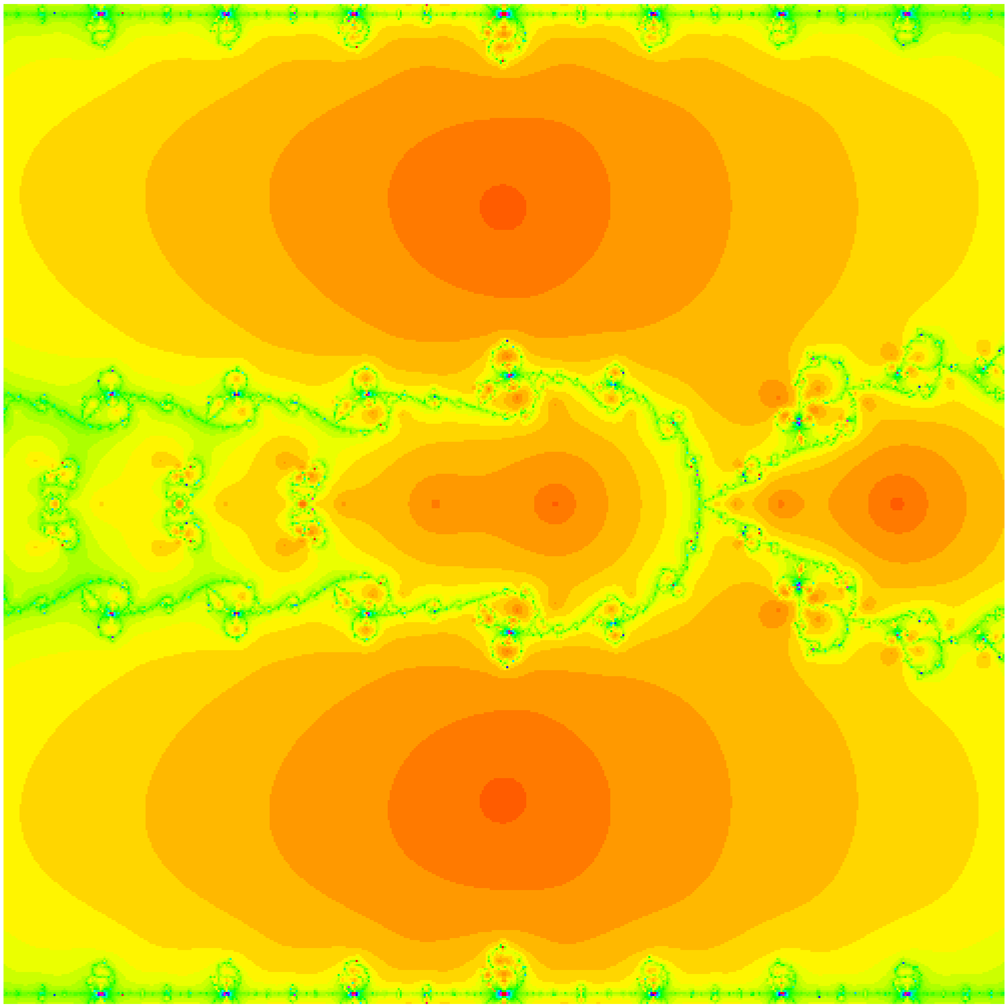

Figure 23. $f(z)=e^{z}\tan z-e^{-z}$, $(bl,ur)=(-4-4i,4+4i)$

$$\pi=4\alpha+4\tan^{-1}(\tanh\alpha) \tag{46}$$

$$f(\alpha)=0\,,\ \alpha=0.412803... \tag{47}$$

22.

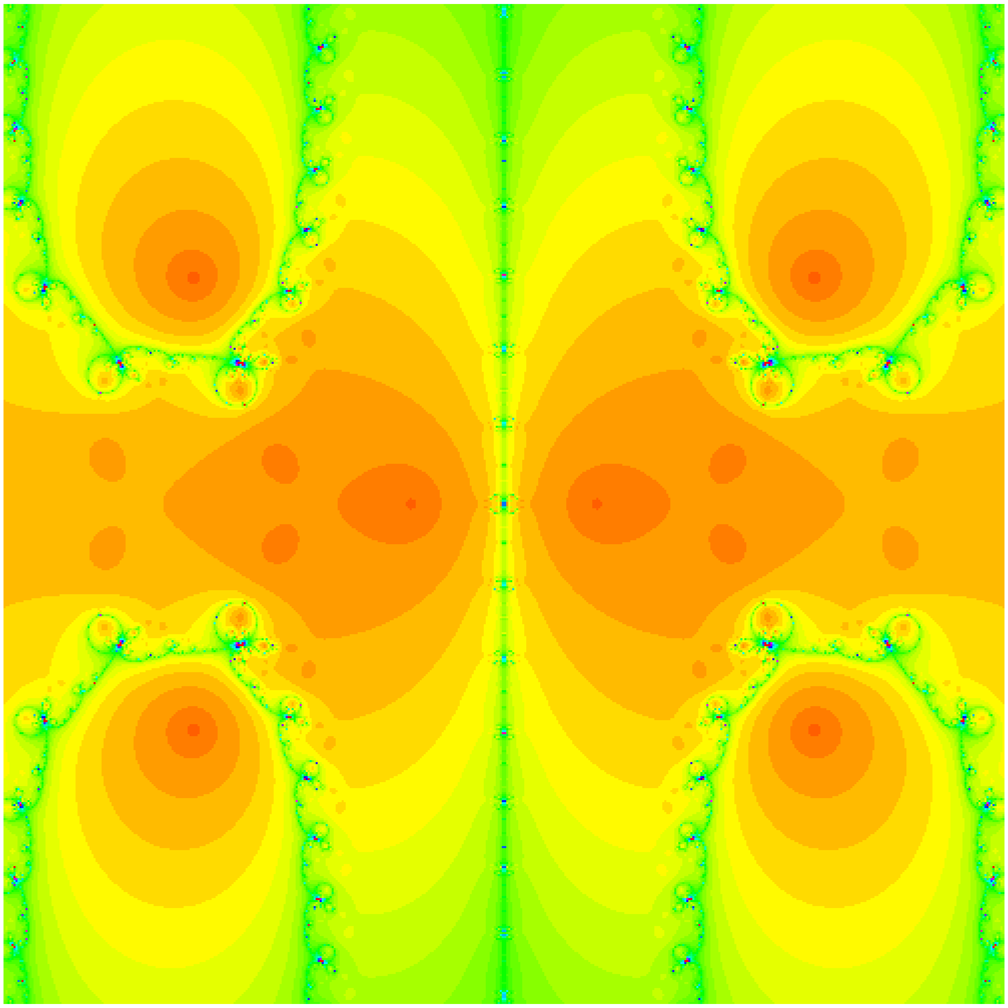

Figure 24. $f(z) = z^2 + (\sin z)^2 - 1, (bl, ur) = (-4-4i, 4+4i)$

$$\pi = 2\alpha + 2\sin^{-1}(\alpha) \tag{48}$$

$$f(\alpha) = 0, \alpha = 0.739085... \tag{49}$$

23.

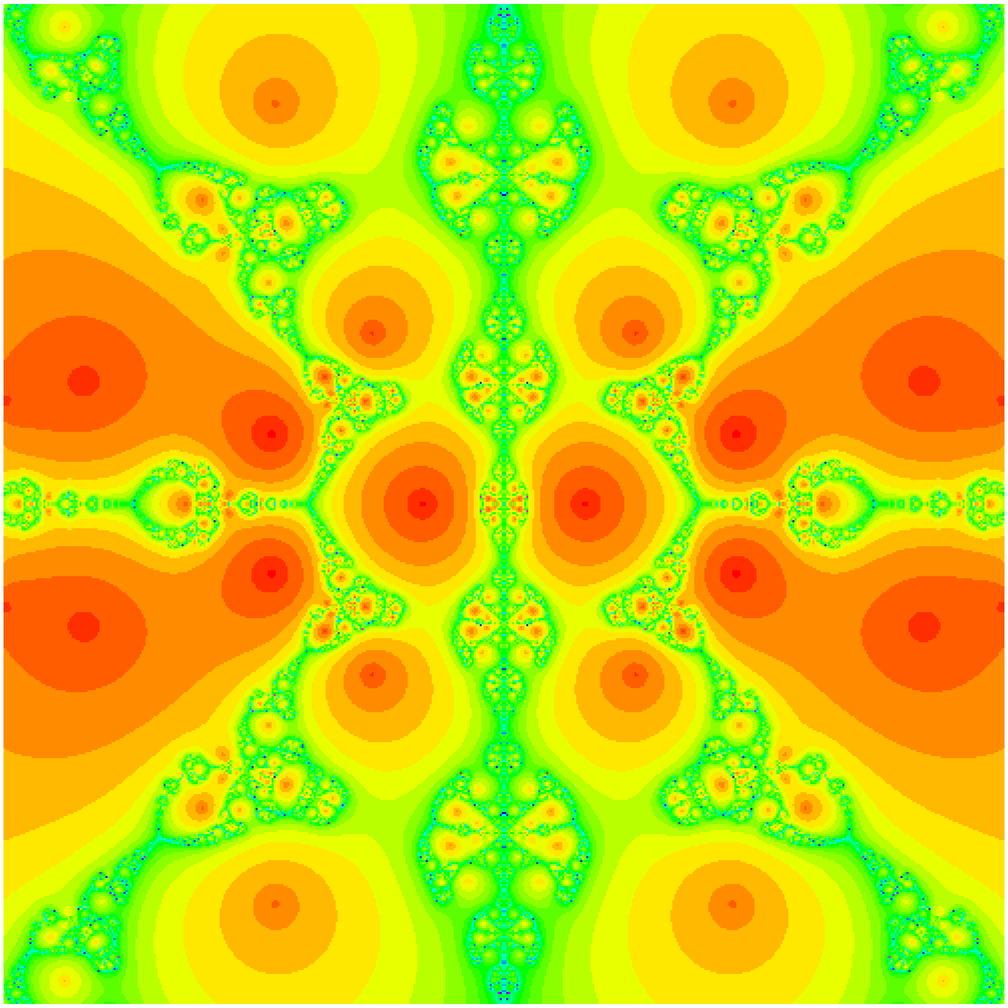

Figure 25. $f(z)=z^2+(\tan z)^2-1$, $(bl,ur)=(-4-4i,4+4i)$

$$\pi=2\sin^{-1}(\alpha)+2\sin^{-1}(\tan\alpha) \quad (50)$$

$$f(\alpha)=0\,,\ \alpha=0.649888... \quad (51)$$

24.

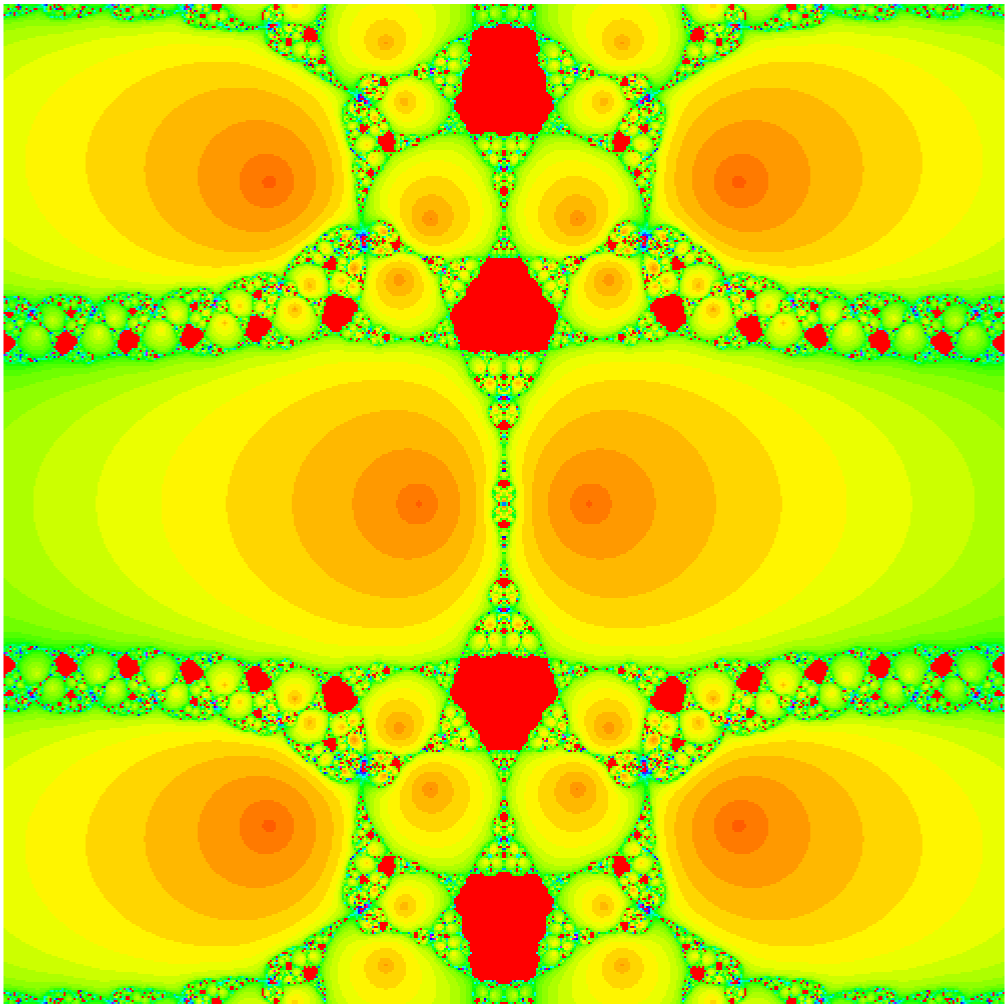

Figure 26. $f(z) = z^2 + (\sinh z)^2 - 1, (bl, ur) = (-4-4i, 4+4i)$

$$\pi = 2\sin^{-1}(\alpha) + 2\sin^{-1}(\sinh \alpha) \quad (52)$$

$$f(\alpha) = 0, \alpha = 0.679807... \quad (53)$$

25.

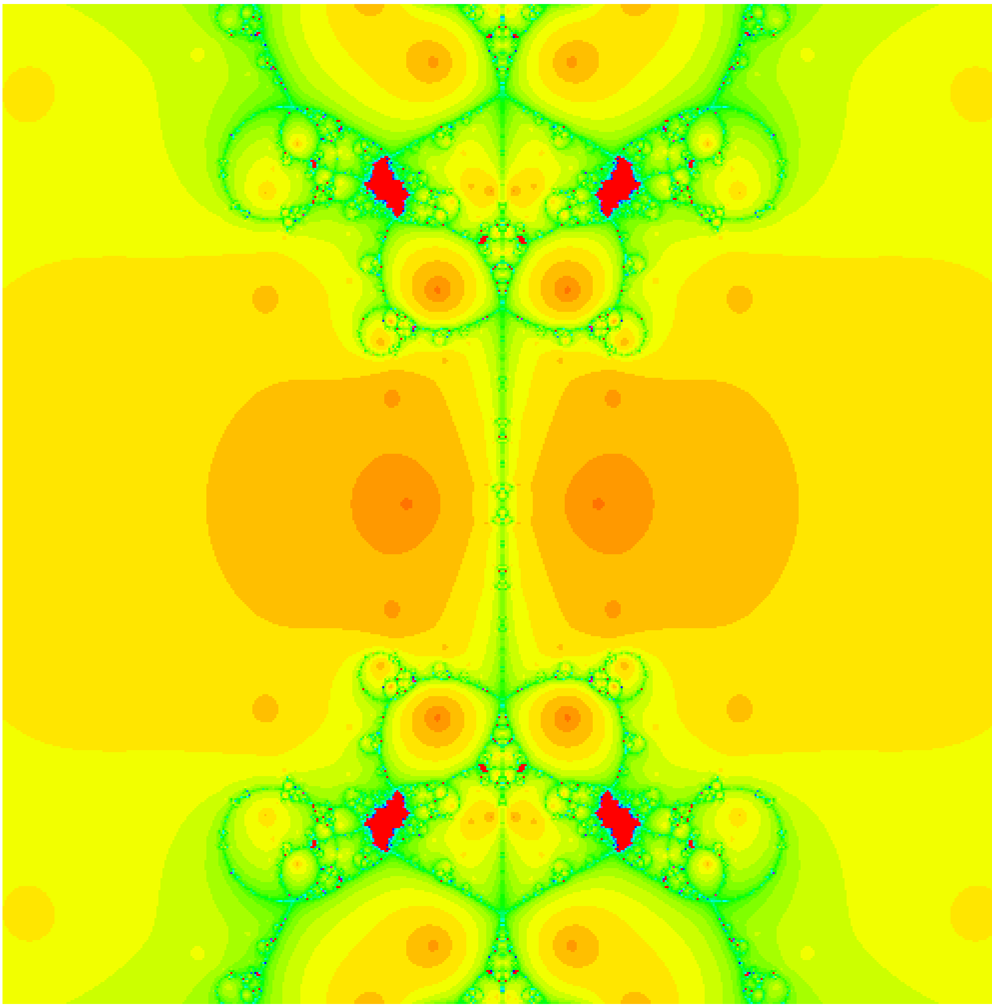

Figure 27. $f(z) = z^2 + (\tanh z)^2 - 1, (bl, ur) = (-4-4i, 4+4i)$

$$\pi = 2\sin^{-1}(\alpha) + 2\sin^{-1}(\tanh\alpha) \tag{54}$$

$$f(\alpha) = 0\,,\ \alpha = 0.765009... \tag{55}$$

26.

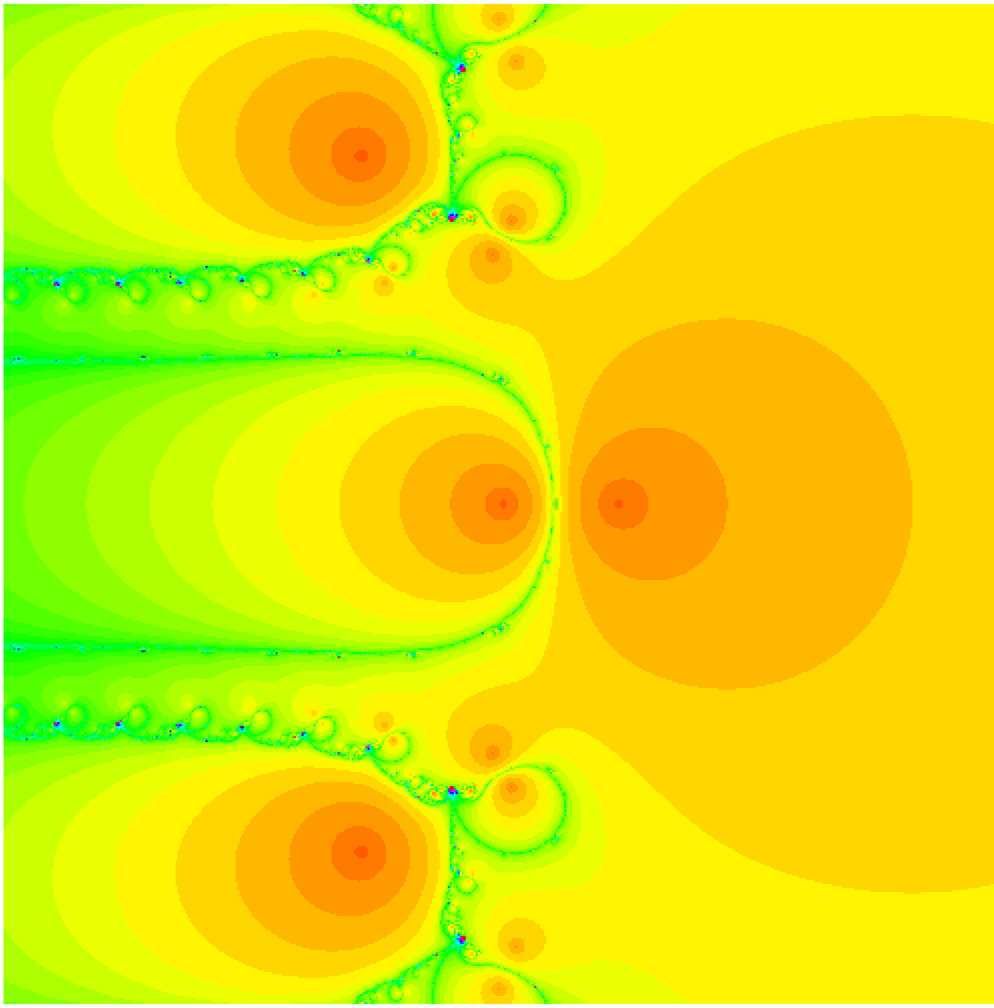

Figure 28. $f\left(z\right)=z^2+e^{-2z}-1, \left(bl,ur\right)=\left(-4-4i,4+4i\right)$

$$\pi=2\sin^{-1}\left(\alpha\right)+2\sin^{-1}\left(e^{-\alpha}\right) \quad (56)$$

$$f\left(\alpha\right)=0\,,\ \alpha=0.916562... \quad (57)$$

27.

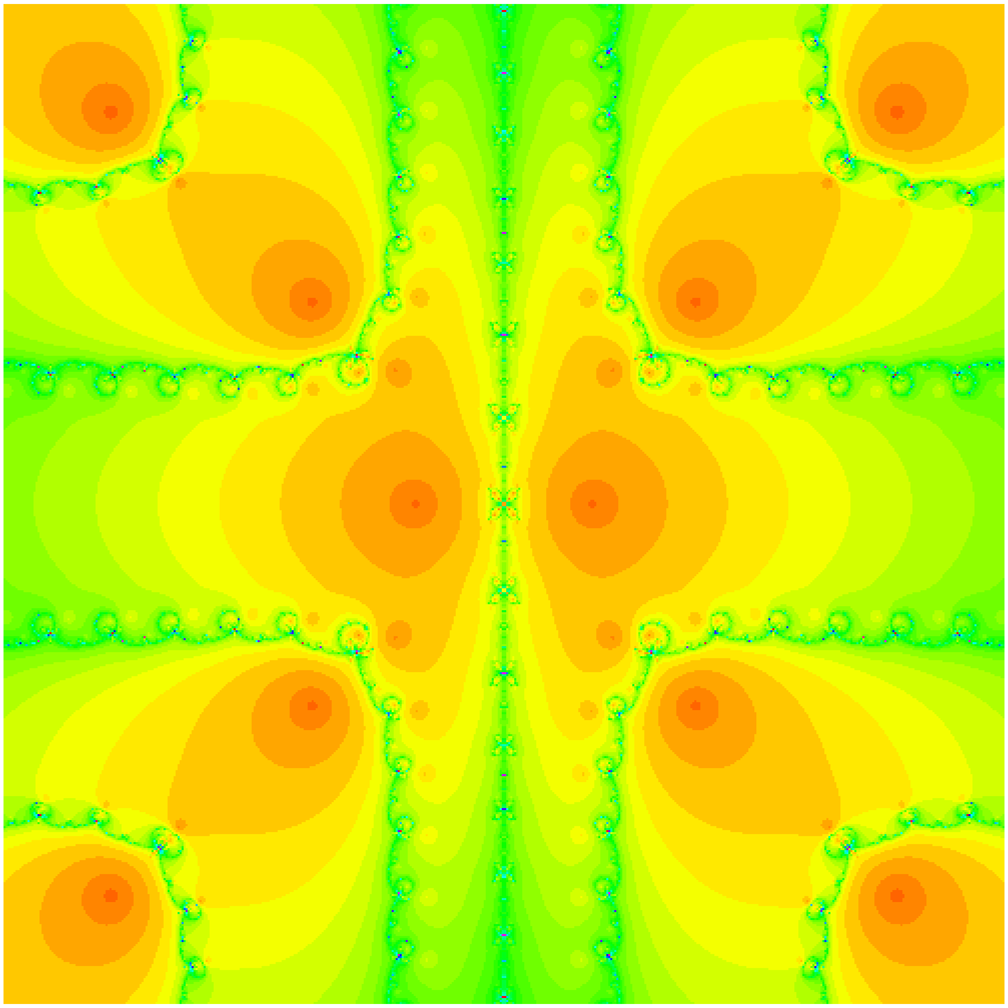

Figure 29. $f\left(z\right)=\left(\sin z\right)^2+\left(\sinh z\right)^2-1$, $\left(bl,ur\right)=\left(-4-4i,4+4i\right)$

$$\pi = 2\alpha + 2\sin^{-1}\left(\sinh\alpha\right) \tag{58}$$

$$f\left(\alpha\right)=0\,,\ \alpha = 0.703290... \tag{59}$$

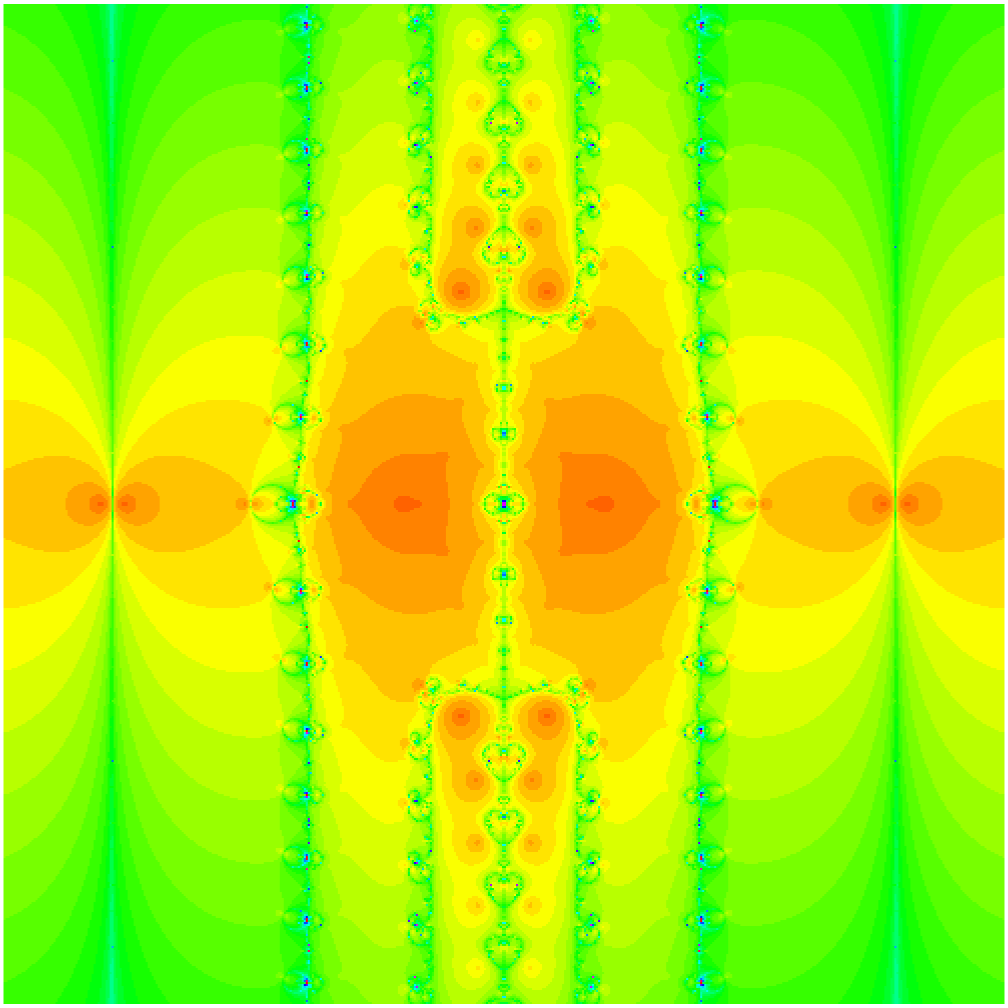

Figure 30. $f(z) = (\sin z)^2 + (\tanh z)^2 - 1, (bl, ur) = (-4-4i, 4+4i)$

$$\pi = 2\alpha + 2\sin^{-1}(\tanh\alpha) \quad (60)$$

$$f(\alpha) = 0, \alpha = 0.825607... \quad (61)$$

29.

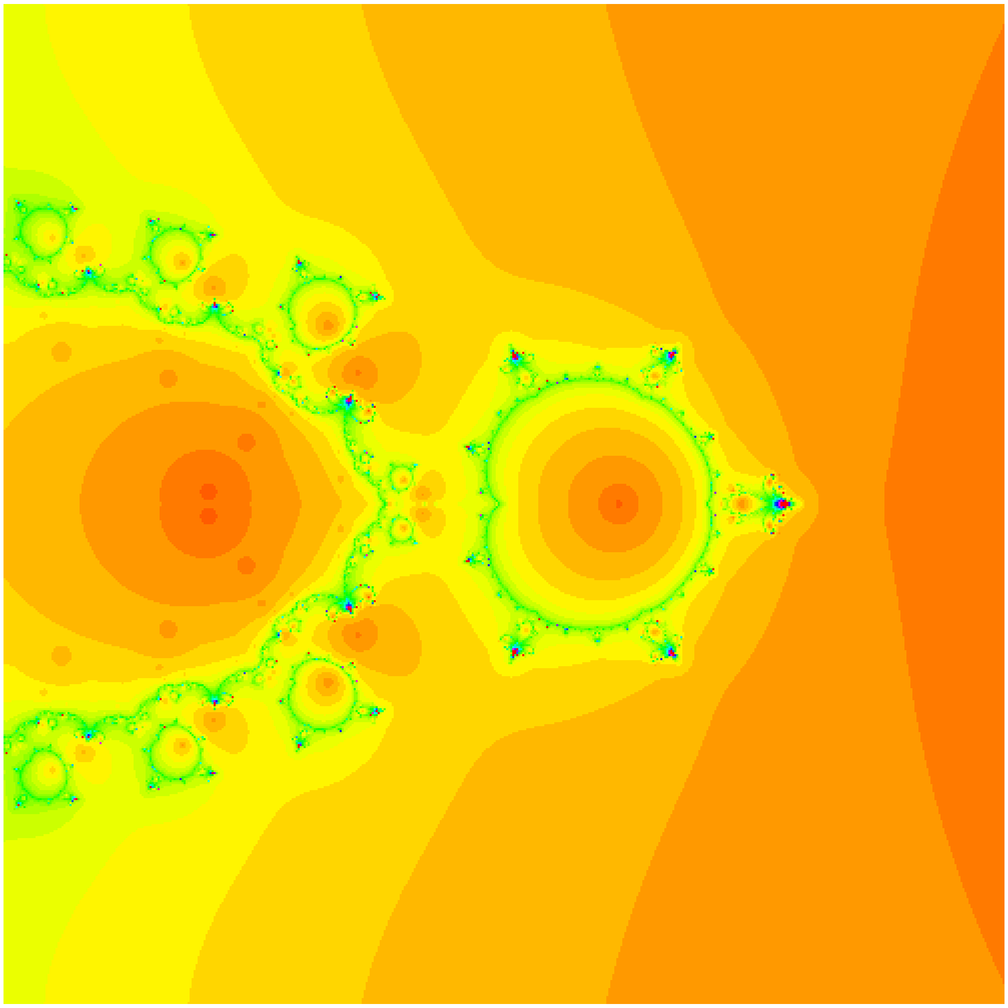

Figure 31. $f(z)=e^{-2z}+(\ln z)^2-1,\ (bl,ur)=(-2-2i,2+2i)$

$$\pi=2\sin^{-1}\left(e^{-2\alpha}\right)-2\sin^{-1}\left(\ln\alpha\right) \tag{62}$$

$$f(\alpha)=0,\ \alpha=0.460363... \tag{63}$$

1. $f\left(x\right)=x\ln x-x-1=0$:

$$x_{n+1}=\frac{1+x_n}{\ln x_n}\,,\; x_1=4\Rightarrow x_n\rightarrow 3.5911... \tag{64}$$

2. $f\left(x\right)=\sin\left(2x\right)-x=0$:

$$x_{n+1}=\frac{2x_n\cos\left(2x_n\right)-\sin\left(2x_n\right)}{2\cos\left(2x_n\right)-1}\,,\; x_1=1\Rightarrow x_n\rightarrow 0.9477... \tag{65}$$

3. $f\left(x\right)=\sin\left(3x\right)-x=0$:

$$x_{n+1}=\frac{3x_n\cos\left(3x_n\right)-\sin\left(3x_n\right)}{3\cos\left(3x_n\right)-1}\,,\; x_1=3/4\Rightarrow x_n\rightarrow 0.7596... \tag{66}$$

4. $f\left(x\right)=\sin\left(5x\right)-x=0$:

$$x_{n+1}=\frac{5x_n\cos\left(5x_n\right)-\sin\left(5x_n\right)}{5\cos\left(5x_n\right)-1}\,,\; x_1=1/2\Rightarrow x_n\rightarrow 0.5191... \tag{67}$$

5. $f\left(x\right)=x^3-x-1=0$:

$$x_{n+1}=\frac{2x_n^3+1}{3x_n^2-1}\,,\; x_1=1\Rightarrow x_n\rightarrow 1.3247... \tag{68}$$

6. $f\left(x\right)=xe^x-2\sin x=0$:

$$x_{n+1}=\frac{e^{x_n}x_n^2-2x_n\cos x_n+2\sin x_n}{e^{x_n}+x_ne^{x_n}-2\cos x_n}\,,\; x_1=1/2\Rightarrow x_n\rightarrow 0.6267... \tag{69}$$

7. $f\left(x\right)=x-\cosh x\ln\left(1+e^{-x}\right)=0$:

$$x_{n+1}=\frac{x_ne^{x_n}\cosh x_n-\left(1+e^{-x_n}\right)\left(x_n\sinh x_n-\cosh x_n\right)\ln\left(1+e^{-x_n}\right)}{1+e^{-x_n}\left(1+\cosh x_n\right)-\sinh x_n\left(1+e^{-x_n}\right)\ln\left(1+e^{-x_n}\right)} \tag{70}$$

$$x_1=1/2\Rightarrow x_n\rightarrow 0.6267...$$

8. $f\left(x\right)=x\left(1+\frac{2}{e^{2x}-1}\right)-3=0$:

$$x_{n+1}=\frac{3e^{4x_n}-6e^{2x_n}-4x_n^2e^{2x_n}+3}{e^{4x_n}-4x_ne^{2x_n}-1}\,,\; x_1=3\Rightarrow x_n\rightarrow 2.9847... \tag{71}$$

9. $f\left(x\right)=x+\ln\left(1-x^2\right)=0$:

$$x_{n+1}=\frac{2x_n^2+\left(1-x_n^2\right)\ln\left(1-x_n^2\right)}{x_n^2+2x_n-1}\,,\ x_1=1/2\Rightarrow x_n\to 0.7145... \tag{72}$$

10. $f\left(x\right)=x+\sin x+x\sin x-1=0$:

$$x_{n+1}=\frac{1-\sin x_n+x\cos x_n+x_n^2\cos x_n}{1+\sin x_n+\cos x_n+x\cos x_n}\,,\ x_1=1/2\Rightarrow x_n\to 0.4203... \tag{73}$$

11. $f\left(x\right)=x+\cos x+x\cos x-1=0$:

$$x_{n+1}=\frac{\left(x_n^2+x_n\right)\sin x_n+\cos x_n-1}{\left(x_n+1\right)\sin x_n-\cos x_n-1}\,,\ x_1=2\Rightarrow x_n\to 1.8817... \tag{74}$$

12. $f\left(x\right)=x+\tan x+x\tan x-1=0$:

$$x_{n+1}=\frac{x_n+x_n^2+\cos x_n\left(\cos x_n-\sin x_n\right)}{1+x_n+\cos x_n\left(\cos x_n+\sin x_n\right)}\,,\ x_1=1/2\Rightarrow x_n\to 0.4026... \tag{75}$$

13. $f\left(x\right)=x+\sinh x+x\sinh x-1=0$:

$$x_{n+1}=\frac{1-\sinh x_n+\left(x_n+x_n^2\right)\cosh x_n}{1+e^{x_n}+x_n\cosh x_n}\,,\ x_1=1/2\Rightarrow x_n\to 0.4084... \tag{76}$$

14. $f\left(x\right)=xe^x-e^{-x}=0$:

$$x_{n+1}=\frac{1+x_n+x_n^2e^{2x_n}}{1+\left(1+x_n\right)e^{2x_n}}\,,\ x_1=1/2\Rightarrow x_n\to 0.4263... \tag{77}$$

15. $f\left(x\right)=x-3\tanh\left(\frac{x}{2}\right)=0$:

$$x_{n+1}=\frac{3\sinh x_n-3x_n}{\cosh x_n-2}\,,\ x_1=5/2\Rightarrow x_n\to 2.5756... \tag{78}$$

16. $f\left(x\right)=\sin x+\sinh x+\sin x\sinh x-1=0$:

$$x_{n+1}=\frac{1-\sin x_n+x_n\cos x_n+\left(x_n\cos x_n-\sin x_n-1\right)\sinh x_n+\left(1+\sin x_n\right)x_n\cosh x_n}{\left(1+\sin x_n\right)\cosh x_n+\left(1+\sinh x_n\right)\cos x_n} \tag{79}$$
$$x_1=1/2\Rightarrow x_n\to 0.4141...$$

17. $f\left(x\right)=e^{-x}+\cos x+e^{-x}\cos x-1=0$:

$$x_{n+1}=\frac{e^{-x_n}\left(1+x_n+x_n\sin x_n+\left(1+x_n\right)\cos x_n\right)+x_n\sin x_n+\cos x_n-1}{\sin x_n+e^{-x_n}\left(1+\sin x_n+\cos x_n\right)} \tag{80}$$
$$x_1=1\Rightarrow x_n\to 1.0627...$$

18. $f\left(x\right)=e^{-x}+\sin x+e^{-x}\sin x-1=0$:

$$x_{n+1}=\frac{e^{-x_n}\left(x_n\cos x_n-\left(1+x_n\right)\left(1+\sin x_n\right)\right)+x_n\cos x_n-\sin x_n+1}{\cos x_n+e^{-x_n}\left(\cos x_n-\sin x_n-1\right)} \tag{81}$$
$$x_1=2\Rightarrow x_n\to 2.2103...$$

19. $f\left(x\right)=\tan x+\sinh x+\tan x\sinh x-1=0$:

$$x_{n+1}=x_n+\frac{1-\sinh x_n-\left(1+\sinh x_n\right)\tan x_n}{1+e^{x_n}+\tan x_n\cosh x_n+\left(1+\sinh x_n\right)\left(\tan x_n\right)^2} \tag{82}$$
$$x_1=1/2\Rightarrow x_n\to 0.3977...$$

20. $f\left(x\right)=\tan x+\cosh x+\tan x\cosh x-1=0$:

$$x_{n+1}=x_n+\frac{1-\cosh x_n-\left(1+\cosh x_n\right)\tan x_n}{1+e^{x_n}+\tan x_n\sinh x_n+\left(1+\cosh x_n\right)\left(\tan x_n\right)^2} \tag{83}$$
$$x_1=5/2\Rightarrow x_n\to 2.5154...$$

21. $f\left(x\right)=e^{x}\tan x-e^{-x}=0$:

$$x_{n+1}=\frac{1+x_n+e^{2x_n}\left(x_n+\left(x_n-1\right)\tan x_n+x_n\left(\tan x_n\right)^2\right)}{1+e^{2x_n}\left(1+\tan x_n+\left(\tan x_n\right)^2\right)} \tag{84}$$
$$x_1=1/2\Rightarrow x_n\to 0.4128...$$

22. $f\left(x\right)=x^2+\left(\sin x\right)^2-1=0$:

$$x_{n+1}=\frac{x_n^2+\left(\cos x_n\right)^2+x_n\sin\left(2x_n\right)}{2x_n+\sin\left(2x_n\right)},\ x_1=3/4\Rightarrow x_n\to 0.7390... \tag{85}$$

23. $f\left(x\right)=x^2+\left(\tan x\right)^2-1=0$:

$$x_{n+1}=x_n+\frac{1-x_n^2-\left(\tan x_n\right)^2}{2x_n+2\tan x_n+2\left(\tan x_n\right)^3},\ x_1=1/2\Rightarrow x_n\to 0.6498... \tag{86}$$

24. $f\left(x\right)=x^2+\left(\sinh x\right)^2-1=0$:

$$x_{n+1}=\frac{x_n\sinh(2x_n)-(\cosh x_n)^2+x_n^2+2}{2x_n+\sinh(2x_n)},\ x_1=1/2\Rightarrow x_n\to 0.6798... \quad (87)$$

25. $f(x)=x^2+(\tanh x)^2-1=0$:

$$x_{n+1}=x_n+\frac{1-x_n^2-(\tanh x_n)^2}{2x_n+2\tanh x_n-2(\tanh x_n)^3},\ x_1=3/4\Rightarrow x_n\to 0.7650... \quad (88)$$

26. $f(x)=x^2+e^{-2x}-1=0$:

$$x_{n+1}=\frac{1+x_n^2-(1+2x_n)e^{-2x_n}}{2x_n-2e^{-2x_n}},\ x_1=1\Rightarrow x_n\to 0.9165... \quad (89)$$

27. $f(x)=(\sin x)^2+(\sinh x)^2-1=0$:

$$x_{n+1}=\frac{x_n\sinh(2x_n)-(\cosh x_n)^2+1+x_n\sin(2x_n)+(\cos x_n)^2}{\sin(2x_n)+\sinh(2x_n)} \quad (90)$$
$$x_1=3/4\Rightarrow x_n\to 0.7032...$$

28. $f(x)=(\sin x)^2+(\tanh x)^2-1=0$:

$$x_{n+1}=x_n+\frac{1-(\sin x_n)^2-(\tanh x_n)^2}{\sin(2x_n)+2\tanh x_n-2(\tanh x)^3},\ x_1=1\Rightarrow x_n\to 0.8256... \quad (91)$$

29. $f(x)=e^{-2x}+(\ln x)^2-1=0$:

$$x_{n+1}=\frac{x_n\left(2x_ne^{-2x_n}+(\ln x_n)^2+e^{-2x_n}-2\ln x_n-1\right)}{2x_ne^{-2x_n}-2\ln x_n},\ x_1=1/2\Rightarrow x_n\to 0.4603... \quad (92)$$

- Remark: formula(20), $J_1(x)$ Bessel function of the firt kind , $N_0(x)$ Bessel function of second kind (also called Neumann function).

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