# Gravitational Shift for Beginners

*This paper, which I wrote in 2006, formulates the equations for gravitational shifts from the relativistic framework of special relativity. First I derive the formulas for the gravitational redshift and then the formulas for the gravitational blueshift.*

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# 1. Gravitational Redshift

Let's consider a star of mass *M* and radius *R* that emits a photon as shown in Fig 1. The photon travels through empty space an arbitrary distance *r* before reaching our planet. We shall also consider an observer located on Earth who measures the frequency $f = f(r)$ of the received photon.

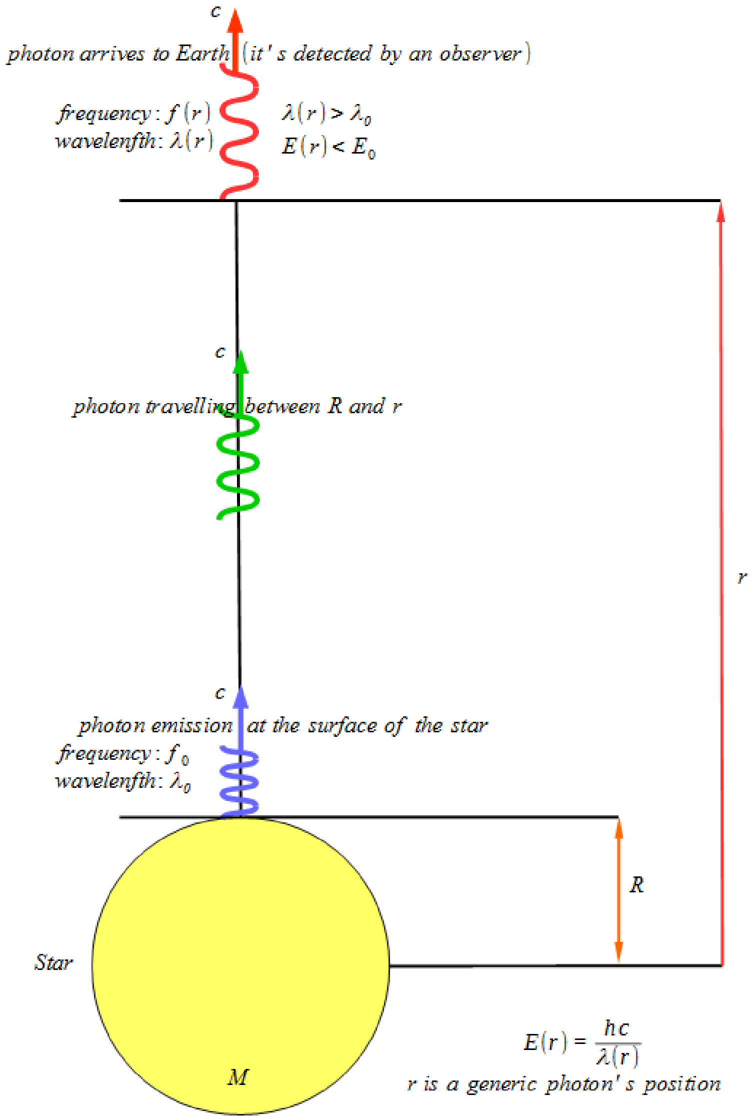


***Fig 1:*** *Gravitational redshift.*

We want to calculate the wavelength shift produce by the star's gravity on the emitted photon. Thus, if the initial frequency of the emitted photon was $f_0$ (on the surface of the star), the final frequency of the photon, just before detection, will be $f(r)$. Thus, we want to find the formula that gives the frequency, $f$, as a function of *r* and $f_0$.

According to the law of conservation of energy we can write

*Conservation of energy* $$E(r) + U(r) = E_0 + U_0 \quad (1.1)$$

Because photons are considered to be massless, their total energy, $E$, is identical to their kinetic energy, *K*.

$E(r) =$ Total (or kinetic) energy of the photon at a distance *r* from the center of the star

$E_0 =$ Total (or kinetic) energy of the photon on the surface of the star

$U(r) =$ Gravitational potential energy of the photon at a distance *r* from the center of the star

$U_0 =$ Gravitational potential energy of the photon on the surface of the star

The energy of the photon on the surface of the Earth can be expressed as

$$E(r) = h f(r) \tag{1.2}$$

The energy of the photon on the surface of the star can be expressed as

$$E_0 = h f_0 \tag{1.3}$$

The Gravitational potential energy is given by

*Gravitational potential energy*

$$U(r) = -\frac{G M m}{r} \tag{1.4}$$

Where *m* is the equivalent mass of the photon and can be derived form the following equation

$$h f = m c^2 \tag{1.5}$$

The first side of this equation is the energy of the photon according to Planck's theory of electromagnetic radiation. The second side of the equation is the same energy according to Einstein's formula of equivalence of mass and energy. Hence, solving this equation for *m* yields

*Equivalent mass of the photon*

$$m = \frac{h f}{c^2} \tag{1.6}$$

Replacing the variable *m* in equation (3.4), by the value found in eq. (3.6) we have

$$U(r) = -\frac{G M h}{c^2} \frac{f}{r} \tag{1.7}$$

This is the potential energy of the photon at a distance *r* from the center of the star. Similarly, the potential energy of the photon on the surface of the star is

$$U_0 = -\frac{G M h}{c^2} \frac{f_0}{R} \tag{1.8}$$

From the equation of conservation of energy, (1.1), and from equations (1.2), (1.3), (1.7) and (1.8) we can write

$$hf(r) - \frac{GMh}{c^2}\frac{f(r)}{r} = hf_0 - \frac{GMh}{c^2}\frac{f_0}{R} \tag{1.9}$$

Dividing by *h* both sides and taking a common factor, $f(r)$ , on the first side of the equation and a common factor, $f_0$ , on the second side, we have

$$f(r)\left(1 - \frac{GM}{c^2}\frac{1}{r}\right) = f_0\left(1 - \frac{GM}{c^2}\frac{1}{R}\right) \tag{1.10}$$

Solving this equation for $f(r)$ yields

$$f(r) = \left(\frac{1 - \frac{GM}{c^2}\frac{1}{R}}{1 - \frac{GM}{c^2}\frac{1}{r}}\right) f_0 \tag{1.11}$$

This formula gives the frequency of the photon as a function of the distance *r* from the center of the star. This is the frequency an observer would measure if gravitational shifts were the only shifts acting in the universe (of course this is not true).

Now if the distance *r* is infinite the previous formula transforms into

$$f(\infty) = f_\infty = \left(1 - \frac{GM}{c^2}\frac{1}{R}\right) f_0 \tag{1.12}$$

Now we define the "special" Schwarzchild radius as

$$R_0 \equiv \frac{GM}{c^2} \tag{1.13}$$

It is worthwhile to observe that, because this approach doesn't use general relativity, the value of the "special" Schwarzchild radius is half of what we get when we apply Einstein's field equations. Inserting this value, the two previous equations become

$$f(r) = \left(\frac{1 - \frac{R_0}{R}}{1 - \frac{R_0}{r}}\right) f_0 \tag{1.14}$$

and

$$f_\infty = \left(1 - \frac{R_0}{R}\right) f_0 \tag{1.15}$$

Now I shall express equations (1.14) and (1.15) in terms of the wavelength of the photon. To achieve that we consider the following formula for the speed of light

$$c = \frac{\lambda(r)}{T(r)} = \lambda(r) f(r) \tag{1.16}$$

Thus

$$f(r) = \frac{c}{\lambda(r)} \tag{1.17}$$

Now I use eq. (1.17) to eliminate *f* from eq. (1.14) and I solve it for $\lambda(r)$

$$\lambda(r) = \left( \frac{1 - \frac{R_0}{r}}{1 - \frac{R_0}{R}} \right) \lambda_0 \tag{1.18}$$

Doing a similar work, eq. (1.15) becomes

$$\lambda_\infty = \frac{1}{\left(1 - \frac{R_0}{R}\right)} \lambda_0 \tag{1.19}$$

Now, we assume that $R_0 < R$ this is, the star is not a black hole. Because $r > R$

$$\left(1 - \frac{R_0}{r}\right) > \left(1 - \frac{R_0}{R}\right) \tag{1.20}$$

Then I divide by $\left(1 - \frac{R_0}{R}\right)$ both sides of inequation (1.20). This gives

$$\left( \frac{1 - \frac{R_0}{r}}{1 - \frac{R_0}{R}} \right) > 1 \tag{1.21}$$

Comparing eq. (1.18) with inequation (1.21) we see that the quantity inside the parenthesis is the same in both expressions, therefore $\lambda(r)$ must be greater than $\lambda_0$ Mathematically

$$\lambda(r) > \lambda_0 \tag{1.22}$$

Because the observed wavelength of the photon (measured by an observer) is greater than its initial wavelength (when it was emitted by the star), the energy of the received photon is less that the energy of the emitted photon. Therefore we say that there is a gravitational red shift.

## 2. Gravitational Blueshift

Because the derivation of the formulas for both shifts is identical (with the difference, in the end, that we solve the equations for different parameters), I shall use a slightly different method to make it more interesting. Specifically I shall use a distance *H* which is the distance from the surface of the star to a distant photon that travels towards the star. In the end this photon is swallowed by the star. I assume that there is an observer on the surface of the star who measures (I don't know how) the frequency and the wavelength of the photon at its arrival.

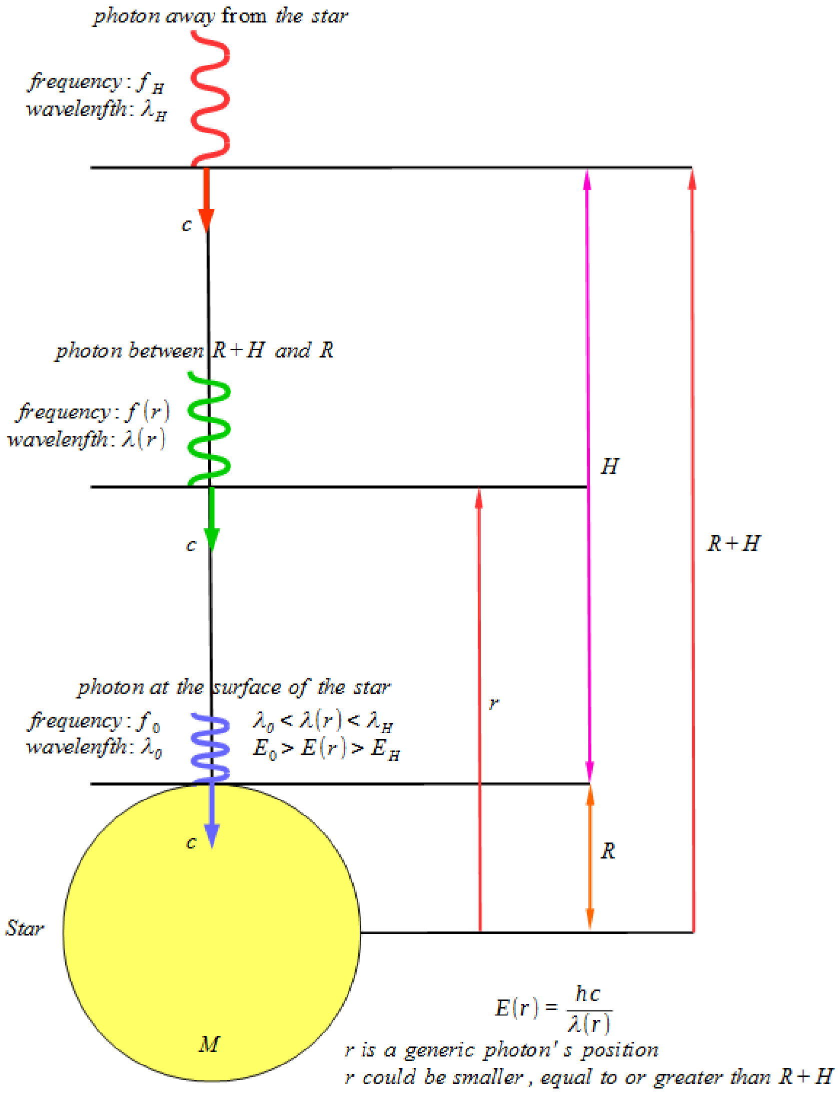


***Fig 2:*** *Gravitational blueshift.*

According to the law of conservation of energy we can write

*Conservation of energy* $$E_H + U_H = E_0 + U_0 \quad (2.1)$$

Because photons are considered to be massless, their total energy *E* is identical to their kinetic energy *K*.

$E_H =$ Total (or kinetic) energy of the photon at a distance $H$ from the surface of the star

$E_0 =$ Total (or kinetic) energy of the photon on the surface of the star

$U_H =$ Gravitational potential energy of the photon at a distance $H$ from the surface of the star

$U_0 =$ Gravitational potential energy of the photon on the surface of the star

The photon energy at position 1 can be expressed as

$$E_H = h f_H \tag{2.2}$$

The photon energy at position 2 can be expressed as

$$E_0 = h f_0 \tag{2.3}$$

The Gravitational potential energy is given by

*Gravitational potential energy*

$$U = -\frac{G M m}{r} \tag{2.4}$$

Where $m$ is the equivalent mass of the photon and can be derived form the following equation

$$h f = m c^2 \tag{2.5}$$

The first side of this equation is the energy of the photon according to Planck's theory of electromagnetic radiation. The second side of the equation is the same energy according to Einstein's formula of equivalence of mass and energy. Hence, solving this equation for $m$ yields

*Equivalent mass of the photon*

$$m = \frac{h f}{c^2} \tag{2.6}$$

Replacing the variable $m$ in equation (2.4), by the value found in eq. (2.6) we have

*Gravitational potential energy*

$$U = -\frac{G M h}{c^2} \frac{f}{r} \tag{2.7}$$

Thus, the potential energy of the photon at a distance $H$ from the surface of the star is

$$U_H = -\frac{G M h}{c^2} \frac{f_H}{R + H} \tag{2.8}$$

Similarly, the potential energy of the photon on the surface of the star is

$$U_0 = -\frac{G M h}{c^2} \frac{f_0}{R} \tag{2.9}$$

From the equation of conservation of energy, (2.1), and from equations (2.2), (2.3), (2.8) and (2.9) we can write

$$h f_H - \frac{G M h}{c^2} \frac{f_H}{R+H} = h f_0 - \frac{G M h}{c^2} \frac{f_0}{R} \tag{2.10}$$

Dividing by *h* both sides and taking a common factor $f_H$ on the first side and a common factor $f_0$ on the second one, we have

$$f_H \left( 1 - \frac{G M}{c^2} \frac{1}{R+H} \right) = f_0 \left( 1 - \frac{G M}{c^2} \frac{1}{R} \right) \tag{2.11}$$

Solving this equation for $f_0$ yields

$$f_0 = \frac{\left( 1 - \frac{G M}{c^2} \frac{1}{R+H} \right)}{\left( 1 - \frac{G M}{c^2} \frac{1}{R} \right)} f_H \tag{2.12}$$

This formula gives the frequency of the photon on the surface of the star as a function of its frequency at a distance *H*.

Now we consider that the distance *H* is infinite. Thus we can take the limit of the previous expression when *H* tends to infinity. Mathematically

$$\lim_{H \to \infty} f_0 = \lim_{H \to \infty} \left( \frac{1 - \frac{G M}{c^2} \frac{1}{R+H}}{1 - \frac{G M}{c^2} \frac{1}{R}} f_H \right) \tag{2.13}$$

The result of this limit is

$$f_0 = \frac{1}{1 - \frac{G M}{c^2} \frac{1}{R}} f_\infty \tag{2.14}$$

Where $f_\infty$ is the frequency of the photon at an infinite distance from the star, in other words: $f_H(\infty) = f_\infty$ . Now we define

$$R_0 \equiv \frac{G M}{c^2} \tag{2.15}$$

It is worthwhile to observe that, because this approach doesn't use general relativity, the value of this radius is half of what we get when we apply Einstein's field equations.

Combining equations (2.14) and (2.15) yields

$$f_0 = \frac{1}{1 - \frac{R_0}{R}} f_\infty \tag{2.16}$$

Now I shall express equations (2.14) and (2.16) in terms of the wavelength of the photon. To achieve that we consider the following formula

$$c = \frac{\lambda}{T} = \lambda f \tag{2.17}$$

Combining equations (2.14) and (2.15) and solving for $\lambda_0$ we get

$$\lambda_0 = \left(1 - \frac{G M}{c^2} \frac{1}{R}\right) \lambda_\infty \tag{2.18}$$

Combining equations (2.16) and (2.15) yields

$$\lambda_0 = \left(1 - \frac{R_0}{R}\right) \lambda_\infty \tag{2.19}$$

Let's interpret this result. If $R > R_0$ then the quotient $R_0 / R$ is less than 1. Mathematically

$$\frac{R_0}{R} < 1 \tag{2.20}$$

This means that that the following inequation must be true

$$\left(1 - \frac{R_0}{R}\right) < 1 \tag{2.21}$$

Thus, in accordance to eq. (2.16) we have the following inequation

$$\lambda_0 < \lambda_\infty \tag{2.22}$$

Where $\lambda_\infty$ is the wavelength of the photon when *H* is infinite. This is $\lambda_H(\infty) = \lambda_\infty$ .
Therefore the wavelength of the photon decreases as it gets closer to the star.
But because the energy of the photon is proportional to the inverse of its wavelength, as the following formula shows,

$$E = \frac{h c}{\lambda} \tag{2.23}$$

the energy of the photon increases as its wavelength decreases. In other words, the energy of the photon increases as it gets closer to the star. This effect is known as blue shift.

From the point of view of the two types of energy involved in the process: potential and kinetic (or total), we can say that, as the photon approaches the star, it loses gravitational potential energy and gains kinetic energy (which is equal to its total relativistic energy). Thus, the energy of the photon on the surface of the star is greater than its energy at infinity. Mathematically we express this fact as follows

$$\underset{\text{Surface}}{\left(E_0 = \frac{hc}{\lambda_0}\right)} > \underset{\text{Infinity}}{\left(E_\infty = \frac{hc}{\lambda_\infty}\right)} \tag{2.24}$$

# 3. Summary of Formulas

The following tables summarize the final formulas derived in the previous sections.

| Red shift $R_0 \equiv \frac{GM}{c^2}$ | Formulas (when $r$ is finite) | Formulas (when $r$ is infinite) |
|---|---|---|
| Frequency measured on Earth | $f(r) = \left(\frac{1-\frac{R_0}{R}}{1-\frac{R_0}{r}}\right) f_0$ | $f_\infty = \left(1-\frac{R_0}{R}\right) f_0$ |
| Wavelength measured on Earth | $\lambda(r) = \left(\frac{1-\frac{R_0}{r}}{1-\frac{R_0}{R}}\right) \lambda_0$ | $\lambda_\infty = \frac{1}{\left(1-\frac{R_0}{R}\right)} \lambda_0$ |

***Table 1:*** *Reshift equations*

| Blue shift $R_0 \equiv \frac{GM}{c^2}$ | Formulas (when $r$ is finite) | Formulas (when $r$ is infinite) |
|---|---|---|
| Frequency on the surface of the star | $f_0(r) = \left(\frac{1-\frac{R_0}{r}}{1-\frac{R_0}{R}}\right) f(r)$ | $f_0 = \frac{1}{1-\frac{R_0}{R}} f_\infty$ |
| Wavelength on the surface of the star | $\lambda_0(r) = \left(\frac{1-\frac{R_0}{R}}{1-\frac{R_0}{r}}\right) \lambda(r)$ | $\lambda_0 = \left(1-\frac{R_0}{R}\right) \lambda_\infty$ |

*Table 2: Blueshift equations*

# Appendix 1
# Nomenclature

I shall use the following nomenclature for the constants and variables used in this paper

$c$ = speed of light in vacuum
$h$ = Planck's constant
$G$ = Gravitational constant (also known as constant of gravitation, constant of gravity, gravitational force constant, universal constant of gravity, universal gravitational constant, Newton's gravitational constant, Newtonian gravitational constant, etc.)
$r$ = distance from the center of the star to an observer on Earth. Also distance from a distant photon to the center of the star
$H$ = distance from the surface of the star to a distant photon that travels towards the star (used in the derivation of blueshifts only)
$m$ = equivalent mass of the photon
$M$ = mass of the star
$R$ = radius of the star
$R_0$ = "special" Swarzchild radius of the star (special relativity's formula)
I denoted this quantity this way to differentiate it from the correct Schwarzchild radius, derived from general relativity, which is generally denoted by $R_S$
$E$ = total relativistic energy of the photon
$K$ = kinetic relativistic energy of the photon
$E(r)$ = total (or kinetic) energy of the photon at a distance *r* from the center of the star or, equivalently, the energy of the photon on the surface of the Earth
$E_0$ = total (or kinetic) energy of the photon on the surface of the star
$U(r)$ = gravitational potential energy of the photon at a distance *r* from the center of the star, or, equivalently, the gravitational potential energy of the photon on the surface of the Earth
$U_0$ = gravitational potential energy of the photon on the surface of the star
$f_0$ = frequency of the photon on the surface of the star
$f_\infty$ = frequency of the photon at an infinite distance from the star
$\lambda_0$ = wavelength of the photon on the surface of the star
$\lambda_\infty$ = wavelength of the photon at an infinite distance from the star
$f(r)$ = frequency of the photon at a distance *r* from the center of the star or, equivalently, the frequency of the photon on the surface of the Earth
$T$ = period (I use *K* for kinetic energy)
$T(r)$ = period of the photon at a distance *r* from the center of the star
$\lambda(r)$ = wavelength of the photon at a distance *r* from the center of the star or, equivalently, the wavelength of the photon on the surface of the Earth