Question 2360: Series For Pi

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Abstract

This note presents some series for pi constant.

1. Introduction: Formulas

Definition of pi:

$$\pi = 4\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 3.1415926535... \tag{1}$$

Series:

$$\pi = \sum_{n=0}^{\infty} 3^{-n} w^n \left(\frac{2}{4n+1} - \frac{1}{4n+3} + \frac{(-1)^n}{2n+1} + u(n-1)\left(\frac{2}{4n-3} + \frac{4}{4n-1} - \frac{2(-1)^n}{2n-1} \right) + u(n-2)\left(\frac{(-1)^n}{2n-3} + \frac{1}{4n-5} \right) \right) \tag{2}$$

$$\pi = 4\sum_{n=0}^{\infty} 3^{-n} w^n \left(\frac{1}{8n+2} + \frac{(-1)^n}{8n+4} - \frac{1}{32n+24} + u(n-1)\left(\frac{1}{8n-6} - \frac{(-1)^n}{4n-2} - \frac{3}{32n-8} \right) + u(n-2)\left(\frac{(-1)^n}{8n-12} - \frac{3}{32n-40} \right) - u(n-3)\frac{1}{32n-72} \right) \tag{3}$$

$$\pi = 4\sum_{n=1}^{\infty} \frac{c_n}{n}\left(\frac{w}{3}\right)^{n/4} = 4\sum_{n=1}^{\infty} \frac{c_n}{n}\left(\frac{3+w}{6}\right)^{n} \tag{4}$$

$$c_1 = 1, c_2 = 2, c_3 = -1, c_4 = 0, c_5 = 1, c_6 = -2 \tag{5}$$

$$c_n = -\left(c_{n-2} + c_{n-4} + c_{n-6}\right), n \ge 7 \tag{6}$$

$$w = -5 + 2\sqrt[3]{3\sqrt{33}+17} - 2\sqrt[3]{3\sqrt{33}-17} \tag{7}$$

$$u(n)=\begin{cases}1 & ,n\geq 0\\ 0 & ,n<0\end{cases} \tag{8}$$

$$\pi=4\sum_{n=1}^{\infty}\frac{c_n}{n}\left(A_n+B_n v+C_n v^2\right) \tag{9}$$

$$v=\frac{3+w}{6} \tag{10}$$

$$A_{n+1}=C_n, B_{n+1}=A_n-C_n, C_{n+1}=B_n-C_n, A_1=0, B_1=1, C_1=0 \tag{11}$$

2. Formulas related with the w number

$$w^3+15w^2+99w-27=0 \tag{12}$$

$$w=-5+\sqrt[3]{272-24\sqrt[3]{272-24\sqrt[3]{272-\dots}}} \tag{13}$$

$$w=-5+2\sqrt[3]{34-6\sqrt[3]{34-6\sqrt[3]{34-\dots}}} \tag{14}$$

$$\frac{1}{w}=\frac{11}{9}+\frac{2}{9}\sqrt[3]{199+3\sqrt{33}}+\frac{2}{9}\sqrt[3]{199-3\sqrt{33}} \tag{15}$$

$$\frac{1}{w}=\frac{11}{9}+\frac{2}{9}\sqrt[3]{398+102\sqrt[3]{398+102\sqrt[3]{398+\dots}}} \tag{16}$$

$$\begin{cases} w=-3+6\lim_{n\to\infty}\frac{u_n}{u_{n+1}} \\ u_{n+3}=u_{n+2}+u_{n+1}+u_n \ ,u_1=1,u_2=1,u_3=2 \\ u_n=\{1,1,2,4,7,13,24,44,\dots\} \end{cases} \tag{17}$$

$$w=\cfrac{a}{10+\cfrac{a}{10+\cfrac{a}{10+\dots}}} \tag{18}$$

$$\frac{1}{w}=\sqrt{\frac{1}{a}+\frac{10}{a}\sqrt{\frac{1}{a}+\frac{10}{a}\sqrt{\frac{1}{a}+\dots}}} \tag{19}$$

$$a=-33+6\pi\left(\int_0^{2\pi}h\left(e^{ix}\right)e^{ix}dx\right)^{-1}, h(z)=\frac{1}{z^3+5z^2+11z-1} \tag{20}$$

$$w^n=A_n w^2+B_n w+C_n \ ,n\in\mathbb{N}\cup\{0\} \tag{21}$$

$$A_{n+1} = -15A_n + B_n, B_{n+1} = -99A_n + C_n, C_{n+1} = 27A_n \tag{22}$$
$$A_0 = B_0 = 0, C_0 = 1 \tag{23}$$

$$x_{n+1} = 3\left(\frac{3+x_n}{6}\right)^4, x_1 = 0 \Rightarrow \lim_{n\to\infty} x_n = w \tag{24}$$
$$x_{n+1} = \frac{27}{99+15x_n + x_n^2}, x_1 = 0 \Rightarrow \lim_{n\to\infty} = w \tag{25}$$
$$x_{n+1} = \frac{27-15x_n^2 - x_n^3}{99}, x_1 = 0 \Rightarrow \lim_{n\to\infty} x_n = w \tag{26}$$
$$x_{n+1} = \frac{27-15x_n^2}{99+x_n^2}, x_1 = 0 \Rightarrow \lim_{n\to\infty} x_n = w \tag{27}$$
$$x_{n+1} = \frac{27-x_n^3}{99+15x_n}, x_1 = 0 \Rightarrow \lim_{n\to\infty} x_n = w \tag{28}$$
$$x_{n+1} = \frac{27+8x_n - 15x_n^2 - x_n^3}{107}, x_1 = 0 \Rightarrow \lim_{n\to\infty} = w \tag{29}$$
$$x_{n+1} = \frac{27+15x_n^2 + 2x_n^3}{99+30x_n + 3x_n^2}, x_1 = 0 \Rightarrow \lim_{n\to\infty} = w \tag{30}$$

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