Mathematical beauty with prime numbers: Elegant sieve-based primality test formula constructed with periodic functions

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Abstract

Formulas which relate basic mathematical constants, operations and prime numbers are presented. Some related ideas are explored.

Introductory note: This text was previously published on Scribd[1].

1 Introduction

The well-known Euler equation is often cited as the example of mathematical beauty. A similarly beautiful formulas (I believe) can be related to the prime numbers. I will present formulas converging for any natural number and I will also construct a smooth primality test function. However, unless some summation of the formulas presented later in this text exists, the results are probably not of practical use.

2 Method

I propose here one realization of the sieve of Eratosthenes constructed with periodic functions. Let me define periodic functions with period L (natural number) which have (within one period) the following properties:

$$Q^L(x) = \begin{cases} 1 & \text{if } x = 0 \text{ or } x = L \\ 0 & \text{if } x \text{ is a natural number different from L (and 0)} \\ & \text{arbitrary value elswhere} \end{cases}$$

Then the function

$$Q\left(x\right) = \sum_{L=2}^{\infty} Q^L\left(x\right)$$

represents a sieve-based primality test where the condition for a natural number $k\,(>0)$ is:

$$k \text{ is prime if and only if } Q\left(k\right) = 1.$$

Many variants of this construction can be thought of. One might start with $L = 1$

$$Q\left(x\right) = \sum_{L=1}^{\infty} Q^L\left(x\right)$$

getting then

$$k \text{ is prime if and only if } Q\left(k\right) = 2.$$

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I am willing to publish any of my ideas presented through free-publishing services in a journal, if someone (an editor) judges them interesting enough. Journals in the "*Current Contents*" database are strongly preferred.

[1] *https://www.scribd.com/document/295973151/Mathematical-beauty-with-prime-numbers-Elegant-sieve-based-primality-test-formula-constructed-with-periodic-functions*

One can also define

$$
\begin{aligned}
Q^{L}(x) &= \begin{cases} L & \text{if } x=0 \text{ or } x=L \\ 0 & \text{if } x \text{ is a natural number different from L (and 0)} \\ & \text{arbitrary value elswhere} \end{cases} \\
Q\left(x\right) &= \sum_{L=2}^{\infty} Q^{L}\left(x\right)
\end{aligned}
$$

leading to the condition

$$k \text{ is prime if and only if } Q\left(k\right)=k.$$

Starting in this case with $L=1$, $Q\left(x\right)=\sum_{L=1}^{\infty} Q^{L}\left(x\right)$, one has

$$k \text{ is prime if and only if } Q\left(k\right)=k+1.$$

A finite (but growing with L) realization can be constructed by truncating the summation (with k a natural number)

$$Q\left(k\right)=\sum_{L=2}^{k} Q^{L}\left(k\right).$$

3 Realization and mathematical beauty

Different realizations of the functions Q^{L} can be proposed. A very elegant one can be written as

$$
\begin{aligned}
Q^{L}\left(k\right) &= \sum_{n=q}^{q+L-1} e^{i\frac{2\pi}{L}nk}, \\
Q\left(k\right) &= \sum_{L=2}^{L_{max}} Q^{L}\left(k\right).
\end{aligned}
$$

I adopt the most natural choice $q=0$ and $L_{max}=\infty$

$$Q\left(k\right)=\sum_{L=2}^{\infty}\sum_{n=0}^{L-1} e^{i\frac{2\pi}{L}nk}$$

where the condition

$$k \text{ is prime if and only if } Q\left(k\right)=k$$

applies. Obviously, many different alternatives exist: one could, for example, take the real part of this expression.

At this stage "beautiful" equalities can be constructed. A naturally elegant expressions can be written as

$$\sum_{L=2}^{\infty}\sum_{n=0}^{L-1} e^{i\frac{2\pi}{L}nk}=k \text{ or } \sum_{L=1}^{\infty}\sum_{n=0}^{L-1} e^{i\frac{2\pi}{L}nk}=k+1 \tag{1}$$

which relate a prime number k (in the "if and only if" sense) with the constants π and e and with the imaginary unit i. Furthermore, we have the operation of multiplication (and division), exponentiation[2] and addition[3] (and subtraction). Present is also the neutral element of multiplication "1" (in the sum limit for the first formula) and the neutral element of addition "zero" (only in sum limits) . In addition, the expressions contains the infinity symbol. Moving everything on one side of the equation one can make appear "zero" also outside the summation limits. In any of these expressions the upper L-summation limit can be changed to k, leading so to finite (but growing) expressions.

[2]and in some sense also "L"-th root (radical), since L is in the denominator of the exponent.

[3]In the first formula the addition is represented by the symbol Σ.

4 Further computations

4.1 Summing geometric series

The internal summation in (1) (with the outer L-summation limit modified $\infty \to k$) corresponds to a geometric series and can be performed

$$\begin{aligned} Q(k) &= \sum_{L=2}^{k} \sum_{n=0}^{L-1} e^{2\pi i \frac{n}{L} k} \\ &= \sum_{L=2}^{k} \left[\sum_{n=0}^{L-1} \left(e^{2\pi i \frac{k}{L}} \right)^n \right] \\ &= \sum_{L=2}^{k} \left\{ \lim_{k' \to k} \left[\frac{\left(e^{2\pi i \frac{k'}{L}} \right)^L - 1}{e^{2\pi i \frac{k}{L}'} - 1} \right] \right\} \\ &= \sum_{L=2}^{k} \left\{ \lim_{k' \to k} \left[\frac{e^{2\pi i k'} - 1}{e^{2\pi i \frac{k'}{L}} - 1} \right] \right\}. \end{aligned}$$

Unfortunately, the geometric series summation formula does not apply to series with quotient 1, which is actually an important situation four our prime-number case. Fortunately however, the summation formula limit corresponds to the correct value. For this reason the limit appears in the expressions above. Since the limit exists for all terms, we can, for a finite series, interchange the order of the summation and the limit[4]

$$Q(k) = \lim_{k' \to k} \left[\left(e^{2\pi i k'} - 1 \right) \sum_{L=2}^{k} \frac{1}{e^{\frac{2\pi i k'}{L}} - 1} \right]. \tag{2}$$

4.2 Alternative expressions for Q^L

The summed expression makes one think of many alternatives of the same nature: alternating function in the numerator making it vanish for every natural number, divided by a function with period L which, in division, provides a finite non-zero limit after $L-1$ "zeros" generated by the numerator. For example

$$Q^L(x) = \frac{1}{L^2} \frac{1 - \cos(2\pi x)}{1 - \cos\left(\frac{2\pi}{L} x\right)}$$

or

$$Q^L(x) = \frac{1}{L^2} \frac{\sin^2(\pi x)}{\sin^2\left(\frac{\pi}{L} x\right)}$$

with "k is prime $\Leftrightarrow Q(k) = 1$".

4.3 Exploiting the sum

The sum in (2) seems to be an interesting object and can be further processed[5]. Let me use the notation $\varphi = 2\pi k'$. If there would be a way to rapidly evaluate

$$\sum_{L=2}^{k} \frac{1}{e^{i\frac{\varphi}{L}} - 1}, \tag{3}$$

then an efficient primality test would result. I doubt any closed finite (i.e. not growing) elementary-function formula can be found:

- If it existed it would be already known.
- The prime numbers are too complicated and "profound" to be "pinioned" by a simple formula.

[4]Mixing little bit the natural numbers with the real numbers, but, hopefully, in an understandable way.
[5]The question is, whether it is worth it...

However, strangely enough, the real part of terms in (3) is equal to $-\frac{1}{2}$ and can be easily summed

$$\begin{aligned}
\sum_{L=2}^{k}\frac{1}{e^{i\frac{\varphi}{L}}-1} &= \sum_{L=2}^{k}\left[\mathrm{Re}\left(\frac{1}{e^{i\frac{\varphi}{L}}-1}\right)+i\mathrm{Im}\left(\frac{1}{e^{i\frac{\varphi}{L}}-1}\right)\right] \\
&= \sum_{L=2}^{k}\left(-\frac{1}{2}\right)+i\sum_{L=2}^{k}\frac{1}{2}\frac{\sin\left(\frac{\varphi}{L}\right)}{\cos\left(\frac{\varphi}{L}\right)-1} \\
&= \frac{1-k}{2}+\frac{i}{2}\sum_{L=2}^{k}\frac{\sin\left(\frac{\varphi}{L}\right)\left[\cos\left(\frac{\varphi}{L}\right)+1\right]}{\left[\cos\left(\frac{\varphi}{L}\right)-1\right]\left[\cos\left(\frac{\varphi}{L}\right)+1\right]} \\
&= \frac{1-k}{2}+\frac{i}{2}\sum_{L=2}^{k}\frac{\sin\left(\frac{\varphi}{L}\right)\left[\cos\left(\frac{\varphi}{L}\right)+1\right]}{\cos^{2}\left(\frac{\varphi}{L}\right)-1} \\
&= \frac{1-k}{2}+\frac{i}{2}\sum_{L=2}^{k}\frac{\sin\left(\frac{\varphi}{L}\right)\left[\cos\left(\frac{\varphi}{L}\right)+1\right]}{1-\sin^{2}\left(\frac{\varphi}{L}\right)-1} \\
&= \frac{1-k}{2}-\frac{i}{2}\sum_{L=2}^{k}\frac{\cos\left(\frac{\varphi}{L}\right)+1}{\sin\left(\frac{\varphi}{L}\right)} \\
&= \frac{1-k}{2}-\frac{i}{2}\left[\sum_{L=2}^{k}\frac{\cos\left(\frac{\varphi}{L}\right)}{\sin\left(\frac{\varphi}{L}\right)}+\sum_{L=2}^{k}\frac{1}{\sin\left(\frac{\varphi}{L}\right)}\right] \\
&= \frac{1-k}{2}-\frac{i}{2}\left[\sum_{L=2}^{k}\cot\left(\frac{\varphi}{L}\right)+\sum_{L=2}^{k}\csc\left(\frac{\varphi}{L}\right)\right].
\end{aligned}$$

Considering the whole expression (2), the summation of the real part does not help us. In proximity of a natural number (especially when $k=mL$, with m natural), only the imaginary part plays a role and the real part vanishes . To see it, one can use the Taylor expansion

$$\begin{aligned}
\lim_{k'\to k}\left[\left(e^{2\pi ik'}-1\right)\sum_{L=2}^{k}\frac{1}{e^{\frac{2\pi ik'}{L}}-1}\right] &= \lim_{\varepsilon\to 0}\left\{\left[e^{2\pi i(k+\varepsilon)}-1\right]\sum_{L=2}^{k}\frac{1}{e^{\frac{2\pi ik'}{L}}-1}\right\} \\
&= \lim_{\varepsilon\to 0}\left[\left(e^{2\pi ik}e^{2\pi i\varepsilon}-1\right)\sum_{L=2}^{k}\frac{1}{e^{\frac{2\pi ik'}{L}}-1}\right] \\
&= \lim_{\varepsilon\to 0}\left\{\left[1\left(1+2\pi i\varepsilon\right)-1\right]\sum_{L=2}^{k}\frac{1}{e^{\frac{2\pi ik'}{L}}-1}\right\} \\
&= \lim_{\varepsilon\to 0}\left\{2\pi i\varepsilon\sum_{L=2}^{k}\left[\mathrm{Re}\left(\frac{1}{e^{\frac{2\pi ik'}{L}}-1}\right)+i\mathrm{Im}\left(\frac{1}{e^{\frac{2\pi ik'}{L}}-1}\right)\right]\right\} \\
&= \lim_{\varepsilon\to 0}\left[2\pi i\varepsilon\sum_{L=2}^{k}\mathrm{Re}\left(\frac{1}{e^{\frac{2\pi ik'}{L}}-1}\right)\right]-\lim_{\varepsilon\to 0}\left[2\pi\varepsilon\sum_{L=2}^{k}\mathrm{Im}\left(\frac{1}{e^{\frac{2\pi ik'}{L}}-1}\right)\right] \\
&= \lim_{\varepsilon\to 0}\left[2\pi i\varepsilon\frac{1-k}{2}\right]-2\pi\lim_{\varepsilon\to 0}\left[\varepsilon\sum_{L=2}^{k}\mathrm{Im}\left(\frac{1}{e^{\frac{2\pi ik'}{L}}-1}\right)\right] \\
&= 0-2\pi\lim_{\varepsilon\to 0}\left\{\varepsilon\sum_{L=2}^{k}\mathrm{Im}\left[\frac{1}{e^{\frac{2\pi i(k+\varepsilon)}{L}}-1}\right]\right\}.
\end{aligned}$$

5 Continuous function

Numerical results suggest that the expression of the type

$$Q\left(x\right)=\sum_{L=2}^{\infty}\sum_{n=0}^{L-1}e^{i\frac{2\pi}{L}nx}$$

does not converge for real arguments different from the natural numbers. If one wants to construct a real-argument real-value continuous function, one needs to do some modifications. A finite expression of the form

$$\sum_{L=2}^{k}\sum_{n=0}^{L-1} e^{2\pi i\frac{n}{L}k}$$

does not provide a natural generalization to real numbers. One can rather keep the infinite summation and transform the summed expression. It is convenient to start with

$$\sum_{L=2}^{\infty}\frac{1}{L}\sum_{n=0}^{L-1} e^{2\pi i\frac{n}{L}x}$$

where the condition "x is prime $\Leftrightarrow Q(x)=1$" applies. The function

$$Q^{L}(x)=\frac{1}{L}\sum_{n=0}^{L-1} e^{2\pi i\frac{n}{L}x}$$

is complex, in the construction I propose, the real part is taken. The new definition stands

$$\begin{aligned}
Q^{L}(x) &= \mathrm{Re}\left(\frac{1}{L}\sum_{n=0}^{L-1} e^{2\pi i\frac{n}{L}x}\right)\\
&= \lim_{x'\to x}\frac{1}{L}\mathrm{Re}\left(\frac{e^{2\pi i x'}-1}{e^{2\pi i\frac{x'}{L}}-1}\right)\\
&= \lim_{x'\to x}\frac{1}{2L}\frac{1-\cos(2\pi x')-\cos\left(\frac{2\pi x'}{L}\right)+\cos(2\pi x')\cos\left(\frac{2\pi x'}{L}\right)+\sin(2\pi x')\sin\left(\frac{2\pi x'}{L}\right)}{1-\cos\left(\frac{2\pi x'}{L}\right)}\\
&= \lim_{x'\to x}\frac{1}{2L}\left[1-\cos(2\pi x')+\frac{\sin(2\pi x')\sin\left(\frac{2\pi x'}{L}\right)}{1-\cos\left(\frac{2\pi x'}{L}\right)}\right].
\end{aligned}$$

This expression is little bit long but one should not panic: it is the real part of the average of L complex unitary vectors. Now I want to profit from the following

$$\begin{aligned}
\lim_{L\to\infty} 1^{L} &= 1,\\
\lim_{L\to\infty} y^{L} &= 0 \text{ for } |y|<1
\end{aligned}$$

to make the function converge. The continuous function can be written as[6]

$$\begin{aligned}
Q_{\mathbb{R}}(x) &= \sum_{L=2}^{\infty}\left[Q^{L}(x)\right]^{L}\\
&= \sum_{L=2}^{\infty}\left[\frac{1}{L}\mathrm{Re}\left(\sum_{n=0}^{L-1} e^{2\pi i\frac{n}{L}x}\right)\right]^{L}.
\end{aligned}\tag{4}$$

Being too lazy to provide rigorous proof, let me provide some arguments:

- If x is far from a natural number, then all the functions $Q^{L}(x)$ can be majorated by the same number $q(x)<1$. Consequently

  $$\sum_{L=2}^{\infty}\left[Q^{L}(x)\right]^{L}$$

  can be majorated by the geometric series

  $$\sum_{L=2}^{\infty} q^{L}$$

  which is convergent. The assumption of the possibility of majoration needs to be thought of. The real part of the complex Q^{L} function can approach one only if *all* complex unitary vectors are close to one. If x is far from a natural number this is not true[7].

[6] I apologize for the redefinition of Q^{L}: now Q^{L} is not "the thing" which is summed over L, but $\left(Q^{L}\right)^{L}$ is summed.

[7] I admit the argumentation is not sufficient.

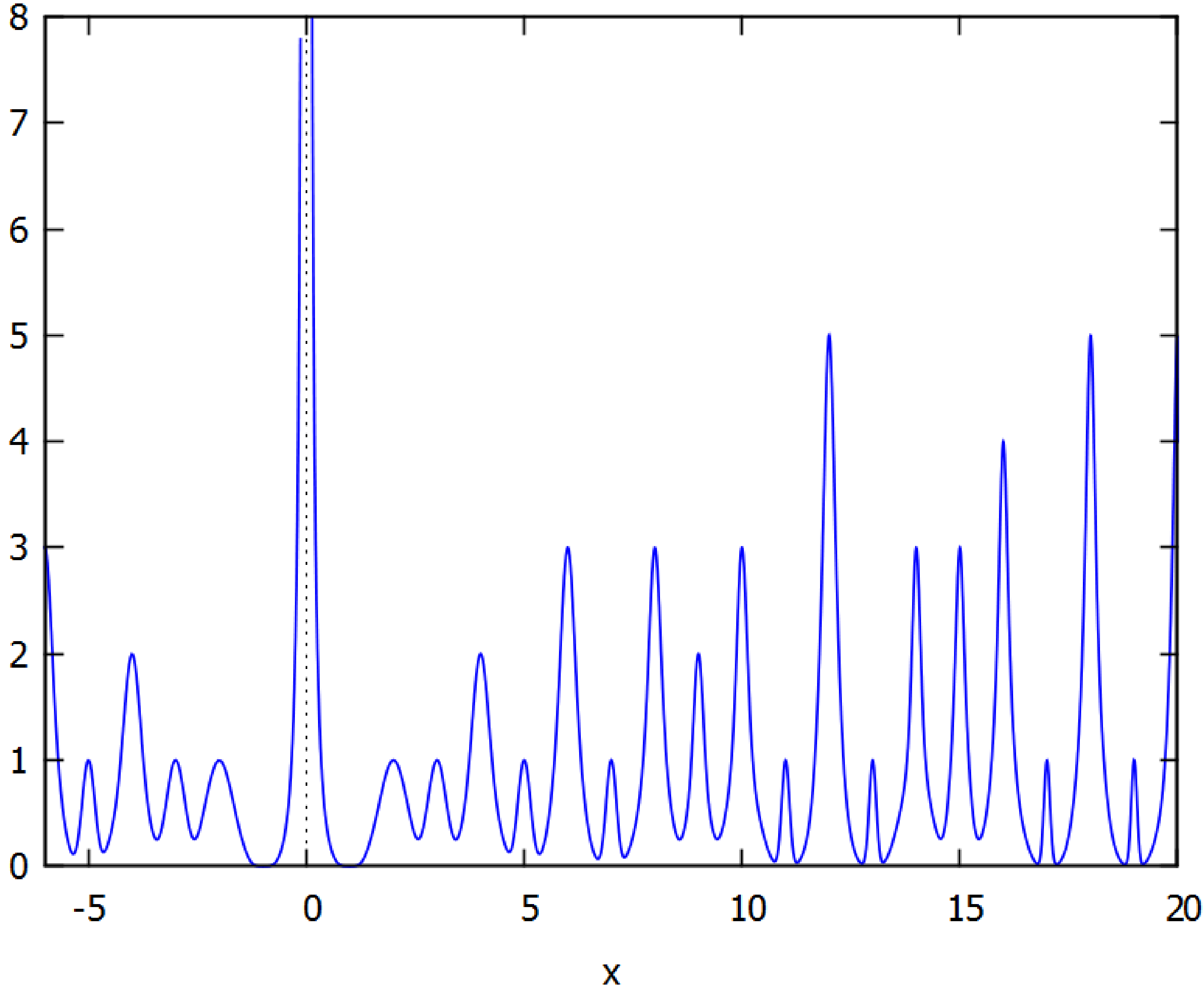

Figure 1: Function $Q_{\mathbb{R}}$ (produced with $L_{max} = 100$).

- If x is close to a natural number, all but finite number of functions $Q^L(x)$ are approaching zero. From the finite number of those approaching one, I can choose the biggest one and use it for majoration based on a geometric series, as in the previous case.

One may notice that a discontinuity could be produced, because rising to a growing power leads to a discontinuity at 1 (consider x^n in the neighborhood of $x = 1$ for $n \to \infty$). This is however not our case: this discontinuity is pushed with increasing L to infinity because the period of the functions Q^L grows to infinity. The only discontinuity that $Q_{\mathbb{R}}(x)$ has is thus situated at $x = 0$. The function can be extended to negative numbers as it is naturally an even function. The graph of $Q_{\mathbb{R}}$ is plotted in Figure 1. Let me summarize the important properties of the function $Q_{\mathbb{R}}$:

- $Q_{\mathbb{R}}$ is defined for every $x \varepsilon \mathbb{R}$ except zero.
- If k is natural number then "k is prime $\Leftrightarrow Q_{\mathbb{R}}(k) = 1$".
- If k is natural number then $Q_{\mathbb{R}}(k)$ is also natural and corresponds to the number of *all* (natural) divisors of k greater than 1.

The numerical computations actually suggest, a much weaker condition is enough for the sum to converge: one can take the square of Q^L instead of taking the L-th power

$$Q_{\mathbb{R}}(x) = \sum_{L=2}^{\infty} \left[Q^L(x)\right]^2 .$$

I study this kind of expression later in Section 6.2.

6 Pushing calculations (too much) - brainstorming

In this section I allow myself "free" computations neglecting the mathematical rules: changing the order of infinite sums, using expansions beyond their convergence domains, etc... This being said, I will make no more comments on this issue. The aim is to satisfy my curiosity and see what interesting formal structures are hidden in the equations.

6.1 Power expansion of cotangent and cosecant

The power expressions for cot and csc are known. However, in order to arrive to the Riemann zeta function, it is convenient to start with $L = 1$ and to switch to an infinite summation

$$Q(k) = \lim_{k' \to k} \left[\left(e^{2\pi i k'} - 1 \right) \sum_{L=1}^{\infty} \frac{1}{e^{\frac{2\pi i k'}{L}} - 1} \right]$$

with "k is prime $\Leftrightarrow Q(k) = 2$". I admit the limit is very suspicious: clearly the function is well defined for any natural number when written in the form

$$Q(k) = \sum_{L=1}^{\infty} \sum_{n=0}^{L-1} e^{2\pi i \frac{n}{L} k}.$$

It seems, however, it is not defined in any neighborhood of it[8]. Therefore, the limit expression resulting from summing up the geometric series might not make sense (limit = approaching the natural number from its neighborhood). This said, let me come back to the subject: after the modifications, the imaginary part of the sum looks like

$$\begin{aligned}
\mathrm{Im}\left(\sum_{L=1}^{\infty} \frac{1}{e^{i\frac{\varphi}{L}} - 1} \right) &= -\frac{1}{2} \left[\sum_{L=1}^{\infty} \cot\left(\frac{\varphi}{L}\right) + \sum_{L=1}^{\infty} \csc\left(\frac{\varphi}{L}\right) \right] \\
&= -\frac{1}{2} \sum_{L=1}^{\infty} \left[\cot\left(\frac{\varphi}{L}\right) + \csc\left(\frac{\varphi}{L}\right) \right] \\
&= -\frac{1}{2} \sum_{L=1}^{\infty} \left\{ \left[\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} B_{2n}}{(2n)!} \left(\frac{\varphi}{L}\right)^{2n-1} \right] + \left[\sum_{n=0}^{\infty} \frac{(-1)^{n+1} 2 \left(2^{2n-1} - 1\right) B_{2n}}{(2n)!} \left(\frac{\varphi}{L}\right)^{2n-1} \right] \right\} \\
&= -\frac{1}{2} \sum_{L=1}^{\infty} \left\{ \sum_{n=0}^{\infty} \left[\frac{(-1)^n 2^{2n} B_{2n}}{(2n)!} \left(\frac{\varphi}{L}\right)^{2n-1} + \frac{(-1)^{n+1} 2 \left(2^{2n-1} - 1\right) B_{2n}}{(2n)!} \left(\frac{\varphi}{L}\right)^{2n-1} \right] \right\} \\
&= -\frac{1}{2} \sum_{L=1}^{\infty} \left\{ \sum_{n=0}^{\infty} \left[\left\{ (-1)^n 2^{2n} + (-1)^{n+1} 2 \left(2^{2n-1} - 1\right) \right\} \frac{B_{2n}}{(2n)!} \left(\frac{\varphi}{L}\right)^{2n-1} \right] \right\} \\
&= -\frac{1}{2} \sum_{L=1}^{\infty} \left\{ \sum_{n=0}^{\infty} \left[\left\{ 2^{2n} - 2 \left(2^{2n-1} - 1\right) \right\} \frac{(-1)^n B_{2n}}{(2n)!} \left(\frac{\varphi}{L}\right)^{2n-1} \right] \right\} \\
&= -\frac{1}{2} \sum_{L=1}^{\infty} \left\{ \sum_{n=0}^{\infty} \left[\left\{ 2^{2n} - 2^{2n} + 2 \right\} \frac{(-1)^n B_{2n}}{(2n)!} \left(\frac{\varphi}{L}\right)^{2n-1} \right] \right\} \\
&= -\sum_{L=1}^{\infty} \left\{ \sum_{n=0}^{\infty} \left[\frac{(-1)^n B_{2n}}{(2n)!} \left(\frac{\varphi}{L}\right)^{2n-1} \right] \right\}.
\end{aligned}$$

[8] Based on numerical observations, this needs to be shown.

Surprisingly, a rather compact expression is obtained. Changing the summation order one has

$$\begin{aligned}
\mathrm{Im}\left(\sum_{L=1}^{\infty}\frac{1}{e^{i\frac{\varphi}{L}}-1}\right) &= -\sum_{n=0}^{\infty}\left\{\sum_{L=1}^{\infty}\left[\frac{(-1)^{n}B_{2n}}{(2n)!}\left(\frac{1}{L}\right)^{2n-1}\varphi^{2n-1}\right]\right\} \\
&= \sum_{n=0}^{\infty}\left[\frac{(-1)^{n+1}B_{2n}}{(2n)!}\varphi^{2n-1}\sum_{L=1}^{\infty}\left(\frac{1}{L}\right)^{2n-1}\right] \\
&= \sum_{n=0}^{\infty}\left[\frac{(-1)^{n+1}B_{2n}}{(2n)!}(2\pi k)^{2n-1}\zeta(2n-1)\right] \\
&= \sum_{n=0}^{\infty}\frac{(-1)^{n+1}(2\pi)^{2n-1}\zeta(2n-1)B_{2n}}{(2n)!}k^{2n-1}.
\end{aligned}$$

It is quite unfortunate that we get an odd number as the Riemann zeta function argument instead of an even one. In the latter case we would avoid the pathology for $n=1$, where $\zeta(1)$ is not defined and an explicit formula formula with Bernoulli numbers would be available. Ignoring all the pathologies, the "formal" summary is

$$\begin{aligned}
Q(k) &= \lim_{k'\to k}\left[\left(e^{2\pi ik'}-1\right)\sum_{L=1}^{\infty}\frac{1}{e^{\frac{2\pi ik'}{L}}-1}\right] \\
&= \lim_{k'\to k}\left\{\left(e^{2\pi ik'}-1\right)\left[\mathrm{Re}\left(\sum_{L=1}^{\infty}\frac{1}{e^{\frac{2\pi ik'}{L}}-1}\right)+i\,\mathrm{Im}\left(\sum_{L=1}^{\infty}\frac{1}{e^{\frac{2\pi ik'}{L}}-1}\right)\right]\right\} \\
&= \lim_{k'\to k}\left\{\left(e^{2\pi ik'}-1\right)\left[-\frac{k'}{2}+i\sum_{n=0}^{\infty}\frac{(-1)^{n+1}(2\pi)^{2n-1}\zeta(2n-1)B_{2n}}{(2n)!}k'^{2n-1}\right]\right\} \\
&\quad k\text{ is natural} \\
&= \lim_{\epsilon\to 0}\left[-2\pi\varepsilon\sum_{n=0}^{\infty}\frac{(-1)^{n+1}(2\pi)^{2n-1}\zeta(2n-1)B_{2n}}{(2n)!}(k+\varepsilon)^{2n-1}\right] \\
&= \lim_{\epsilon\to 0}\left[\varepsilon\sum_{n=0}^{\infty}\frac{(-1)^{n}(2\pi)^{2n}\zeta(2n-1)B_{2n}}{(2n)!}(k+\varepsilon)^{2n-1}\right].
\end{aligned}$$

6.2 Exploring "alternative" sine-based expression

Numerical results (= increasing L_{max}) about the summation

$$Q(x)=\sum_{L=1}^{\infty}\frac{1}{L^{2}}\frac{\sin^{2}(\pi x)}{\sin^{2}\left(\frac{\pi}{L}x\right)} \tag{5}$$

are ambiguous. Their optimistic interpretation is: the sum converges and is worth to be studied. I start by putting the L-independent term in front of the summation symbol

$$Q(k) = \lim_{k'\to k}\left[\sin^{2}(\pi k')\sum_{L=1}^{\infty}\frac{1}{L^{2}}\frac{1}{\sin^{2}\left(\frac{\pi}{L}k'\right)}\right].$$

Further, one has

$$\begin{aligned}
\frac{1}{\sin^{2}(\alpha)} &= \csc^{2}(\alpha) \\
&= \sum_{k=-\infty}^{+\infty}\frac{1}{(k\pi+\alpha)^{2}}.
\end{aligned}$$

So[9]

$$
\begin{aligned}
Q(k) &= \lim_{k'\to k}\left[\sin^2(\pi k')\sum_{L=1}^{\infty}\frac{1}{L^2}\sum_{q=-\infty}^{+\infty}\frac{1}{\left(q\pi+\frac{\pi}{L}k'\right)^2}\right]\\
&= \lim_{k'\to k}\left[\sin^2(\pi k')\sum_{L=1}^{\infty}\frac{1}{L^2}\sum_{q=-\infty}^{+\infty}\frac{1}{\left(\frac{q\pi L+\pi k'}{L}\right)^2}\right]\\
&= \lim_{k'\to k}\left[\sin^2(\pi k')\sum_{L=1}^{\infty}\frac{L^2}{L^2}\sum_{q=-\infty}^{+\infty}\frac{1}{\left(q\pi L+\pi k'\right)^2}\right]\\
&= \lim_{k'\to k}\left[\frac{\sin^2(\pi k')}{\pi^2}\sum_{L=1}^{\infty}\sum_{q=-\infty}^{+\infty}\frac{1}{\left(qL+k'\right)^2}\right]\\
&= \lim_{k'\to k}\left[\frac{\sin^2(\pi k')}{\pi^2}\sum_{q=-\infty}^{+\infty}\sum_{L=1}^{\infty}\frac{1}{\left[q\left(L+\frac{k'}{q}\right)\right]^2}\right]\\
&= \lim_{k'\to k}\left[\frac{\sin^2(\pi k')}{\pi^2}\sum_{q=-\infty}^{+\infty}\frac{1}{q^2}\sum_{L=1}^{\infty}\frac{1}{\left(L+\frac{k'}{q}\right)^2}\right]\\
&= \lim_{k'\to k}\left\{\frac{\sin^2(\pi k')}{\pi^2}\sum_{q=-\infty}^{+\infty}\frac{1}{q^2}\left[\zeta\left(2,\frac{k'}{q}\right)-\frac{1}{\frac{k'^2}{q^2}}\right]\right\}\\
&= \lim_{k'\to k}\left\{\frac{\sin^2(\pi k')}{\pi^2}\sum_{q=-\infty}^{+\infty}\frac{1}{q^2}\left[\zeta\left(2,\frac{k'}{q}\right)-\frac{q^2}{k'^2}\right]\right\}\\
&= \lim_{k'\to k}\left\{\frac{\sin^2(\pi k')}{\pi^2}\sum_{q=-\infty}^{+\infty}\left[\frac{\zeta\left(2,\frac{k'}{q}\right)}{q^2}-\frac{1}{k'^2}\right]\right\}\\
&= \lim_{k'\to k}\frac{\sin^2(\pi k')}{\pi^2}\left\{\sum_{q=1}^{+\infty}\left[\frac{\zeta\left(2,\frac{k'}{q}\right)}{q^2}+\frac{\zeta\left(2,-\frac{k'}{q}\right)}{q^2}-\frac{2}{k'^2}\right]+\left[\lim_{q\to 0}\frac{\zeta\left(2,\frac{k'}{q}\right)}{q^2}-\frac{1}{k'^2}\right]\right\}.
\end{aligned}
$$

Expression is rather ugly, not very simple and limits appear. But, surprisingly enough, numerical computation suggest that this expression is a "good" one. It seems the expression 5 indeed converges, limits exist and the numerator and the denominator are finite in the neighborhood of a natural number. They can be calculated for $k+\varepsilon$ (k being natural) and their ratio taken. The true question now is: how quickly the expression converges with respect to q_{max} and how does it correlate with the size of the natural number k one wants to check for primality. I use WxMaxima and observe (on a standard computer):

- If I check for primality of the 1000-th prime number $k=7919$ and I take $\varepsilon=10^{-6}$, $q_{max}=10^4$, then the test works and takes couple of seconds.
- The test with the previous settings is successful for $k=9931$.
- The test with the previous settings fails for $k=10067$ (the next prime number after $k=9931$).

The correlation seems to be quite unfortunate: I summed over L, but it seems the q variable took exactly its function, the test works if $k\lesssim q_{max}$, no better. Nobody expects to get a summation-free expression. One however hopes to trade the "unfortunate" L-summation for a different summation that would not strongly depend of the size of k. This is the only case I managed to sum over L: the trade happened, the profit is zero. Let me remind you that for this case the condition "k is prime $\Leftrightarrow Q(k)=2$" applies. The WxMaxima code is in Appendix.

6.3 Partial fraction expansion of cotangent and cosecant

Haven found additional expansions for cot and csc I come back to investigating the sum

$$\sum_{L=1}^{\infty}\cot\left(\pi\frac{2k}{L}\right)+\csc\left(\pi\frac{2k}{L}\right).$$

[9] ζ stands for the Hurwitz zeta function.

One has

$$\pi\cot(\pi x)=\frac{1}{x}+\sum_{n=1}^{\infty}\frac{2x}{x^2-n^2},\ \pi\csc(\pi x)=\frac{1}{x}+\sum_{n=1}^{\infty}\frac{(-1)^n 2x}{x^2-n^2},$$

so

$$\cot(\pi x)=\frac{1}{\pi}\left(\frac{1}{x}+\sum_{n=1}^{\infty}\frac{2x}{x^2-n^2}\right),\ \csc(\pi x)=\frac{1}{\pi}\left(\frac{1}{x}+\sum_{n=1}^{\infty}\frac{(-1)^n 2x}{x^2-n^2}\right).$$

This implies

$$\begin{aligned}
\operatorname{Im}\left(\sum_{L=1}^{\infty}\frac{1}{e^{i\frac{\varphi}{L}}-1}\right) &= -\frac{1}{2}\sum_{L=1}^{\infty}\left[\cot\left(\pi\frac{2k}{L}\right)+\csc\left(\pi\frac{2k}{L}\right)\right]\\
&= -\frac{1}{2}\sum_{L=1}^{\infty}\left[\frac{1}{\pi}\left(\frac{1}{\frac{2k}{L}}+\sum_{n=1}^{\infty}\frac{2\frac{2k}{L}}{\left(\frac{2k}{L}\right)^2-n^2}\right)+\frac{1}{\pi}\left(\frac{1}{\frac{2k}{L}}+\sum_{n=1}^{\infty}\frac{(-1)^n 2\frac{2k}{L}}{\left(\frac{2k}{L}\right)^2-n^2}\right)\right]\\
&= -\frac{1}{2\pi}\sum_{L=1}^{\infty}\left[\frac{L}{k}+\sum_{n=1}^{\infty}\frac{\frac{4}{L}k}{\frac{4}{L^2}k^2-n^2}+\sum_{n=1}^{\infty}\frac{(-1)^n\frac{4}{L}k}{\frac{4}{L^2}k^2-n^2}\right]\\
&= -\frac{1}{2\pi}\sum_{L=1}^{\infty}\left[\frac{L}{k}+\frac{4k}{L}\left(\sum_{n=1}^{\infty}\frac{1}{\frac{4}{L^2}k^2-n^2}+\sum_{n=1}^{\infty}\frac{(-1)^n}{\frac{4}{L^2}k^2-n^2}\right)\right]\\
&= -\frac{1}{2\pi}\sum_{L=1}^{\infty}\left[\frac{L}{k}+\frac{8k}{L}\sum_{n=1}^{\infty}\frac{1}{\frac{4}{L^2}k^2-(2n)^2}\right]\\
&= -\frac{1}{2\pi}\sum_{L=1}^{\infty}\left[\frac{L}{k}+\frac{8k}{L}\sum_{n=1}^{\infty}\frac{1}{\frac{4}{L^2}k^2-4n^2}\right]\\
&= -\frac{1}{2\pi}\sum_{L=1}^{\infty}\left[\frac{L}{k}+\frac{2k}{L}\sum_{n=1}^{\infty}\frac{1}{\frac{1}{L^2}k^2-n^2}\right]\\
&= -\frac{1}{2\pi}\sum_{L=1}^{\infty}\left[\frac{L}{k}+\frac{2k}{L}\sum_{n=1}^{\infty}\frac{1}{\frac{k^2-L^2n^2}{L^2}}\right]\\
&= -\frac{1}{2\pi}\sum_{L=1}^{\infty}\left[\frac{L}{k}+\frac{2k}{L}\sum_{n=1}^{\infty}\frac{L^2}{k^2-L^2n^2}\right]\\
&= -\frac{1}{2\pi}\sum_{L=1}^{\infty}\left[\frac{L}{k}+2kL\sum_{n=1}^{\infty}\frac{1}{k^2-L^2n^2}\right]\\
&= -\frac{1}{2\pi}\sum_{L=1}^{\infty}\left[\frac{L}{k}+2kL\sum_{n=1}^{\infty}\frac{1}{k^2\left(1-\frac{L^2n^2}{k^2}\right)}\right]\\
&= -\frac{1}{2\pi}\sum_{L=1}^{\infty}\left[\frac{L}{k}+2\frac{L}{k}\sum_{n=1}^{\infty}\frac{1}{1-\frac{L^2n^2}{k^2}}\right]\\
&= -\frac{1}{2\pi}\sum_{L=1}^{\infty}\left[\frac{L}{k}\left(1+2\sum_{n=1}^{\infty}\frac{1}{1-\frac{L^2n^2}{k^2}}\right)\right]\\
&= -\frac{1}{2\pi k}\sum_{L=1}^{\infty}\left(L+2L\sum_{n=1}^{\infty}\frac{1}{1-\frac{L^2n^2}{k^2}}\right).
\end{aligned}$$

A compact expression is obtained, but I do not see how it could be processed any further. One would like to switch the summations, in order to get rid of the L-sum. It is clear that, to test an entire number k, one needs to sum $L\sim k$ terms, which is bad (if we want to get an efficient primality test). But presumably the sum of the type

$$\sum_{m=1}^{\infty}\frac{m}{1-a^2m^2}\leftrightarrow\sum_{L=1}^{\infty}\frac{L}{1-\frac{n^2}{k^2}L^2}$$

diverge for any a (probably becoming too close to a harmonic series with increasing m). It seems the summation oven n can be done, but this is "one step forward and one step back", returning to an expression summed over different period lengths. The following formula may be true[10]

$$\sum_{n=0}^{\infty}\frac{1}{1-a^2n^2}=\frac{1}{2}+\frac{1}{2a}\pi\cot\left(\frac{\pi}{a}\right),$$

so

$$\begin{aligned}
\mathrm{Im}\left(\sum_{L=1}^{\infty}\frac{1}{e^{i\frac{\varphi}{L}}-1}\right) &= -\frac{1}{2\pi k}\sum_{L=1}^{\infty}\left(L+2L\sum_{n=1}^{\infty}\frac{1}{1-\frac{L^2n^2}{k^2}}\right)\\
&= -\frac{1}{2\pi k}\sum_{L=1}^{\infty}\left(L-2L+2L\sum_{n=0}^{\infty}\frac{1}{1-\frac{L^2n^2}{k^2}}\right)\\
&= -\frac{1}{2\pi k}\sum_{L=1}^{\infty}\left[2L\left(\sum_{n=0}^{\infty}\frac{1}{1-\frac{L^2}{k^2}n^2}\right)-L\right]\\
&= -\frac{1}{2\pi k}\sum_{L=1}^{\infty}\left\{2L\left[\frac{1}{2}+\frac{1}{2\frac{L}{k}}\pi\cot\left(\frac{\pi}{\frac{L}{k}}\right)\right]-L\right\}\\
&= -\frac{1}{2\pi k}\sum_{L=1}^{\infty}\left\{2L\left[\frac{1}{2}+\frac{k}{2L}\pi\cot\left(\frac{\pi k}{L}\right)\right]-L\right\}\\
&= -\frac{1}{2\pi k}\sum_{L=1}^{\infty}\left[k\pi\cot\left(\frac{\pi k}{L}\right)\right]\\
&= -\frac{1}{2}\sum_{L=1}^{\infty}\cot\left(\frac{\pi k}{L}\right).
\end{aligned}$$

Expression is nice, but summation over period lengths remains. A very small step "forward" can be claimed: instead two trigonometric functions, only one appears. This could be probably achieved using some standard trigonometric identities. To get a finite formula I apply the cutoff $L_{max}=k$. The primality test then looks like

$$\begin{aligned}
\lim_{k'\to k}\left[\left(e^{2\pi ik'}-1\right)\sum_{L=1}^{k}\frac{1}{e^{\frac{2\pi ik'}{L}}-1}\right] &= -2\pi\lim_{\varepsilon\to 0}\left\{\varepsilon\sum_{L=1}^{k}\mathrm{Im}\left[\frac{1}{e^{\frac{2\pi ik'}{L}}-1}\right]\right\}\\
&= \pi\lim_{\varepsilon\to 0}\varepsilon\sum_{L=1}^{k}\cot\left[\frac{\pi\left(k+\varepsilon\right)}{L}\right].
\end{aligned}$$

It is certainly a nice expression, with the criterion "k is prime $\Leftrightarrow Q\left(k\right)=k+1$".

6.4 Removing limit for n-summed expression

In the expression with the sine function

$$Q^L\left(x\right)=\frac{1}{L^2}\frac{\sin^2\left(\pi x\right)}{\sin^2\left(\frac{\pi}{L}x\right)}$$

the Chebyshev polynomials of the second kind U_n can be used to remove the limit. The procedure stands

$$\begin{aligned}
\frac{1}{L^2}\frac{\sin^2\left(\pi x\right)}{\sin^2\left(\frac{\pi}{L}x\right)} &= \left[\frac{\sin\left(L\frac{\pi x}{L}\right)}{L\sin\left(\frac{\pi x}{L}\right)}\right]^2\\
&= \left[\frac{\sin\left(\frac{\pi x}{L}\right)U_{L-1}\left[\cos\left(\frac{\pi x}{L}\right)\right]}{L\sin\left(\frac{\pi x}{L}\right)}\right]^2\\
&= \frac{1}{L^2}U_{L-1}^2\left[\cos\left(\frac{\pi x}{L}\right)\right]
\end{aligned}$$

Getting so

$$\begin{aligned}
Q\left(k\right) &= \lim_{k'\to k}\sum_{L}\frac{1}{L^2}\frac{\sin^2\left(\pi k'\right)}{\sin^2\left(\frac{\pi}{L}k'\right)}\\
&= \sum_{L}\frac{1}{L^2}U_{L-1}^2\left(\cos\frac{\pi k}{L}\right).
\end{aligned}$$

[10] Suggested by Wolfram Mathematica online "Series Calculator".

I doubt this helps in any way, but it is nice to see the limit disappear.

6.5 Three sums reordered

One can take the real part of the initial expression

$$\sum_{L=2}^{\infty}\sum_{n=0}^{L-1} e^{i\frac{2\pi}{L}nk}$$

and try to proceed with three sums (B_j refers to the Bernoulli polynomials)

$$\begin{aligned}
\mathrm{Re}\left[\sum_{L=2}^{\infty}\sum_{n=0}^{L-1}\exp\left(i\frac{2\pi}{L}nk\right)\right] &= \sum_{L=2}^{\infty}\sum_{n=0}^{L-1}\cos\left(\frac{2\pi}{L}nk\right) \\
&= \sum_{L=2}^{\infty}\sum_{n=0}^{L-1}\sum_{j=0}^{\infty}\frac{(-1)^j}{(2j)!}\left(\frac{2\pi}{L}nk\right)^{2j} \\
&= \sum_{j=0}^{\infty}\frac{(-1)^j(2\pi)^{2j}}{(2j)!}k^{2j}\sum_{L=2}^{\infty}\sum_{n=0}^{L-1}\frac{1}{L^{2j}}n^{2j} \\
&= \sum_{j=0}^{\infty}\frac{(-1)^j(2\pi)^{2j}}{(2j)!}k^{2j}\sum_{L=2}^{\infty}\frac{1}{L^{2j}}\sum_{n=0}^{L-1}n^{2j} \\
&= \sum_{j=0}^{\infty}\frac{(-1)^j(2\pi)^{2j}}{(2j)!}k^{2j}\sum_{L=2}^{\infty}\frac{1}{L^{2j}}\frac{B_{2j+1}(L)-B_{2j+1}(0)}{2j+1} \\
&= \sum_{L=2}^{\infty}\left[B_1(L)-B_1(0)\right]+\sum_{j=1}^{\infty}\frac{(-1)^j(2\pi)^{2j}}{(2j)!}k^{2j}\sum_{L=2}^{\infty}\frac{1}{L^{2j}}\frac{B_{2j+1}(L)}{2j+1} \\
&= \sum_{L=2}^{\infty}\left[L-\frac{1}{2}+\frac{1}{2}\right]+\sum_{j=1}^{\infty}\frac{(-1)^j(2\pi)^{2j}}{(2j)!}k^{2j}\sum_{L=2}^{\infty}\frac{1}{L^{2j}}\frac{B_{2j+1}(L)}{2j+1} \\
&= \sum_{L=2}^{\infty}L+\sum_{j=1}^{\infty}\frac{(-1)^j(2\pi)^{2j}}{(2j)!}k^{2j}\sum_{L=2}^{\infty}\frac{1}{L^{2j}}\frac{B_{2j+1}(L)}{2j+1} \\
&= \sum_{L=2}^{\infty}L+\sum_{j=1}^{\infty}\frac{(-1)^j(2\pi)^{2j}}{(2j)!(2j+1)}k^{2j}\sum_{L=2}^{\infty}\frac{B_{2j+1}(L)}{L^{2j}}
\end{aligned}$$

One however arrives to the pathology $\sum_{L=2}^{\infty}L$.

More might be expected from squaring the "normalized" expression (i.e. with the $1/L$ factor), the numerical analysis suggests it converges for positive real numbers (as already mentioned). Formal procedure looks like

$$
\begin{aligned}
\sum_{L=2}^{\infty}\left[\frac{1}{L}\sum_{n=0}^{L-1}\cos\left(\frac{2\pi}{L}nk\right)\right]^2 &= \sum_{L=2}^{\infty}\frac{1}{L^2}\left[\sum_{n=0}^{L-1}\cos\left(\frac{2\pi}{L}nk\right)\right]^2 \\
& \left(\sum_{w}a_w\right)^2=\sum_{q,w}a_qa_w \\
&= \sum_{L=2}^{\infty}\frac{1}{L^2}\sum_{n=0}^{L-1}\sum_{m=0}^{L-1}\left[\cos\left(\frac{2\pi}{L}nk\right)\cos\left(\frac{2\pi}{L}mk\right)\right] \\
&= \sum_{L=2}^{\infty}\frac{1}{L^2}\sum_{n=0}^{L-1}\sum_{m=0}^{L-1}\left[\cos(ak)\cos(bk)\right] \\
& \text{with } a=\frac{2\pi}{L}n \text{ and } b=\frac{2\pi}{L}m \\
&= \sum_{L=2}^{\infty}\frac{1}{L^2}\sum_{n=0}^{L-1}\sum_{m=0}^{L-1}\sum_{j=0}^{\infty}\frac{(-1)^j}{(2j)!}\left[\sum_{q=0}^{j}\binom{2j}{2q}a_{n,L}^{2(j-q)}b_{m,L}^{2q}\right]k^{2j} \\
&= \sum_{j=0}^{\infty}\frac{(-1)^j}{(2j)!}k^{2j}\sum_{q=0}^{j}\binom{2j}{2q}\sum_{L=2}^{\infty}\frac{1}{L^2}\sum_{n=0}^{L-1}\sum_{m=0}^{L-1}\left(\frac{2\pi}{L}n\right)^{2(j-q)}\left(\frac{2\pi}{L}m\right)^{2q} \\
&= \sum_{j=0}^{\infty}\frac{(-1)^j}{(2j)!}k^{2j}\sum_{q=0}^{j}\binom{2j}{2q}(2\pi)^{2(j-q)}(2\pi)^{2q}\sum_{L=2}^{\infty}\frac{1}{L^2}\sum_{n=0}^{L-1}\sum_{m=0}^{L-1}\frac{1}{L^{2(j-q)}}n^{2(j-q)}\frac{1}{L^{2q}}m^{2q} \\
&= \sum_{j=0}^{\infty}\frac{(-1)^j}{(2j)!}k^{2j}\sum_{q=0}^{j}\binom{2j}{2q}(2\pi)^{2j}\sum_{L=2}^{\infty}\frac{1}{L^2}\frac{1}{L^{2j}}\sum_{n=0}^{L-1}n^{2(j-q)}\sum_{m=0}^{L-1}m^{2q} \\
&= \sum_{j=0}^{\infty}\frac{(-1)^j(2\pi)^{2j}}{(2j)!}k^{2j}\sum_{q=0}^{j}\binom{2j}{2q}\times \\
&\quad \sum_{L=2}^{\infty}\frac{1}{L^2}\frac{1}{L^{2j}}\frac{B_{2(j-q)+1}(L)-B_{2(j-q)+1}(0)}{2(j-q)+1}\frac{B_{2q+1}(L)-B_{2q+1}(0)}{2q+1} \\
&= \sum_{j=0}^{\infty}\frac{(-1)^j(2\pi)^{2j}}{(2j)!}k^{2j}\sum_{q=0}^{j}\binom{2j}{2q}\frac{(2\pi)^{2j}}{(2j-2q+1)(2q+1)}\times \\
&\quad \sum_{L=2}^{\infty}\frac{\left[B_{2(j-q)+1}(L)-B_{2(j-q)+1}\right]\left[B_{2q+1}(L)-B_{2q+1}\right]}{L^{2(1+j)}} \\
&= \sum_{j=0}^{\infty}\lambda(j)k^{2j}
\end{aligned}
$$

where

$$
\lambda(j)=\sum_{q=0}^{j}\binom{2j}{2q}\frac{(-1)^j(2\pi)^{2j}}{(2j-2q+1)(2q+1)(2j)!}\sum_{L=2}^{\infty}\frac{\left[B_{2(j-q)+1}(L)-B_{2(j-q)+1}\right]\left[B_{2q+1}(L)-B_{2q+1}\right]}{L^{2(1+j)}}.
$$

Three (too much) sums remain and, in addition, numerical computations suggest a divergent behavior.

Many other possibilities could be investigated. For example, one could from the beginning interchange the L and n summation realizing that the two double sums

$$
\sum_{L=1}^{\infty}\sum_{n=0}^{L-1},\ \sum_{n=0}^{\infty}\sum_{L=n+1}^{\infty}
$$

cover the same region in the L,n plane. Loosing my hopes about possible better convergence properties of such and similar approaches, I stop my investigations here.

7 Conclusion

I addressed two points in this text: beauty and efficiency. I believe I succeeded in finding beautiful formulas, but I did not managed to convert them into an efficient primality test (this being presumably related to the L-summation). I think I made clear that the expressions of the form

$$\sum_L g\left(\frac{\varphi}{L}\right),$$

where g is an exponential or a trigonometric function, play an important role in the prime-number mathematics.

Looking on the internet I discovered a huge amount of existing mathematical information, tools and methods which could be used in this domain and may allow for further treatment of the above expressions. As a mathematical hobbyist, I would need to invest more time and energy to further explore this topic than I actually can.

I apologize for too detailed computations. I appreciate when an author is very detailed, so try to be I.

Appendix

WxMaxima code for primality test with Hurwitz zeta function (sine-based expression)

```
load ("bffac")$
prec : 30$
hwzInSum(k,q) := (bfhzeta(2,k/q,prec) + bfhzeta(2,-k/q,prec))/q^2 - 2/k^2;
hwzSum(k,qMax) := sum(hwzInSum(k,q),q,1,qMax);
hwzPrime(k,eps,qMax) := ( sin(%pi*(k+eps))^2/%pi^2
        * (bfhzeta(2,(k+eps)/eps,prec)/eps^2 - 1/(k+eps)^2
        + hwzSum(k+eps,qMax)) );
float(hwzPrime(9931,0.000001,10000));
```