Unit-Jacobian Coordinate Transformations: The Superior Consequence of the Little-Known Einstein-Schwarzschild Coordinate Condition

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Abstract

Because the Einstein equation can't uniquely determine the metric, it must be supplemented by additional metric constraints. Since the Einstein equation can be derived in a purely special-relativistic context, those constraints (which can't be generally covariant) should be Lorentz-covariant; moreover, for the effect of the constraints to be natural from the perspective of observational and empirical physical scientists, they should also constrain the general coordinate transformations (which are compatible with the unconstrained Einstein equation) so that the constrained transformations manifest a salient feature of the Lorentz transformations. The little-known Einstein-Schwarzschild coordinate condition, which requires the metric's determinant to have its -1 Minkowski value, thereby constrains coordinate transformations to have unit Jacobian, and for that reason causes tensor densities to transform as true tensors, which is a salient feature of the Lorentz transformations. The Einstein-Schwarzschild coordinate condition also allows the static Schwarzschild solution's singular radius to be exactly zero; though another coordinate condition that allows zero Schwarzschild radius exists, it isn't Lorentz-covariant.

Introduction

The Einstein equation is under-determined because the Bianchi identity implies that all four components of the covariant divergence of the ten-component Einstein tensor must vanish, so the Einstein equation by itself *could at most determine only six of the ten components of the gravitational metric-tensor field* $g_{\mu\nu}$ [1]. That makes it necessary to specify four *additional* function constraints on the ten-function metric-tensor field $g_{\mu\nu}$. Those four additional function constraints on the metric-tensor field $g_{\mu\nu}$ can't be generally covariant, but the theoretical-physics nature of the Einstein equation strongly suggests that those constraints *must respect special relativity:* the weak-field (linearized) form of the Einstein equation is the unique special-relativistic extension of static Newtonian gravitational theory in a setting *strictly analogous* to that of Maxwell's equation for *four-vector* A^μ in terms of the divergence-free *four-vector* j^μ, except that for weak-field gravitation it *must* be for *symmetric second-rank tensor* $g_{\mu\nu}$ in terms of the divergence-free *symmetric second-rank tensor* $T^{\mu\nu}$ [2], and the iterative *corrections* to the weak-field Einstein tensor are *uniquely* a matter of pursuing special-relativistic gravitational-field stress-energy self-consistency in the context of preserving an overall variational principle [3]; i.e., the Einstein equation can be viewed as being *special-relativistic to its core*.

Thus the four additional function constraints on $g_{\mu\nu}$ ought to be Lorentz-covariant. Moreover, for the *effect* of the constraints *to be natural from the perspective of observational and empirical physical scientists*, they should *as well constrain* the general coordinate transformations (which of course are compatible with the *unconstrained* Einstein equation) *so that the resulting constrained coordinate transformations manifest a salient feature of the Lorentz transformations*.

The *only widely-recognized* Lorentz-covariant four-vector constraint on $g_{\mu\nu}$ is the so-called "harmonic" coordinate condition [4],

$$g^{\mu\nu}\Gamma^\lambda_{\mu\nu} = 0. \tag{1a}$$

The purported *natural* upshot of the Eq. (1a) coordinate condition is that it allegedly constrains the coordinate four-vector x^μ to be harmonic in the generally covariant sense that [5],

$$g^{\alpha\beta}x^\mu_{;\alpha;\beta} = 0. \tag{1b}$$

In "support" of Eq. (1b), it is pointed out that if ϕ is a general invariant, then [6],

$$g^{\alpha\beta}\phi_{;\alpha;\beta} = g^{\alpha\beta}\partial_\alpha\partial_\beta\phi - g^{\alpha\beta}\Gamma^\lambda_{\alpha\beta}\partial_\lambda\phi, \tag{1c}$$

so that if the Eq. (1a) coordinate condition holds, then,

$$g^{\alpha\beta}\phi_{;\alpha;\beta} = g^{\alpha\beta}\partial_\alpha\partial_\beta\phi. \tag{1d}$$

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However, Eq. (1b) *fails to follow from* Eq. (1d) *because the four components of x^μ obviously aren't general invariants*. In fact, although x^μ is a four-vector under *Lorentz* transformations, *it isn't a covariant entity of any kind under general coordinate transformations*.

In a desperate attempt to "rescue" Eq. (1b), however, we note that dx^μ *does* transform as a four-vector under general coordinate transformations, so that *formally*, at least, we can calculate that,

$$dx^\mu_{;\alpha;\beta} = \partial_\beta\left(\partial_\alpha\left(dx^\mu\right) + \Gamma^\mu_{\alpha\kappa}dx^\kappa\right) + \Gamma^\mu_{\beta\sigma}\left(\partial_\alpha\left(dx^\sigma\right) + \Gamma^\sigma_{\alpha\kappa}dx^\kappa\right) - \Gamma^\tau_{\beta\alpha}\left(\partial_\tau\left(dx^\mu\right) + \Gamma^\mu_{\tau\kappa}dx^\kappa\right). \quad (1e)$$

If we now make the desperate assumption that the undefined entity $\partial_\gamma\left(dx^\lambda\right)$ is equal to $\partial_\gamma\left(x^\lambda\right) = \delta^\lambda_\gamma$, then Eq. (1e) becomes,

$$dx^\mu_{;\alpha;\beta} = \Gamma^\mu_{\alpha\beta} + \left[\partial_\beta\left(\Gamma^\mu_{\alpha\kappa}\right) + \Gamma^\mu_{\beta\sigma}\Gamma^\sigma_{\alpha\kappa} - \Gamma^\tau_{\beta\alpha}\Gamma^\mu_{\tau\kappa}\right]dx^\kappa. \quad (1f)$$

By using the Eq. (1a) coordinate condition in concert with the questionable Eq. (1f), we "calculate" that,

$$g^{\alpha\beta}dx^\mu_{;\alpha;\beta} = \left[g^{\alpha\beta}\partial_\beta\left(\Gamma^\mu_{\alpha\kappa}\right) + g^{\alpha\beta}\Gamma^\mu_{\beta\sigma}\Gamma^\sigma_{\alpha\kappa}\right]dx^\kappa \neq 0. \quad (1g)$$

Therefore *not even* the desperate assumption that the undefined entity $\partial_\gamma\left(dx^\lambda\right)$ is equal to $\partial_\gamma\left(x^\lambda\right) = \delta^\lambda_\gamma$ can in any sense "rescue" the proposition that the Eq. (1a) coordinate condition implies Eq. (1b). Thus to call the Eq. (1a) coordinate condition "harmonic" *is to express a grievous misconception*.

Since the widely-recognized Lorentz four-vector coordinate condition given by Eq. (1a) produces *no* discernibly *natural* upshot, it is reasonable to examine the little-known *remaining* Lorentz four-vector coordinate condition *which follows from requiring that the contraction of two of the indices of $\Gamma^\nu_{\lambda\mu}$ yields zero*, namely $\Gamma^\nu_{\lambda\nu} = 0$. Since,

$$\Gamma^\nu_{\lambda\nu} = \tfrac{1}{2}g^{\nu\mu}\left(\partial_\lambda g_{\nu\mu} + \partial_\nu g_{\lambda\mu} - \partial_\mu g_{\lambda\nu}\right) = \tfrac{1}{2}g^{\nu\mu}\left(\partial_\lambda g_{\mu\nu}\right) = \partial_\lambda \ln\left(\left(-\det\left(g_{\mu\nu}\right)\right)^{\frac{1}{2}}\right), \quad (2a)$$

imposition of,

$$\Gamma^\nu_{\lambda\nu} = 0, \quad (2b)$$

implies that $\left(-\det\left(g_{\mu\nu}\right)\right)^{\frac{1}{2}}$ *is a constant*; substituting $\eta_{\mu\nu}$ for $g_{\mu\nu}$ yields the value *unity* for that constant, which is *equivalent* to the little-known Lorentz-invariant Einstein-Schwarzschild coordinate condition [7],

$$\det\left(g_{\mu\nu}\right) = -1. \quad (2c)$$

Since $\left(-\det\left(g_{\mu\nu}\right)\right)^{\frac{1}{2}} d^4x$ *is invariant under general coordinate transformations*, the Eq. (2c) *constraint on $g_{\mu\nu}$* causes the infinitesimal space-time four-volume d^4x *itself* to be invariant under *those coordinate transformations* which the imposition of the Eq. (2c) coordinate condition (or, equivalently, which the imposition of the Eq. (2b) coordinate condition) *actually allows*.

The coordinate transformations $\bar{x}^\mu(x^\nu)$ which the Eq. (2c) Einstein-Schwarzschild coordinate condition *actually allows* obviously *must* satisfy the unit-Jacobian coordinate transformation constraint [7],

$$\det\left(\partial_\nu \bar{x}^\mu\right) = \pm 1, \quad (3a)$$

which clearly parallels the well-known constraint,

$$\det\left(\Lambda^\mu_\nu\right) = \pm 1, \quad (3b)$$

that is satisfied by the Lorentz transformations. Thus the coordinate transformations which the Einstein-Schwarzschild coordinate condition *allows* clearly manifest *a salient feature* of the Lorentz transformations.

It is apparent that the Eq. (3a) unit-Jacobian coordinate transformation constraint on the coordinate transformations which are allowed by the Eq. (2c) Einstein-Schwarzschild coordinate condition turns all *tensor densities* of the *general* coordinate transformations into *true tensors* of the coordinate transformations which are *allowed* by the Einstein-Schwarzschild coordinate condition. This *lack of any distinction between tensor densities and true tensors* is *as well*, of course, *a salient feature of the Lorentz transformations*. We have above *already seen in particlar* that infinitesimal space-time four-volumes d^4x, which are scalar *densities* of the *general* coordinate transformations, are *true invariants* of the coordinate transformations which are *allowed* by the Eq. (2c) Einstein-Schwarzschild coordinate condition—of course the d^4x *are very well-known indeed to as well be true invariants of the Lorentz transformations*.

Therefore the Einstein-Schwarzschild coordinate condition of Eq. (2c) *definitely* constrains the *general* coordinate transformations in such a way that the resulting constrained transformations of Eq. (3a) *manifest some of the salient features of the Lorentz transformations*, which makes the Einstein-Schwarzschild coordinate condition *natural* from the perspective of observational and empirical physical scientists.

We next turn to Schwarzschild's 1916 static, spherically-symmetric empty-space (aside from at the origin $r = 0$) metric solution of the Einstein equation which adheres to the Einstein-Schwarzschild coordinate condition $\det(g_{\mu\nu}) = -1$ of Eq. (2c) [8].

The static, isotropic empty-space metric in Einstein-Schwarzschild coordinates

The static, spherically-symmetric empty-space (except at $r = 0$) Einstein-equation solution for the Einstein-Schwarzschild coordinate condition $\det(g_{\mu\nu}) = -1$ of Eq. (2c) is [8],

$$ds^2 = (1 - (r_S/R(r;r_0)))\,(cdt)^2 - \frac{(r/R(r;r_0))^4}{(1 - (r_S/R(r;r_0)))}(dr)^2 - (R(r;r_0)/r)^2\,(r)^2\left[(d\theta)^2 + (\sin\theta d\phi)^2\right], \quad (4a)$$

where,

$$r_S \stackrel{\text{def}}{=} 2GM/c^2, \quad R(r;r_0) \stackrel{\text{def}}{=} \left(r^3 + (r_0)^3\right)^{\frac{1}{3}}, \quad (4b)$$

and r_0 is an arbitrary constant of integration of the Einstein equation. When the arbitrary constant of integration r_0 is set equal to r_S, the metric of Eq. (4a) has *no* singularities or time-dilation infinities at *positive* values of r [8],

$$ds^2 = (1 - (r_S/R(r;r_S)))\,(cdt)^2 - \frac{(r/R(r;r_S))^4}{(1 - (r_S/R(r;r_S)))}(dr)^2 - (R(r;r_S)/r)^2\,(r)^2\left[(d\theta)^2 + (\sin\theta d\phi)^2\right]. \quad (4c)$$

Eq. (4c) is Schwarzschild's 1916 solution [8] produced by the Einstein-Schwarzschild coordinate condition $\det(g_{\mu\nu}) = -1$ of Eq. (2c) and the setting of the arbitrary constant of Einstein-equation integration r_0 to r_S, which eliminates all metric singularities and time-dilation infinities at *positive* values of r.

It is of interest to now examine two coordinate conditions *which are algebraically more simply expressed* than is the Eq. (2c) Einstein-Schwarzschild coordinate condition, but in whose favor *there doesn't exist the kind of physical motivation* which was discussed at length for the Einstein-Schwarzschild coordinate condition in the preceding section.

To facilitate the transformation of the Eq. (4c) 1916 Schwarzschild solution to these *other* choices of coordinate condition, we note that the *transformed* static, spherically-symmetric metric must have the *generic* form,

$$\begin{gathered} ds^2 = \bar{f}_0(\bar{r})(cdt)^2 - \bar{f}_\odot(\bar{r})(d\bar{r})^2 - \bar{f}_\angle(\bar{r})\bar{r}^2[(d\theta)^2 + (\sin\theta d\phi)^2] = \\ \bar{f}_0(\bar{r}(r))(cdt)^2 - \bar{f}_\odot(\bar{r}(r))(d\bar{r}(r)/dr)^2(dr)^2 - \bar{f}_\angle(\bar{r}(r))(\bar{r}(r)/r)^2 r^2[(d\theta)^2 + (\sin\theta d\phi)^2]. \end{gathered} \quad (5a)$$

Eq. (4c) and the last line of Eq. (5a) imply *three transformation relations* which are mediated by *the radius transformation function* $\bar{r}(r)$,

$$\bar{f}_0(\bar{r}(r)) = (1 - (r_S/R(r;r_S)))\,,\ \bar{f}_\odot(\bar{r}(r)) = \frac{(r/R(r;r_S))^4}{(1 - (r_S/R(r;r_S)))\,(d\bar{r}(r)/dr)^2},\ \bar{f}_\angle(\bar{r}(r)) = \frac{(R(r;r_S))^2}{(\bar{r}(r))^2}. \quad (5b)$$

The *first* algebraically simple coordinate condition which we shall now consider is the Lorentz-*noncovariant* "standard" coordinate condition,

$$\bar{f}_\angle(\bar{r}) = 1, \quad (6a)$$

which from the *third* transformation relation of Eq. (5b) allows us to immediately *solve* for the needed *radius transformation function* $\bar{r}(r)$ and also its *derivative* $d\bar{r}(r)/dr$,

$$\bar{r}(r) = R(r;r_S) = \left(r^3 + (r_S)^3\right)^{\frac{1}{3}} \text{ and } d\bar{r}(r)/dr = r^2/\left(r^3 + (r_S)^3\right)^{\frac{2}{3}} = (r/R(r;r_S))^2, \quad (6b)$$

where we have used Eq. (4b) in the case that $r_0 = r_S$. The Schwarzschild metric of Eq. (4c) *doesn't* have infinite time dilation or singularities for *positive* r, *but it does have both* at $r = 0$, and we clearly see from Eq. (6b) that $r = 0$ in the Einstein-Schwarzschild coordinates *corresponds to* $\bar{r} = r_S > 0$ in "standard" coordinates. Thus the Schwarzschild solution in "standard" coordinates *is* indeed afflicted by infinite time dilation and metric singularities at a *positive* value of the "standard" coordinate $\bar{r}$, namely at $\bar{r} = r_S$.

We can obtain the metric singularities of the Schwarzschild solution in "standard" coordinates in full detail by taking note that Eq. (6b) also immediately yields the *inverse* $r(\bar{r})$ of $\bar{r}(r)$,

$$r(\bar{r}) = \left(\bar{r}^3 - (r_S)^3\right)^{\frac{1}{3}}, \tag{6c}$$

and that Eq. (6b) in conjunction with Eq. (6c) furthermore yields,

$$R(r(\bar{r}); r_S) = \bar{r}. \tag{6d}$$

Since, of course,

$$\bar{f}_0(\bar{r}) = \bar{f}_0(\bar{r}(r(\bar{r}))) \text{ and } \bar{f}_\odot(\bar{r}) = \bar{f}_\odot(\bar{r}(r(\bar{r}))), \tag{6e}$$

the first and second transformation relations of Eq. (5b) in conjunction with Eqs. (6b) and (6d) allow us to deduce that,

$$\bar{f}_0(\bar{r}) = (1 - (r_S/\bar{r})) \text{ and } \bar{f}_\odot(\bar{r}) = (1 - (r_S/\bar{r}))^{-1}, \tag{6f}$$

while from Eq. (6a) we of course have that $\bar{f}_\angle(\bar{r}) = 1$. Therefore from the first line of Eq. (5a) the Schwarzschild solution metric in "standard" coordinates is given by,

$$ds^2 = (1 - (r_S/\bar{r}))(cdt)^2 - (1 - (r_S/\bar{r}))^{-1}(d\bar{r})^2 - \bar{r}^2[(d\theta)^2 + (\sin\theta d\phi)^2]. \tag{6g}$$

The Eq. (6g) Schwarzschild solution metric produced by the Eq. (6a) Lorentz-*noncovariant* "standard" coordinate condition is algebraically *considerably simpler* than the Eq. (4c) Schwarzschild solution metric produced by the carefully physically-motivated Eq. (2c) Einstein-Schwarzschild coordinate condition, but that algebraic simplicity comes at a high price in infinite time dilation and metric singularity at the *positive* radius $\bar{r} = r_S > 0$. The Eq. (4c) Schwarzschild solution metric produced by the Eq. (2c) Einstein-Schwarzschild coordinate condition, of course has *no* such issues at any *positive* value of r.

The *second* algebraically simple coordinate condition which we consider is the Lorentz-*noncovariant* "alternative standard" coordinate condition,

$$\bar{f}_\odot(\bar{r}) = 1, \tag{7a}$$

which from the *second* transformation relation of Eq. (5b) allows us to solve for $d\bar{r}(r)/dr$,

$$d\bar{r}(r)/dr = (1 - (r_S/R(r; r_S)))^{-\frac{1}{2}} (r/R(r; r_S))^2 = (1 - (r_S/R(r; r_S)))^{-\frac{1}{2}} dR(r; r_S)/dr, \tag{7b}$$

where the *second* equality of Eq. (7b) follows from the first equality and Eq. (4b) with $r_0 = r_S$ because $R(r; r_S) = \left(r^3 + (r_S)^3\right)^{\frac{1}{3}}$ implies that,

$$dR(r; r_S)/dr = r^2/\left(r^3 + (r_S)^3\right)^{\frac{2}{3}} = (r/R(r; r_S))^2.$$

We now integrate that *second* form of Eq. (7b) to obtain $\bar{r}(r)$ as,

$$\bar{r}(r) = \bar{r}(0) + \int_0^r (1 - (r_S/R(r'; r_S)))^{-\frac{1}{2}} (dR(r'; r_S)/dr')\, dr'. \tag{7c}$$

In Eq. (7c) we change the variable of integration from r' to $R = R(r'; r_S)$ to obtain,

$$\bar{r}(r) = \bar{r}(0) + \int_{r_S}^{R(r; r_S)} (1 - (r_S/R))^{-\frac{1}{2}}\, dR. \tag{7d}$$

Changing the variable of integration once more from R to $u = \cosh^{-1}\left(\sqrt{R/r_S}\right)$ implies that $R = r_S \cosh^2(u)$, $dR = 2r_S \cosh(u)\sinh(u)du$ and $(1 - (r_S/R))^{-\frac{1}{2}} = \cosh(u)/\sinh(u)$. Thus Eq. (7d) becomes,

$$\begin{gathered}\bar{r}(r) = \bar{r}(0) + r_S \int_0^{\cosh^{-1}\left(\sqrt{R(r; r_S)/r_S}\right)} (\cosh(2u) + 1)\, du = \\ \bar{r}(0) + (R(r; r_S))^{\frac{1}{2}}(R(r; r_S) - r_S)^{\frac{1}{2}} + r_S \ln\left((R(r; r_S)/r_S)^{\frac{1}{2}} + ((R(r; r_S)/r_S) - 1)^{\frac{1}{2}}\right),\end{gathered} \tag{7e}$$

where $R(r; r_S) = \left(r^3 + (r_S)^3\right)^{\frac{1}{3}}$. Since the Eq. (7b) expression for $d\bar{r}(r)/dr$ *is positive for* $r > 0$, $\bar{r}(r)$ *increases monotonically with* r for $r > 0$. So putting the arbitrary constant $\bar{r}(0)$ in Eq. (7e) to *zero* implies that the Schwarzschild solution with the Eq. (7a) coordinate condition has *no* singularities or time-dilation infinities for *positive* values of $\bar{r}$ *because* the Eq. (4c) Schwarzschild solution has no singularities or time-dilation infinities for *positive* values of r. (But notwithstanding that with $\bar{r}(0) = 0$ the Eq. (7e) expression for $\bar{r}(r)$ definitely *possesses* a unique inverse $r(\bar{r})$ for $\bar{r} \geq 0$, Eq. (7e) makes it clear that that inverse $r(\bar{r})$ *isn't* amenable to being worked out *analytically* and explicitly displayed, so *neither* is the Schwarzschild solution *metric* amenable to being *explicitly displayed* for the coordinate condition $\bar{f}_{\odot}(\bar{r}) = 1$ of Eq. (7a).)

However *unlike* the "alternative standard" coordinate condition $\bar{f}_{\odot}(\bar{r}) = 1$ of Eq. (7a), the "standard" coordinate condition $\bar{f}_{\angle}(\bar{r}) = 1$ of Eq. (6a) *does not allow* the Schwarzschild solution to have *no* singularities or time dilation infinities at *positive* values of the radius $\bar{r}$; instead the "standard" coordinate condition *definitely* places a singularity and a time dilation infinity at *positive* $\bar{r} = r_S > 0$, as is apparent from the "standard" Schwarzschild solution metric given by Eq. (6g).

Likewise, it is well-known that the so-called "isotropic" coordinate condition $\bar{f}_{\odot}(\bar{r}) = \bar{f}_{\angle}(\bar{r})$ and also the so-called "harmonic" coordinate condition $g^{\mu\nu}\Gamma^{\lambda}_{\mu\nu} = 0$ *both do not allow* the Schwarzschild solution to have *no* singularities or time dilation infinities at *positive* values of the radius [9]. Such *unavoidable* occurrence of Schwarzschild-solution singularities or time-dilation infinities at *positive* radius values transparently *isn't a feature of gravitational physics*, but rather *is as straightforward an indication as could possibly exist that the coordinate conditions which give rise to those unphysical anomalies must be shunned*. We have, of course, already argued at length in the preceding section that the Einstein-Schwarzschild coordinate condition $\det(g_{\mu\nu}) = -1$ of Eq. (2c) stands head and shoulders *above other coordinate conditions in terms of naturalness for observational and empirical physical scientists* because *it is Lorentz-covariant* and because the unit-Jacobian coordinate transformations which it allows *manifest a salient feature of the Lorentz transformations*, namely that all tensor densities (such as the scalar-density infinitesimal space-time four-volumes d^4x) *transform as true tensors*. That the "alternative standard" coordinate condition $\bar{f}_{\odot}(\bar{r}) = 1$ of Eq. (7a) *as well* happens *not* to give rise to an *unavoidable* Schwarzschild-solution singularity or time-dilation infinity at a *positive* value of $\bar{r}$ *still can't justify* simply *disregarding* the fact that *that* coordinate condition is Lorentz-*noncovariant*.

References

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