The Simple Infinite Set

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Abstract

Many have suggested that the infinite set has a fundamental problem. The usual complaint rails against the actually infinite which (say critics of various finitist persuasions) unjustifiably goes *beyond* the finite. Here we identify the exact opposite. The problem of the *infinite* set defined to have an identity (content) that is specified and restricted to be *forever finite*.

Set theory is taken at its word. The existence of the infinite set and the representation of irrational reals as infinite sets of terms is accepted. In this context, it is shown that the standard definition of the infinite countable set is *inconsistent* with the existence of its own classic convergents of construction. If the set *is* infinite then it must be quite unlike that which set theory asserts it to be.

Set theory found itself in some trouble over a century ago trusting an *unrestricted* anthropic comprehension. But serious doubt is cast on the validity of infinite sets which have been defined by a comprehension which overly-*restricts* their content.

Keywords

Set Theory; Infinite Set Content.

Contents

1. Set Totality Theorem

The standard definition of a countable infinite set is inconsistent
with the foundation of its own infinite sequence of construction.

Definitions

[A] Let the set S be an infinite countable set of elements:

$$S = \{s_1 , s_2 \dots s_k \dots\} \quad (\forall k \text{ finite})$$

[B] Let the partial collections or *convergents* of construction of S be the infinite sequence of finite sets:

$$^1S , \quad ^2S \quad \dots \quad ^kS \quad \dots$$

$$\{s_1\}, \{s_1 , s_2\} \dots \{s_1 , s_2 \dots s_k \dots\} \dots$$

[C] By standard definition and assumption, every element of the countable infinite set $\{s_1 , s_2, s_3 \dots\}$ is indexed by a natural number:

$$z \in \{s_1 , s_2 , s_3 \dots\} \rightarrow \exists k \ (z = s_k \in \{s_1 , s_2 \dots s_k\}) \tag{1}$$

Theorem Proof

The set S is a proper superset of every one of its indexed convergents:

$$S \supset \cdots \ ^kS \dots \supset \ ^3S \supset \ ^2S \supset \ ^1S \tag{2a}$$

$$\rightarrow \quad S \supset (\ ^kS \ \forall k) \tag{2b}$$

Expanding the element content of S and its convergents:

$$\rightarrow \quad \{s_1 , s_2 , s_3 \dots\} \supset (\{s_1 , s_2 \dots s_k\} \ \forall k) \tag{3}$$

By the meaning of the proper superset relation:

$$\rightarrow \quad \exists z \in \{s_1 , s_2 , s_3 \dots\} \ (z \notin \{s_1 , s_2 \dots s_k\} \ \forall k) \tag{4}$$

Not every element of $\{s_1 , s_2 , s_3 \dots\}$ is indexed by a natural number.

Contradiction between (1) and (4).

QED.

2 Taking Set Theory at its Word

Are we misled by the Set Totality theorem ? One simple way to get closer to the argument is to consider a sequence of finite cases.

Let S be an infinite countable set and ${}^{1}S$, ${}^{2}S$ ... ${}^{k}S$ its first k convergents of construction. We (should) have no hesitation in accepting the following sequence of implications:

$$S \supset {}^{1}S$$
$$\rightarrow \quad \exists z \in S \ \left(z \notin {}^{1}S \right)$$

$$S \supset {}^{2}S \supset {}^{1}S$$
$$\rightarrow \quad \exists z \in S \ \left(z \notin {}^{2}S \ \& \ z \notin {}^{1}S \right)$$
$$\vdots$$
$$S \supset {}^{k}S \supset \cdots \supset {}^{2}S \supset {}^{1}S$$
$$\rightarrow \quad \exists z \in S \ \left(z \notin {}^{k}S \ \& \ldots \& \ z \notin {}^{2}S \ \& \ z \notin {}^{1}S \right)$$

That is, for the case of k finite, no matter how large;

$$\exists z \in S \ \left(z \notin {}^{i}S \ \ \forall i = 1, k \right)$$

What we understand from each small case (finite k) is unambiguously shown. But it is irrelevant that the case *is small*[1]. There is no reason to think that the result would *not* apply in the infinite case. Taking set theory at its word – that the infinite collection is *nothing more than* a natural totality of all finitely indexed instances – why would the result be any different if we let the index domain be extended to *all* natural numbers ? In fact mathematical induction over the above sequence of implications can be used to derive $\exists z \in S \ \left(z \notin {}^{i}S \ \ \forall i \right)$.

The Set Totality theorem has been described for the case of any simple infinite but countable set S and each of its finite convergents. This means it must apply to the archetype infinite set of ZF set theory $\{ 0 , 1 , 2 , \ \ldots \ \}$ and to the classic general real quantity expressed as an infinite set of bits.

[1] From a turn of phrase used by Prof James Franklin (UNSW) [1]

3. As Applied to an Irrational Real

The standard definition of a countable infinite set is inconsistent with the foundation of its own infinite sequence of construction.

Definitions

[A] Let an irrational real λ_1 on the unit interval $[0,1]$ be specified as an infinite binary series of terms λ_{1k} :

$$\lambda_1 = \frac{1}{2^1}[0/1] + \frac{1}{2^2}[0/1] + \cdots + \frac{1}{2^k}[0/1] + \cdots$$
$$= \lambda_{11} + \lambda_{12} + \cdots + \lambda_{1k} + \cdots$$

The real λ_1 can be *represented* by an infinite subset of the included elements of the binary base set $\left\{\frac{1}{2^1}, \frac{1}{2^2}, \frac{1}{2^3} \ldots\right\}$:

$$\lambda_1 \sim \{\lambda_{11}, \lambda_{12} \ldots \lambda_{1k} \ldots\} \quad (\forall k \text{ such that } \lambda_{1k} = \frac{1}{2^k} \neq 0)$$

[B] Let the infinite sequence of rational reals:

$${}^1\lambda_1, \quad {}^2\lambda_1 \quad \ldots \quad {}^k\lambda_1 \ \ldots$$

be the partial sums or *convergents* of construction of λ_1 , similarly represented as (finite) subsets of the binary base set:

$$\{\lambda_{11}\}, \{\lambda_{11}, \lambda_{12}\} \ldots \{\lambda_{11}, \lambda_{12}, \ldots \lambda_{1k}\} \ldots$$

[C] By standard definition and assumption, every element of the countable infinite set $\{\lambda_{11}, \lambda_{12}, \lambda_{13} \ldots\}$ is indexed by a natural number:

$$z \in \{\lambda_{11}, \lambda_{12}, \lambda_{13} \ldots\} \to \exists k \ (z = \lambda_{1k} \in \{\lambda_{11}, \lambda_{12}, \ldots \lambda_{1k}\}) \qquad (1)$$

Theorem Proof

The real λ_1 is a superset of each and every one of its indexed convergents:

$$\lambda_1 \supset \ldots {}^k\lambda_1 \ldots \supseteq {}^3\lambda_1 \supseteq {}^2\lambda_1 \supseteq {}^1\lambda_1 \qquad (2a)$$

$$\to \quad \lambda_1 \supset ({}^k\lambda_1 \ \forall k) \qquad (2b)$$

Expanding the element content of λ_1 and its convergents:

$$\to \quad \{\lambda_{11}, \lambda_{12}, \lambda_{13} \ldots\} \supset (\{\lambda_{11}, \lambda_{12}, \ldots \lambda_{1k}\} \ \forall k) \qquad (3)$$

By the meaning of the proper superset relation:

$$\to \quad \exists z \in \{\lambda_{11}, \lambda_{12}, \lambda_{13} \ldots\} \ (z \notin \{\lambda_{11}, \lambda_{12}, \ldots \lambda_{1k}\} \ \forall k) \qquad (4)$$

Not every element of $\{\lambda_{11}, \lambda_{12}, \lambda_{13} \ldots\}$ is indexed by a natural number.

Contradiction between (1) and (4).

QED.

4 The Objection Domain

If we accept the existence of the irrational real λ_1 represented as an infinite set of series terms, it is a superset of every one of its finite convergents ${}^{k}\lambda_{1}$. If this were not the case λ_1 would be rational. This establishes (2) of the theorem. Now of course the set of elements $\{ \lambda_{11} , \lambda_{12} , \lambda_{13} \ldots \}$ always contains *some* element in addition to the elements of any $\{ \lambda_{11} , \lambda_{12} , \ldots \lambda_{1k} \}$. A proper observation is that this can be satisfied by a *different* indexed element for each ${}^{k}\lambda_{1}$ and this promotes a sceptical critique that the inference at (4) of the theorem is invalid.

The observation is correct but the objection wrong.
The Set Totality theorem does *not* seek to find element(s) of the convergents that reside as-it-were closer and closer to the ultimate content of λ_1 , as if pursuing some prey across the plains of the potentially infinite, one-foot-after-the-other. It uses the extant understanding that λ_1 is a fixed set. And it is a superset of *any* of the indexed convergents. One will not and cannot locate a k for which this is not true. This means that it *is true* $\forall k$. The meaning of the superset relation then provides the content implication for the sets of terms that correspond to the set and its convergents.

The objection fails because it is asserting that $\forall k$ means *other than* a totality of all finite index values. It says in effect that only the *potential* infinite is permitted. This disowns the leap that mathematical induction is able to take to the conclusion $\forall k$ and is a rejection of set theory's leap to the actually infinite set.

On the other hand, the theorem has traction because it accepts the representation of λ_1 as an *actually* infinite set and the quantification $\forall$ as exhaustive and complete over *all* finite values of the index and therefore decisive for the proposition. It is so abundantly clear that the set representation of an irrational real λ_1 is a proper superset of *all* of its finitely indexed convergents … that one is left wondering just where the objection can obtain any support at all, other than by appeal to definition and authority.

A key perspective arising from the theorem is that the projection or leap from the convergents of construction to the fixed infinite set is matched by a simultaneous projection or leap of the set's content. The indexed sequence of set convergents is always synchronized with the *content* of the sequence. Given the formation of the infinite set, why would we imagine that the corresponding *content* of the sequence could have any imperative other than to take that same leap ?

The act of making the set *actually infinite* has of necessity forced its identity beyond the finitely indexed realm. The leap to the infinite cannot yield a set that is otherwise.

5. Quite Unlike that which Set Theory Asserts

There are significant consequences arising from the Set Totality theorem.
On the one hand, it would be concluded by many a finitist that it is invalid nonsense to assert the existence of the single set totality of (say) all the natural numbers. Most attacks on infinite set theory come from this direction. And many paradoxes and ridiculi proffered in support arise from the inconsistency identified by the theorem.

On the other hand, one might accept that reals such as $^1/_{2\pi}$ exist, with a commensurate representation as infinite sets. In this case, by the Set Totality theorem, an *infinite* set representing that real must contain at least one element *other than* those contained in any of the finitely indexed convergents of its construction. And it is clear what these [non-indexed] elements must be:

$$\{\lambda_{11}\},\ \{\lambda_{11},\lambda_{12}\}\ \ldots \quad \curvearrowright \quad \{\ \lambda_{11},\lambda_{12}\ \ldots [0]\ \}$$

$$\{{}^1\lambda_1\},\ \{{}^1\lambda_1,\ {}^2\lambda_1\}\ \ldots \quad \curvearrowright \quad \{\ {}^1\lambda_1,\ {}^2\lambda_1\ \ldots [\lambda_1]\}$$

$$\{\ \},\{0\},\quad \{0,1\}\quad \ldots \quad \curvearrowright \quad \{0,1,\quad 2,\quad \ldots [\omega]\}$$

The ultimate set of the last line above, the archetype infinite set of set theory, must *necessarily* contain a transfinite element. It is somewhat ironic that if we embrace the fixed *actually* infinite set, the formation of the set forces the presence of the transfinite *in the set*. The theorem shows us that the set necessarily contains a transfinite element - that is, an element not equal to any finite natural number. So let us call this element ω. No longer do we have to define ω <u>*as*</u> $\{0, 1,\ 2,\ \ldots\}$. In this sense the Set Totality theorem provides a kind of heuristic proof that the transfinite exists, albeit by assuming that we *can* form the set. Firstly, the set itself can be described as transfinite, because it exists other than as one of the finitely indexed convergent sets in the sequence of its construction. But the set is (correspondingly) also transfinite because its identity (content) is not all finitely indexed *per* that sequence of construction.

If infinite sets are to be accepted and contradiction is to be avoided, the above sets (or any sets that contain such sequences of elements) do not exist *without* also containing the relevant [non-indexed] limit elements. And this has great consequence. Infinite sets are quite unlike that which set theory asserts they be. To cut to the chase and to generalise the implications:

> The rationals can be *listed*.
>
> But the rationals cannot be formed into a single infinite *set* without that set containing *all reals*. The <u>set</u> of all *rationals* <u>is</u> the set of all *reals*.

6. A Good Idea at the Time

It is easy to imagine that one defines, or granting a construction, extracts or creates the collection of finite numbers. 0, 1, 2, 3, ... Supported by familiarity and a supposed integrity of definition, one might feel that it cannot be a mistake to conceptualize the idea that there exists a totality or *single set* of all but only such numbers. Such a single set seems to be in complete harmony with the element contributions from the construction, so what could possibly be wrong with it ? It is a set Cantor [2,P86] would have identified as

> a collection into a whole of definite and separate objects
> of our intuition or our thought.

And the extant definition of the countable infinite set *does* continue to be supported essentially on the basis of our *intuition or thought*. It is a conceptualization that sits easily and lightly in the mind's eye, perhaps as did the idea of a *set of all sets* once upon a time. But as with the historical troubles of set theory, it is naïve to assume automatically that what *we* prefer, imagine or assert can just *be so*. It is known that the *unrestricted* formation of very large sets can lead to contradiction. But the formation of *actually infinite* sets *restricted* to contain *only finitely indexed elements* is also contradictory.

Aristotle , Gauss and many a philosopher, church father and lay thinker, known and unknown, have rejected the idea of the completed infinite sequence of natural numbers. And they are of course correct, in that each is understanding the natural numbers as an infinite *list*. And as a list 0, 1, 2, 3, ... *has no* natural maximum.

Cantor was right to explore the infinite as a meaningful mathematical concept. But who could resist the weight of history, the opinions of such giants as Aristotle and Gauss or the logic itself. In this context, there was only one solution: the never-ending sequence without maximum was to be maintained and placed *inside* a single fixed collection, the set. And as a new principle of generation, this *single object* was to be made the first *transfinite* ordinal. It seemed like a good idea at the time. Indeed, it was a great idea. But the Set Totality theorem is proof that this infinite set is inconsistent. That in the fullness of its identity the *set* of all natural numbers must itself contain at least one element that is *not* a natural number. The transfinite makes its appearance in the set *by the act* of set formation itself ... and in a sense, it is placed where all intuition always said it would be when imagining a list; juxtaposed with but beyond the great chasm at the *transfinite end* of the endless *natural* sequence.

There remains today an attitude concerning the infinite that:

> Infinite totalities do not exist in any sense of the word (i.e. either really or ideally). More precisely, any mention, or purported mention, of infinite totalities is, literally, meaningless. Nevertheless, we should act as if infinite totalities really existed.

These words are from Abraham Robinson [3, P230], a student of Abraham Fraenkel. They express a typically modern *depending-on-the-company-one-keeps* nominalism. It is a duplicitous, somewhat imprecise and certainly far from bold understanding of the mathematical infinite.

And it is a nonsensical understanding that does mathematics no service. To the realist, for whom the infinite holds no automatic terrors, the actually infinite is neither more nor less meaningful or real than the tangents to a circle and vanishing points, negative numbers and their roots or the existence of the real $1/_{2\pi}$ and its commensurate expression as a sum of discrete finitely defined rationals.

The difficulty here is not with the idea of the set, or even the actual infinite. The problem is with the contradictory schizomorphic *actually infinite set* constrained to be *forever finite*.

Whether he saw it or not, this hybrid infinite set is surely a candidate for Hermann Weyl's *inner instability of the foundations* when he wrote [4] in 1920:

> The antinomies of set theory are usually regarded as border skirmishes that concern only the remotest provinces of the mathematical empire … (but) every earnest and honest reflection must lead to the realization that the troubles … (are) symptoms (of an) inner instability of the foundations upon which the structure of the empire rests.

The very existence of the Set Totality theorem should give us pause, because it uses simple every-day logic. A logic happily used and accepted in other contexts as transparent and definitive. Given the implications that rest upon the theorem's efficacy – and *because* it is so simple – one would hope and expect that a refutation is not argued merely on the grounds that it *is* simple, challenges definition or just cannot be right.

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This paper, *The Simple Infinite Set*, is an edited extract of a larger paper, which in turn is extracted from an unpublished book. The author of these works can be contacted at kenseton@gmail.com