A FINITE FIELD ANALOGUE FOR APPELL SERIES F_3

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ABSTRACT. In this paper we introduce a finite field analogue for the Appell series F_3 and give some reduction formulae and certain generating functions for this function over finite fields.

1. INTRODUCTION

Let q be a power of a prime. $\mathbb{F}_q$ and $\widehat{\mathbb{F}}_q^*$ are denoted as the finite field of q elements and the group of multiplicative characters of $\mathbb{F}_q^*$ respectively. Then the domain of all characters χ of $\mathbb{F}_q^*$ can be extended to $\mathbb{F}_q$ by setting $\chi(0) = 0$ for all characters. Let $\overline{\chi}$ and ε denote the inverse of χ and the trivial character respectively. For more details about characters, please see [3] and [9, Chapter 8].

Greene in 1987 developed the theory of hypergeometric functions over finite fields and proved a number of transformation and summation identities for hypergeometric functions over finite fields which are analogues to those in the classical case [7] (see [2] for the definition of the hypergeometric functions). Greene in [7] introduced the notation

$$ {}_2F_1\left(\begin{matrix} A, B \\ C \end{matrix}\middle| x\right)^G = \varepsilon(x)\frac{BC(-1)}{q}\sum_y B(y)\overline{B}C(1-y)\overline{A}(1-xy) $$

for $A, B, C \in \widehat{\mathbb{F}}_q$ and $x \in \mathbb{F}_q$ and defined the finite field analogue of the binomial coefficient as

$$ \binom{A}{B}^G = \frac{B(-1)}{q}J(A, \overline{B}), $$

where $J(\chi, \lambda)$ is the Jacobi sum given by

$$ J(\chi, \lambda) = \sum_u \chi(u)\lambda(1-u). $$

See [5, 6, 11] for more information about the finite field analogue of the hypergeometric functions.

In this paper, for the sake of simplicity, we use the notation

$$ \binom{A}{B} = q\binom{A}{B}^G = B(-1)J(A, \overline{B}) $$

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and define the finite field analogue of the classic Gauss hypergeometric function as

$$ {}_2F_1\left(\begin{matrix}A, B\\ C\end{matrix}\middle| x\right) = q \cdot {}_2F_1\left(\begin{matrix}A, B\\ C\end{matrix}\middle| x\right)^G = \varepsilon(x)BC(-1)\sum_y B(y)\overline{B}C(1-y)\overline{A}(1-xy). $$

Then

$$ {}_2F_1\left(\begin{matrix}A, B\\ C\end{matrix}\middle| x\right) = \frac{1}{q-1}\sum_\chi \binom{A\chi}{\chi}\binom{B\chi}{C\chi}\chi(x) $$

for any $A, B, C \in \widehat{\mathbb{F}}_q$ and $x \in \mathbb{F}_q$. Similarly, the finite field analogue of the generalized hypergeometric function for any $A_0, A_1, \cdots, A_n, B_1, \cdots, B_n \in \widehat{\mathbb{F}}_q$ and $x \in \mathbb{F}_q$ is defined by

$$ {}_{n+1}F_n\left(\begin{matrix}A_0, A_1, \cdots, A_n\\ B_1, \cdots, B_n\end{matrix}\middle| x\right) = \frac{1}{q-1}\sum_\chi \binom{A_0\chi}{\chi}\binom{A_1\chi}{B_1\chi}\cdots\binom{A_n\chi}{B_n\chi}\chi(x). $$

In our notations one of Greene's theorems is as follows.

Theorem 1.1. (See [7, Theorem 4.9])*For any characters $A, B, C \in \widehat{\mathbb{F}_q}$, we have*

$$ {}_2F_1\left(\begin{matrix}A, B\\ C\end{matrix}\middle| 1\right) = A(-1)\binom{B}{\overline{A}C}. \tag{1.1} $$

The results in the following proposition follows readily from some properties of Jacobi sums.

Proposition 1.1. (See [7, (2.6), (2.7), (2.8) and (2.13)]) *If $A, B \in \widehat{\mathbb{F}}_q$, then*

$$ \binom{A}{B} = \binom{A}{A\overline{B}}, \tag{1.2} $$

$$ \binom{A}{B} = \binom{B\overline{A}}{B}B(-1), \tag{1.3} $$

$$ \binom{A}{B} = \binom{\overline{B}}{\overline{A}}AB(-1), \tag{1.4} $$

$$ \binom{\varepsilon}{A} = -A(-1) + (q-1)\delta(A). \tag{1.5} $$

where $\delta(\chi)$ is a function on characters given by

$$ \delta(\chi) = \begin{cases} 1 & \text{if } \chi = \varepsilon \\ 0 & \text{otherwise} \end{cases}. $$

Among these interesting double hypergeometric functions in the field of hypergeometric functions, Appell's four functions may be the most important functions. Below are three of them.

$$F_1(a;b,b';c;x,y)=\sum_{m,n\geq 0}\frac{(a)_{m+n}(b)_m(b')_n}{m!n!(c)_{m+n}}x^my^n,\ |x|<1,\ |y|<1,$$

$$F_2(a;b,b';c,c';x,y)=\sum_{m,n\geq 0}\frac{(a)_{m+n}(b)_m(b')_n}{m!n!(c)_m(c')_n}x^my^n,\ |x|+|y|<1,$$

$$F_3(a,a';b,b';c;x,y)=\sum_{m,n\geq 0}\frac{(a)_m(a')_n(b)_m(b')_n}{m!n!(c)_{m+n}}x^my^n,\ |x|<1,\ |y|<1.$$

See [1, 2, 4, 12] for more detailed material about Appell's functions.

Inspired by Greene's work, Li *et al* in [10] gave a finite field analogue of the Appell series F_1 and obtained some transformation and reduction formulas and the generating functions for the function over finite fields. In that paper, the finite field analogue of the Appell series F_1 was given by

$$F_1(A;B,B';C;x,y)=\varepsilon(xy)AC(-1)\sum_u A(u)\overline{A}C(1-u)\overline{B}(1-ux)\overline{B'}(1-uy).$$

Motivated by the work of Greene [7] and Li *et al* [10], the author *et al* in [8] introduced a finite field analogue of the Appell series F_2 which is

$$\begin{aligned}&F_2(A;B,B';C,C';x,y)\\&\quad=\varepsilon(xy)BB'CC'(-1)\sum_{u,v}B(u)B'(v)\overline{B}C(1-u)\overline{B'}C'(1-v)\overline{A}(1-ux-vy)\end{aligned}$$

and deduced certain transformation and reduction formulas and the generating functions for this function over finite fields.

In this paper we will give a finite field analogue for the Appell series F_3. Since the Appell series F_3 has the following double integral representation [2, Chapter IX]:

$$\begin{aligned}F_3(a,a';b,b';c;x,y)=&\frac{\Gamma(c)}{\Gamma(b)\Gamma(b')\Gamma(c-b-b')}\\&\cdot\int\int u^{b-1}v^{b'-1}(1-u-v)^{c-b-b'-1}(1-ux)^{-a}(1-vy)^{-a'}dudv,\end{aligned}$$

where the double integral is taken over the triangle region $\{(u,v)|u\geq 0,v\geq 0,u+v\leq 1\}$, we now give the finite field analogue of F_3 in the form:

$$F_3(A,A';B,B';C;x,y)=\varepsilon(xy)BB'(-1)\sum_{u,v}B(u)B'(v)C\overline{BB'}(1-u-v)\overline{A}(1-ux)\overline{A'}(1-vy),$$

where $A,A',B,B',C\in\widehat{\mathbb{F}}_q,\ x,y\in\mathbb{F}_q$ and each sum ranges over all the elements of $\mathbb{F}_q$. In the above definition, the factor $\frac{\Gamma(c)}{\Gamma(b)\Gamma(b')\Gamma(c-b-b')}$ is dropped to obtain simpler results. We choose the factor $\varepsilon(xy)\cdot BB'(-1)$ to get a better expression in terms of binomial coefficients.

From the definition of $F_3(A,A';B,B';C;x,y)$ we know that

$$F_3(A,A';B,B';C;x,y)=F_3(A',A;B',B;C;y,x)\tag{1.6}$$

for any $A,B,B',C,C'\in\widehat{\mathbb{F}}_q$ and $x,y\in\mathbb{F}_q$.

The aim of this paper is to give several reduction formulas and certain generating functions for the Appell series F_3 over finite fields. The fact that the Appell series F_3 does not have a single integral representation but has a double one leds us to giving a finite field analogue for the Appell series F_3 which is more complicated than that for F_1. Consequently, the results on the reduction formulas and the generating functions for the Appell series F_3 over finite fields are also quite complicated.

We give two other expressions for $F_3(A, A'; B, B'; C; x, y)$ in the next section. Several reduction formulae for $F_3(A, A'; B, B'; C; x, y)$ is given in Section 3. The last section is devoted to deducing certain generating functions for $F_3(A, A'; B, B'; C; x, y)$.

2. Other expressions for $F_3(A, A'; B, B'; C; x, y)$

In this section we give two other expressions for $F_3(A, A'; B, B'; C; x, y)$.

Theorem 2.1. *For any $A, A', B, B', C, \in \widehat{\mathbb{F}}_q$ and $x, y \in \mathbb{F}_q$, we have*

$$F_3(A, A'; B, B'; C; x, y) = \frac{1}{(q-1)^2} \sum_{\chi,\lambda} \binom{\overline{A}}{\chi} \binom{\overline{A'}}{\lambda} \binom{C\overline{BB'}}{C\overline{B'}\chi} \binom{C\overline{B'}\chi}{C\chi\lambda} \chi(x)\lambda(y)$$
$$+ \varepsilon(y) B'(-1) \binom{\overline{A}}{B'\overline{C}} B'\overline{C}(x)\overline{A'}(1-y),$$

where each sum ranges over all multiplicative characters of $\mathbb{F}_q$.

In order to prove Theorem 2.1 we need an auxiliary result.

Proposition 2.1. (Binomial theorem, see [7, (2.5)]) *For any character $A \in \widehat{\mathbb{F}}_q$ and $x \in \mathbb{F}_q$, we have*

$$A(1+x) = \delta(x) + \frac{1}{q-1} \sum_{\chi} \binom{A}{\chi} \chi(x),$$

where the sum ranges over all multiplicative characters of $\mathbb{F}_q$ and $\delta(x)$ is a function on $\mathbb{F}_q$ given by

$$\delta(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}.$$

We are now in the position to show Theorem 2.1.

Proof of Theorem 2.1. When $v \neq 1$, it is known from the binomial theorem that

$$C\overline{BB'}\left(1 - \frac{u}{1-v}\right) = \delta(u) + \frac{1}{q-1} \sum_{\mu} \binom{C\overline{BB'}}{\mu} \mu(-u)\overline{\mu}(1-v).$$

Then

$$C\overline{BB'}(1-u-v) = C\overline{BB'}(1-v)C\overline{BB'}\left(1 - \frac{u}{1-v}\right) \tag{2.1}$$
$$= \delta(u)C\overline{BB'}(1-v) + \frac{1}{q-1} \sum_{\mu} \binom{C\overline{BB'}}{\mu} \mu(-u)C\overline{BB'}\overline{\mu}(1-v).$$

It is easy to see from the binomial theorem that

$$\overline{A}(1-ux)=\delta(ux)+\frac{1}{q-1}\sum_{\chi}\binom{A}{\chi}\chi(-ux), \tag{2.2}$$

$$\overline{A'}(1-vy)=\delta(vy)+\frac{1}{q-1}\sum_{\lambda}\binom{A'}{\lambda}\lambda(-vy). \tag{2.3}$$

Applying (2.1)–(2.3) and using that fact that $\delta(u)B(u)=\delta(ux)B(u)\varepsilon(xy)=\delta(vy)B'(v)\varepsilon(xy)=0$, [7, (1.15)] and (1.2) yield

$$\begin{aligned}
&\varepsilon(xy)\sum_{u\in\mathbb{F}_q,v\neq 1}B(u)B'(v)C\overline{BB'}(1-u-v)\overline{A}(1-ux)\overline{A'}(1-vy)\\
&=\frac{1}{(q-1)^3}\sum_{u\in\mathbb{F}_q,v\neq 1}B(u)B'(v)\sum_{\mu}\binom{C\overline{BB'}}{\mu}\mu(-u)C\overline{BB'}\overline{\mu}(1-v)\\
&\cdot\sum_{\chi}\binom{A}{\chi}\chi(-ux)\sum_{\lambda}\binom{A'}{\lambda}\lambda(-vy)\\
&=\frac{1}{(q-1)^3}\sum_{u\in\mathbb{F}_q,v\in\mathbb{F}_q}B(u)B'(v)\sum_{\mu}\binom{C\overline{BB'}}{\mu}\mu(-u)C\overline{BB'}\overline{\mu}(1-v)\\
&\cdot\sum_{\chi}\binom{A}{\chi}\chi(-ux)\sum_{\lambda}\binom{A'}{\lambda}\lambda(-vy)\\
&=\frac{1}{(q-1)^3}\sum_{\chi}\binom{A}{\chi}\chi(-x)\sum_{\lambda}\binom{A'}{\lambda}\lambda(-y)\sum_{\mu}\binom{C\overline{BB'}}{\mu}\mu(-1)\\
&\cdot\sum_{u\in\mathbb{F}_q}B\chi\mu(u)\sum_{v\in\mathbb{F}_q}B'\lambda(v)C\overline{BB'}\overline{\mu}(1-v)\\
&=\frac{B(-1)}{(q-1)^2}\sum_{\chi}\binom{A}{\chi}\binom{C\overline{BB'}}{\overline{B\chi}}\chi(x)\sum_{\lambda}\binom{A'}{\lambda}\lambda(-y)\sum_{v}B'\lambda(v)C\overline{B'}\chi(1-v)\\
&=\frac{BB'(-1)}{(q-1)^2}\sum_{\chi,\lambda}\binom{A}{\chi}\binom{A'}{\lambda}\binom{C\overline{BB'}}{C\overline{B'}\chi}\binom{C\overline{B'}\chi}{C\chi\lambda}\chi(x)\lambda(y).
\end{aligned}$$

On the other hand, by (2.2), the fact that $\varepsilon(xy)C\overline{B'}(u)\delta(ux)=0$ and [7, (1.15)]

$$\begin{aligned}
&\varepsilon(xy)\sum_{u\in\mathbb{F}_q,v=1}B(u)B'(v)C\overline{BB'}(1-u-v)\overline{A}(1-ux)\overline{A'}(1-vy)\\
&=\varepsilon(xy)C\overline{BB'}(-1)\overline{A'}(1-y)\sum_{u\in\mathbb{F}_q}C\overline{B'}(u)\overline{A}(1-ux)\\
&=\frac{\varepsilon(y)C\overline{BB'}(-1)\overline{A'}(1-y)}{q-1}\sum_{\chi}\binom{A}{\chi}\chi(-x)\sum_{u\in\mathbb{F}_q}C\overline{B'}\chi(u),
\end{aligned}$$

$$= \varepsilon(y)B(-1)\binom{\overline{A}}{B'\overline{C}}B'\overline{C}(x)\overline{A'}(1-y).$$

Therefore,

$$\begin{aligned}F_3(A,A';B,B';C;x,y) &= \varepsilon(xy)BB'(-1)\left(\sum_{u\in\mathbb{F}_q, v\neq 1}+\sum_{u\in\mathbb{F}_q, v=1}\right)\\ &= \frac{1}{(q-1)^2}\sum_{\chi,\lambda}\binom{\overline{A}}{\chi}\binom{\overline{A'}}{\lambda}\binom{C\overline{BB'}}{C\overline{B'}\chi}\binom{C\overline{B'}\chi}{C\chi\lambda}\chi(x)\lambda(y)\\ &\quad+\varepsilon(y)B'(-1)\binom{\overline{A}}{B'\overline{C}}B'\overline{C}(x)\overline{A'}(1-y).\end{aligned}$$

This completes the proof of Theorem 2.1. □

Corollary 2.2. *For any $A, A', B, B', C, \in \widehat{\mathbb{F}_q}$ and $x, y \in \mathbb{F}_q$, we have*

$$F_3(A,A';B,B';C;x,1) = B'C(-1)\,{}_3F_2\left(\begin{matrix}A, B, \overline{A'B'}C\\ C\overline{B'}, C\overline{A'}\end{matrix}\,\middle|\, x\right), \tag{2.4}$$

$$F_3(A,A';B,B';C;1,y) = BC(-1)\,{}_3F_2\left(\begin{matrix}A', B', \overline{AB}C\\ C\overline{B}, C\overline{A}\end{matrix}\,\middle|\, y\right). \tag{2.5}$$

Proof. We first show (2.4). It follows from Theorem 2.1, (1.3) and (1.1) that

$$\begin{aligned}F_3(A,A';B,B';C;x,1) &= \frac{1}{(q-1)^2}\sum_{\chi,\lambda}\binom{\overline{A}}{\chi}\binom{\overline{A'}}{\lambda}\binom{C\overline{BB'}}{C\overline{B'}\chi}\binom{C\overline{B'}\chi}{C\chi\lambda}\chi(x)\lambda(1)\\ &= \frac{C(-1)}{(q-1)^2}\sum_{\chi}\binom{\overline{A}}{\chi}\binom{C\overline{BB'}}{C\overline{B'}\chi}\chi(-x)\sum_{\lambda}\binom{A'\lambda}{\lambda}\binom{B'\lambda}{C\chi\lambda}\lambda(1)\\ &= \frac{C(-1)}{q-1}\sum_{\chi}\binom{\overline{A}}{\chi}\binom{C\overline{BB'}}{C\overline{B'}\chi}\chi(-x)\,{}_2F_1\left(\begin{matrix}A', B'\\ C\chi\end{matrix}\,\middle|\, 1\right)\\ &= \frac{A'C(-1)}{q-1}\sum_{\chi}\binom{\overline{A}}{\chi}\binom{C\overline{BB'}}{C\overline{B'}\chi}\binom{B'}{C\overline{A'}\chi}\chi(-x)\\ &= \frac{B'C(-1)}{q-1}\sum_{\chi}\binom{A\chi}{\chi}\binom{B\chi}{C\overline{B'}\chi}\binom{\overline{A'B'}C\chi}{C\overline{A'}\chi}\chi(x)\\ &= B'C(-1)\,{}_3F_2\left(\begin{matrix}A, B, \overline{A'B'}C\\ C\overline{B'}, C\overline{A'}\end{matrix}\,\middle|\, x\right),\end{aligned}$$

which proves (2.4). (2.5) follows readily from (2.4) and (1.6). □

The following theorem gives a third expression for $F_3(A, A'; B, B'; C; x, y)$.

Theorem 2.3. *For any $A, A', B, B', C, \in \widehat{\mathbb{F}}_q$ and $x, y \in \mathbb{F}_q$, we have*

$$\begin{aligned} F_3(A, A'; B, B'; C; x, y) &= \frac{1}{(q-1)^2} \sum_{\chi,\lambda} \binom{\overline{A}}{\chi}\binom{\overline{A'}}{\lambda}\binom{C\overline{BB'}}{C\chi\lambda}\binom{\overline{BB'}\,\overline{\chi}\,\overline{\lambda}}{\overline{B'\lambda}}\chi(x)\lambda(y) \\ &+ \frac{B'(-1)}{q-1}\sum_{\chi\lambda=\overline{BB'}}\binom{\overline{A}}{\chi}\binom{\overline{A'}}{\lambda}\chi(x)\lambda(-y), \end{aligned}$$

where the sum in the second term of the right side ranges over the region $\chi, \lambda \in \widehat{\mathbb{F}}_q, \chi\lambda = \overline{BB'}$.

Proof. It follows from the binomial theorem that

$$\sum_{\lambda}\binom{\overline{A'}}{\lambda}\lambda(-y) = (q-1)(\overline{A'}(1-y) - \delta(y)) = (q-1)\varepsilon(y)\overline{A'}(1-y). \tag{2.6}$$

According to [7, (2.15)], we have

$$\binom{A}{B}\binom{C}{A} = \binom{C}{B}\binom{C\overline{B}}{A\overline{B}} - (q-1)B(-1)\delta(A) + (q-1)AB(-1)\delta(B\overline{C})$$

for any $A, B, C \in \widehat{\mathbb{F}}_q$. Then

$$\begin{aligned} \binom{C\overline{BB'}}{C\overline{B'}\chi}\binom{C\overline{B'}\chi}{C\chi\lambda} = \binom{C\overline{BB'}}{C\chi\lambda}\binom{\overline{BB'}\,\overline{\chi}\,\overline{\lambda}}{\overline{B'\lambda}} - (q-1)C\chi\lambda(-1)\delta(C\overline{B'}\chi) \\ + (q-1)B'\lambda(-1)\delta(BB'\chi\lambda). \end{aligned}$$

Using the above identity in Theorem 2.1 and by (2.6), we get

$$\begin{aligned} F_3(A, A'; B, B'; C; x, y) &= \frac{1}{(q-1)^2} \sum_{\chi,\lambda} \binom{\overline{A}}{\chi}\binom{\overline{A'}}{\lambda}\binom{C\overline{BB'}}{C\chi\lambda}\binom{\overline{BB'}\,\overline{\chi}\,\overline{\lambda}}{\overline{B'\lambda}}\chi(x)\lambda(y) \\ &- \frac{C(-1)B'\overline{C}(-x)}{q-1}\binom{\overline{A}}{B'\overline{C}}\sum_{\lambda}\binom{\overline{A'}}{\lambda}\lambda(-y) \\ &+ \frac{B'(-1)}{q-1}\sum_{\chi\lambda=\overline{BB'}}\binom{\overline{A}}{\chi}\binom{\overline{A'}}{\lambda}\chi(x)\lambda(-y) \\ &+ \varepsilon(y)B'(-1)\binom{\overline{A}}{B'\overline{C}}B'\overline{C}(x)\overline{A'}(1-y) \\ &= \frac{1}{(q-1)^2} \sum_{\chi,\lambda} \binom{\overline{A}}{\chi}\binom{\overline{A'}}{\lambda}\binom{C\overline{BB'}}{C\chi\lambda}\binom{\overline{BB'}\,\overline{\chi}\,\overline{\lambda}}{\overline{B'\lambda}}\chi(x)\lambda(y) \\ &+ \frac{B'(-1)}{q-1}\sum_{\chi\lambda=\overline{BB'}}\binom{\overline{A}}{\chi}\binom{\overline{A'}}{\lambda}\chi(x)\lambda(-y), \end{aligned}$$

which ends the proof of Theorem 2.3. □

Actually, from Theorem 2.3 we can also deduce (1.6), (2.4) and (2.5).

3. Reduction formulae

In this section we give some reduction formulae for $F_3(A,A';B,B';C;x,y)$. In order to derive these formulae we need some auxiliary results.

Proposition 3.1. (See [7, Corollary 3.16 and Theorem 3.15]) *For any $A,B,C,D\in\widehat{\mathbb{F}}_q$ and $x\in\mathbb{F}_q$, we have*

$$ {}_2F_1\left(\begin{matrix}\varepsilon, B\\ C\end{matrix}\middle| x\right)=\binom{B}{C}\varepsilon(x)-\overline{C}(x)\overline{B}C(1-x), \tag{3.1}$$

$$\begin{aligned} {}_2F_1\left(\begin{matrix}A, \varepsilon\\ C\end{matrix}\middle| x\right)=&\binom{C}{A}A(-1)\overline{C}(x)\overline{A}C(1-x)-C(-1)\varepsilon(x)\\ &+(q-1)A(-1)\delta(1-x)\delta(\overline{A}C),\end{aligned} \tag{3.2}$$

$$\begin{aligned} {}_2F_1\left(\begin{matrix}A, B\\ A\end{matrix}\middle| x\right)=&\binom{B}{A}\varepsilon(x)\overline{B}(1-x)-\overline{A}(-x)\\ &+(q-1)A(-1)\delta(1-x)\delta(B),\end{aligned} \tag{3.3}$$

$$\begin{aligned} {}_3F_2\left(\begin{matrix}A, B, C\\ D, B\end{matrix}\middle| x\right)=&\binom{C\overline{D}}{B\overline{D}}{}_2F_1\left(\begin{matrix}A, C\\ D\end{matrix}\middle| x\right)-BD(-1)\overline{B}(x)\binom{A\overline{B}}{\overline{B}}\\ &+(q-1)BD(-1)\delta(C\overline{D})\varepsilon(x)\overline{A}(1-x).\end{aligned} \tag{3.4}$$

From the definition of $F_3(a,a';b,b';c;x,y)$ we know that

$$F_3(a,0;b,b';c;x,y)=F_3(a,a';b,0;c;x,y)={}_2F_1\left(\begin{matrix}a, b\\ c\end{matrix}\middle| x\right),$$

$$F_3(0,a';b,b';c;x,y)=F_3(a,a';0,b';c;x,y)={}_2F_1\left(\begin{matrix}a, b'\\ c'\end{matrix}\middle| y\right).$$

We now give finite field analogues of the above identities.

Theorem 3.1. *Let $A,B,B',C,C'\in\widehat{\mathbb{F}}_q$ and $x,\ y\in\mathbb{F}_q$. If $y\neq 0$, then*

(3.5)
$$\begin{aligned} F_3(A,\varepsilon;B,B';C;x,y)=&CB'(-1)\binom{B\overline{C}}{B'}{}_2F_1\left(\begin{matrix}A, B\\ C\end{matrix}\middle| x\right)-\overline{C}B'(x)\delta(y-1)B'(-1)\binom{\overline{A}}{\overline{C}B'}\\ &-B'(-1)\overline{C}(y)\overline{B'}C(1-y){}_2F_1\left(\begin{matrix}A, B\\ C\overline{B'}\end{matrix}\middle| -\frac{x(1-y)}{y}\right)\\ &+(q-1)\varepsilon(x)C(-1)\delta(B\overline{C})\overline{A}(1-x);\end{aligned}$$

if $x\neq 0$, then

(3.6)
$$\begin{aligned} F_3(\varepsilon,A';B,B';C;x,y)=&CB(-1)\binom{B'\overline{C}}{\overline{B}}{}_2F_1\left(\begin{matrix}A', B'\\ C\end{matrix}\middle| y\right)-\overline{C}B(y)\delta(x-1)B(-1)\binom{\overline{A'}}{\overline{C}B}\\ &-B(-1)\overline{C}(x)\overline{B}C(1-x){}_2F_1\left(\begin{matrix}A', B'\\ C\overline{B}\end{matrix}\middle| -\frac{y(1-x)}{x}\right)\\ &+(q-1)\varepsilon(y)C(-1)\delta(B'\overline{C})\overline{A'}(1-y).\end{aligned}$$

Proof. We first prove (3.5). It is easily known from (3.1) and (3.4) that

$$
\begin{aligned}
{}_2F_1\left(\begin{matrix}\varepsilon, B'\\ C\chi\end{matrix}\bigg| y\right) &= \binom{B'}{C\chi} - \overline{C}\overline{\chi}(y)\overline{B'}C\chi(1-y),\\
{}_3F_2\left(\begin{matrix}A, C\overline{B'}, B\\ C, C\overline{B'}\end{matrix}\bigg| x\right) &= \binom{B\overline{C}}{\overline{B'}}{}_2F_1\left(\begin{matrix}A, B\\ C\end{matrix}\bigg| x\right) - B'(-1)\overline{C}B'(x)\binom{A\overline{C}B'}{\overline{C}B'}\\
&\quad + (q-1)B'(-1)\delta(B\overline{C})\varepsilon(x)\overline{A}(1-x).
\end{aligned}
$$

From Theorem 2.1, the above two identities and (1.3) we deduce that

$$
\begin{aligned}
F_3(A,\varepsilon;B,B';C;x,y) &= \frac{C(-1)}{(q-1)^2}\sum_{\chi}\binom{\overline{A}}{\chi}\binom{C\overline{B}\overline{B'}}{C\overline{B'}\chi}\chi(-x)\sum_{\lambda}\binom{\lambda}{\lambda}\binom{B'\lambda}{C\chi\lambda}\lambda(y)\\
&\quad + B'(-1)\binom{\overline{A}}{B'\overline{C}}B'\overline{C}(x)\varepsilon(1-y)\\
&= \frac{C(-1)}{q-1}\sum_{\chi}\binom{\overline{A}}{\chi}\binom{C\overline{B}\overline{B'}}{C\overline{B'}\chi}\chi(-x){}_2F_1\left(\begin{matrix}\varepsilon, B'\\ C\chi\end{matrix}\bigg| y\right)\\
&\quad + B'(-1)\binom{\overline{A}}{B'\overline{C}}B'\overline{C}(x)\varepsilon(1-y)\\
&= \frac{CB'(-1)}{q-1}\sum_{\chi}\binom{A\chi}{\chi}\binom{B\chi}{C\overline{B'}\chi}\binom{C\overline{B'}\chi}{C\chi}\chi(x)\\
&\quad - \frac{B'(-1)\overline{C}(y)\overline{B'}C(1-y)}{q-1}\sum_{\chi}\binom{A\chi}{\chi}\binom{B\chi}{C\overline{B'}\chi}\chi\left(-\frac{x(1-y)}{y}\right)\\
&\quad + B'(-1)\binom{\overline{A}}{B'\overline{C}}B'\overline{C}(x)\varepsilon(1-y)\\
&= CB'(-1){}_3F_2\left(\begin{matrix}A, C\overline{B'}, B\\ C, C\overline{B'}\end{matrix}\bigg| x\right)\\
&\quad - B'(-1)\overline{C}(y)\overline{B'}C(1-y){}_2F_1\left(\begin{matrix}A, B\\ C\overline{B'}\end{matrix}\bigg| -\frac{x(1-y)}{y}\right)\\
&\quad + B'(-1)\binom{\overline{A}}{B'\overline{C}}B'\overline{C}(x)\varepsilon(1-y)\\
&= CB'(-1)\binom{B\overline{C}}{\overline{B'}}{}_2F_1\left(\begin{matrix}A, B\\ C\end{matrix}\bigg| x\right) - C(-1)\overline{C}B'(x)\binom{A\overline{C}B'}{\overline{C}B'}\\
&\quad + (q-1)\varepsilon(x)C(-1)\delta(B\overline{C})\overline{A}(1-x) + B'(-1)\binom{\overline{A}}{B'\overline{C}}B'\overline{C}(x)\varepsilon(1-y)\\
&\quad - B'(-1)\overline{C}(y)\overline{B'}C(1-y){}_2F_1\left(\begin{matrix}A, B\\ C\overline{B'}\end{matrix}\bigg| -\frac{x(1-y)}{y}\right),
\end{aligned}
$$

which is equivalent to (3.5). (3.6) follows from (3.5) and (1.6). This completes the proof of Theorem 3.1. □

Theorem 3.2. *Let $A, B, B', C, C' \in \widehat{\mathbb{F}}_q$ and $x,\ y \in \mathbb{F}_q$. If $y \neq 0$, then*

$$\begin{aligned}F_3(A,A';B,\varepsilon;C;x,y) &= -C(-1)\varepsilon(y){}_2F_1\left(\begin{matrix}A,B\\C\end{matrix}\middle|x\right)+\binom{\overline{A}}{A'\overline{C}}\binom{\overline{B}C}{A'}A'\overline{C}(x)\delta(1-y)\\&+A'(-1)\overline{C}(y)\overline{A'}C(1-y)\binom{A'B\overline{C}}{A'}{}_2F_1\left(\begin{matrix}A,B\\C\overline{A'}\end{matrix}\middle|-\frac{x(1-y)}{y}\right)\\&+(q-1)\varepsilon(x)\overline{C}(y)\overline{A'}C(1-y)\delta(A'B\overline{C})\overline{A}\left(1+\frac{x(1-y)}{y}\right);\end{aligned}$$

if $x \neq 0$, then

$$\begin{aligned}F_3(A,A';\varepsilon,B';C;x,y) &= -C(-1)\varepsilon(x){}_2F_1\left(\begin{matrix}A',B'\\C\end{matrix}\middle|y\right)+\binom{\overline{A'}}{A\overline{C}}\binom{\overline{B'}C}{A}A\overline{C}(y)\delta(1-x)\\&+A(-1)\overline{C}(x)\overline{A}C(1-x)\binom{AB'\overline{C}}{A}{}_2F_1\left(\begin{matrix}A',B'\\C\overline{A}\end{matrix}\middle|-\frac{y(1-x)}{x}\right)\\&+(q-1)\varepsilon(y)\overline{C}(x)\overline{A}C(1-x)\delta(AB'\overline{C})\overline{A'}\left(1+\frac{y(1-x)}{x}\right).\end{aligned}$$

Proof. It follows from (3.2) and (3.4) that

$$\begin{aligned}{}_2F_1\left(\begin{matrix}A',\varepsilon\\C\chi\end{matrix}\middle|y\right) &= \binom{C\chi}{A'}A'(-1)\overline{C}\overline{\chi}(y)\overline{A'}C\chi(1-y)-C\chi(-1)\varepsilon(y)\\&\quad+(q-1)A'(-1)\delta(1-y)\delta(\overline{A'}C\chi),\\{}_3F_2\left(\begin{matrix}A,C,B\\C\overline{A'},C\end{matrix}\middle|-\frac{x(1-y)}{y}\right) &= \binom{A'B\overline{C}}{A'}{}_2F_1\left(\begin{matrix}A,B\\C\overline{A'}\end{matrix}\middle|-\frac{x(1-y)}{y}\right)\\&\quad-A'(-1)C(y)\overline{C}(x(y-1))\binom{A\overline{C}}{\overline{C}}\\&\quad+(q-1)A'(-1)\delta(A'B\overline{C})\varepsilon(x(1-y))\overline{A}\left(1+\frac{x(1-y)}{y}\right).\end{aligned}$$

We deduce from Theorem 2.1, the above identities and (1.3) that

$$\begin{aligned}F_3(A,A';B,\varepsilon;C;x,y) &= \frac{1}{(q-1)^2}\sum_{\chi}\binom{A\chi}{\chi}\binom{B\chi}{C\chi}\chi(-x)\sum_{\lambda}\binom{A'\lambda}{\lambda}\binom{\lambda}{C\chi\lambda}\lambda(y)\\&\quad+\varepsilon(y)\binom{\overline{A}}{\overline{C}}\overline{C}(x)\overline{A'}(1-y)\\&=\frac{1}{q-1}\sum_{\chi}\binom{A\chi}{\chi}\binom{B\chi}{C\chi}\chi(-x){}_2F_1\left(\begin{matrix}A',\varepsilon\\C\chi\end{matrix}\middle|y\right)\\&\quad+\varepsilon(y)\binom{\overline{A}}{\overline{C}}\overline{C}(x)\overline{A'}(1-y)\end{aligned}$$

$$= -\frac{C(-1)\varepsilon(y)}{q-1}\sum_{\chi}\binom{A\chi}{\chi}\binom{B\chi}{C\chi}\chi(x) + \varepsilon(y)\binom{\overline{A}}{\overline{C}}\overline{C}(x)\overline{A'}(1-y)$$
$$+ \frac{A'(-1)\overline{C}(y)\overline{A'}C(1-y)}{q-1}\sum_{\chi}\binom{A\chi}{\chi}\binom{B\chi}{C\chi}\binom{C\chi}{C\overline{A'}\chi}\chi\left(-\frac{x(1-y)}{y}\right)$$
$$+ A'(-1)\binom{AA'\overline{C}}{A'\overline{C}}\binom{BA'\overline{C}}{A'}A'\overline{C}(-x)\delta(1-y)$$
$$= -C(-1)\varepsilon(y){}_2F_1\left(\begin{matrix}A, B\\ C\end{matrix}\middle| x\right) + \varepsilon(y)\binom{\overline{A}}{\overline{C}}\overline{C}(x)\overline{A'}(1-y)$$
$$+ A'(-1)\overline{C}(y)\overline{A'}C(1-y){}_3F_2\left(\begin{matrix}A, C, B\\ C\overline{A'}, C\end{matrix}\middle| -\frac{x(1-y)}{y}\right)$$
$$+ A'(-1)\binom{AA'\overline{C}}{A'\overline{C}}\binom{BA'\overline{C}}{A'}A'\overline{C}(-x)\delta(1-y)$$
$$= -C(-1)\varepsilon(y){}_2F_1\left(\begin{matrix}A, B\\ C\end{matrix}\middle| x\right) + \binom{\overline{A}}{A'\overline{C}}\binom{\overline{B}C}{A'}A'\overline{C}(x)\delta(1-y)$$
$$+ A'(-1)\overline{C}(y)\overline{A'}C(1-y)\binom{A'B\overline{C}}{A'}{}_2F_1\left(\begin{matrix}A, B\\ C\overline{A'}\end{matrix}\middle| -\frac{x(1-y)}{y}\right)$$
$$+ (q-1)\varepsilon(x)\overline{C}(y)\overline{A'}C(1-y)\delta(A'B\overline{C})\overline{A}\left(1+\frac{x(1-y)}{y}\right),$$

which proves the first identity. The second identity follows from the first identity and (1.6). This concludes the proof of Theorem 3.2. □

4. Generating functions

In this section, we establish some generating functions for $F_3(A, A'; B, B'; C; x, y)$.

Theorem 4.1. *For any $A, A', B, B', C \in \widehat{\mathbb{F}}_q$ and $x, t \in \mathbb{F}_q^*\backslash\{1\}$, $y \in \mathbb{F}_q$, we have*

$$\frac{1}{q-1}\sum_{\theta}\binom{A\theta}{\theta}F_3(A\theta, A'; B, B'; C; x, y)\theta(t)$$
$$= \overline{A}(1-t)F_3\left(A, A'; B, B'; C; \frac{x}{1-t}, y\right)$$
$$- \overline{A}(-t)B(-1)\overline{C}(x)\overline{B}C(1-x){}_2F_1\left(\begin{matrix}A', B'\\ \overline{B}C\end{matrix}\middle| -\frac{y(1-x)}{x}\right).$$

Proof. It follows from (3.3) and (1.4) that

$$ {}_2F_1\left(\begin{matrix}\overline{B'\lambda}, B\overline{C\lambda}\\ \overline{B'\lambda}\end{matrix}\middle| x\right) = \binom{B\overline{C\lambda}}{\overline{B'\lambda}}\overline{B}C\lambda(1-x) - B'\lambda(-x)$$
$$= \binom{B'\lambda}{\overline{B}C\lambda}BB'C(-1)\overline{B}C\lambda(1-x) - B'\lambda(-x).$$

Then, by (1.3) and [7, (2.11)]

$$
\begin{aligned}
(4.1)\qquad & \sum_{\chi,\lambda}\binom{\overline{A'}}{\lambda}\binom{C\overline{B}\overline{B'}}{C\overline{B'}\chi}\binom{C\overline{B'}\chi}{C\chi\lambda}\chi(-x)\lambda(y)\\
&= CB'(-1)\sum_{\lambda}\binom{A'\lambda}{\lambda}\lambda(-y)\sum_{\chi}\binom{B\chi}{C\overline{B'}\chi}\binom{C\overline{B'}\chi}{C\chi\lambda}\chi(x)\\
&= CB'(-1)\overline{C}(x)\sum_{\lambda}\binom{A'\lambda}{\lambda}\lambda\left(-\frac{y}{x}\right)\sum_{\chi}\binom{B\overline{C\lambda}\chi}{\overline{B'}\lambda\chi}\binom{\overline{B'\lambda}\chi}{\chi}\chi(x)\\
&= (q-1)CB'(-1)\overline{C}(x)\sum_{\lambda}\binom{A'\lambda}{\lambda}\lambda\left(-\frac{y}{x}\right){}_2F_1\left(\left.\begin{matrix}\overline{B'\lambda}, B\overline{C\lambda}\\ \overline{B'}\lambda\end{matrix}\right| x\right)\\
&= (q-1)B(-1)\overline{C}(x)\overline{B}C(1-x)\sum_{\lambda}\binom{A'\lambda}{\lambda}\binom{B'\lambda}{\overline{B}C\lambda}\lambda\left(-\frac{y(1-x)}{x}\right)\\
&\quad-(q-1)\overline{C}(-x)B'(x)\sum_{\lambda}\binom{A'\lambda}{\lambda}\lambda(y)\\
&= (q-1)^2B(-1)\overline{C}(x)\overline{B}C(1-x){}_2F_1\left(\left.\begin{matrix}A', B'\\ \overline{B}C\end{matrix}\right| -\frac{y(1-x)}{x}\right)\\
&\quad-(q-1)^2\overline{C}(-x)B'(x)\varepsilon(y)\overline{A'}(1-y),
\end{aligned}
$$

where in the second step we have used the substitution $\chi\to\overline{C\lambda}\chi$. It is easily known from Theorem 2.1 that

(4.2)

$$
\begin{aligned}
&\sum_{\chi,\lambda}\binom{\overline{A}}{\chi}\binom{\overline{A'}}{\lambda}\binom{C\overline{B}\overline{B'}}{C\overline{B'}\chi}\binom{C\overline{B'}\chi}{C\chi\lambda}\chi\left(\frac{x}{1-t}\right)\lambda(y)\\
&= (q-1)^2\left(F_3\left(A, A'; B, B'; C; \frac{x}{1-t}, y\right)-B'(-1)\varepsilon(y)B'\overline{C}(x)\overline{B'}C(1-t)\overline{A'}(1-y)\binom{\overline{A}}{B'\overline{C}}\right).
\end{aligned}
$$

It follows from (3.3) that

$$
\begin{aligned}
(4.3)\qquad \sum_{\theta}\binom{A\theta}{\theta}\binom{A\chi\theta}{A\theta}\theta(t) &= (q-1){}_2F_1\left(\left.\begin{matrix}A, A\chi\\ A\end{matrix}\right| t\right)\\
&= (q-1)\binom{A\chi}{A}\overline{A}\overline{\chi}(1-t)-(q-1)\overline{A}(-t),
\end{aligned}
$$

$$
\begin{aligned}
(4.4)\qquad \sum_{\theta}\binom{A\theta}{\theta}\binom{AB'\overline{C}\theta}{A\theta}\theta(t) &= (q-1){}_2F_1\left(\left.\begin{matrix}A, AB'\overline{C}\\ A\end{matrix}\right| t\right)\\
&= (q-1)\binom{AB'\overline{C}}{A}\overline{AB'}C(1-t)-(q-1)\overline{A}(-t).
\end{aligned}
$$

By (1.2), (1.4) and (4.3), we have

$$\sum_{\chi,\lambda}\binom{\overline{A'}}{\lambda}\binom{C\overline{B}\overline{B'}}{C\overline{B'}\chi}\binom{C\overline{B'}\chi}{C\chi\lambda}\chi(-x)\lambda(y)\sum_{\theta}\binom{A\theta}{\theta}\binom{A\chi\theta}{A\theta}\theta(t)$$
$$=(q-1)\overline{A}(1-t)\sum_{\chi,\lambda}\binom{\overline{A}}{\chi}\binom{\overline{A'}}{\lambda}\binom{C\overline{B}\overline{B'}}{C\overline{B'}\chi}\binom{C\overline{B'}\chi}{C\chi\lambda}\chi\left(\frac{x}{1-t}\right)\lambda(y)$$
$$-(q-1)\overline{A}(-t)\sum_{\chi,\lambda}\binom{\overline{A'}}{\lambda}\binom{C\overline{B}\overline{B'}}{C\overline{B'}\chi}\binom{C\overline{B'}\chi}{C\chi\lambda}\chi(-x)\lambda(y). \tag{4.5}$$

Substituting (4.1) and (4.2) into (4.5), then applying Theorem 2.1, (1.2), (1.3), (4.4) and (4.5) in the left side and cancelling some terms gives

$$\frac{1}{q-1}\sum_{\theta}\binom{A\theta}{\theta}F_3(A\theta,A';B,B';C;x,y)\theta(t)$$
$$=\frac{1}{(q-1)^3}\sum_{\chi,\lambda}\binom{\overline{A'}}{\lambda}\binom{C\overline{B}\overline{B'}}{C\overline{B'}\chi}\binom{C\overline{B'}\chi}{C\chi\lambda}\chi(-x)\lambda(y)\sum_{\theta}\binom{A\theta}{\theta}\binom{A\chi\theta}{A\theta}\theta(t)$$
$$+\frac{1}{q-1}\varepsilon(y)C(-1)B'\overline{C}(x)\overline{A'}(1-y)\sum_{\theta}\binom{A\theta}{\theta}\binom{AB'\overline{C}\theta}{A\theta}\theta(t)$$
$$=\overline{A}(1-t)F_3\left(A,A';B,B';C;\frac{x}{1-t},y\right)$$
$$-\overline{A}(-t)B(-1)\overline{C}(x)\overline{B}C(1-x)\,{}_2F_1\left(\left.\begin{matrix}A',B'\\ \overline{B}C\end{matrix}\right|-\frac{y(1-x)}{x}\right).$$

which is exactly the right side. This finishes the proof of Theorem 4.1. □

From Theorem 4.1 and (1.6) we can easily deduce another generating function for $F_3(A,A';B,B';C;x,y)$.

Theorem 4.2. *For any $A,A',B,B',C\in\widehat{\mathbb{F}}_q$ and $y,t\in\mathbb{F}_q^*\backslash\{1\}$, $x\in\mathbb{F}_q$, we have*

$$\frac{1}{q-1}\sum_{\theta}\binom{A'\theta}{\theta}F_3(A,A'\theta;B,B';C;x,y)\theta(t)$$
$$=\overline{A'}(1-t)F_3\left(A,A';B,B';C;x,\frac{y}{1-t}\right)$$
$$-\overline{A'}(-t)B'(-1)\overline{C}(y)\overline{B'}C(1-y)\,{}_2F_1\left(\left.\begin{matrix}A,B\\ \overline{B'}C\end{matrix}\right|-\frac{x(1-y)}{y}\right).$$

We also establish two other generating functions for $F_3(A,A';B,B';C;x,y)$.

Theorem 4.3. *For any $A, A', B, B', C \in \widehat{\mathbb{F}}_q$ and $x, t \in \mathbb{F}_q^*\backslash\{1\}$, $y \in \mathbb{F}_q$, we have*

$$\frac{1}{q-1}\sum_{\theta}\binom{BB'\overline{C}\theta}{\theta}F_3(A, A'; B\theta, B'; C; x, y)\theta(t)$$
$$= \overline{B}(1-t)F_3\left(A, A'; B, B'; C; \frac{x}{1-t}, y\right) + \overline{BB'}C(-t)F_3(A, A'; C\overline{B'}, B'; C; x, y)$$
$$- \overline{B'}(y)\overline{B}(-t)\overline{C}B'\left(-\frac{x}{t}\right)\binom{\overline{A}}{\overline{C}B'}\binom{\overline{A'}}{\overline{B'}}.$$

Proof. It is easily known from Theorem 2.1 that

$$F_3(A, A'; C\overline{B'}, B'; C; x, y) = \frac{1}{(q-1)^2}\sum_{\chi,\lambda}\binom{\overline{A}}{\chi}\binom{\overline{A'}}{\lambda}\binom{\varepsilon}{C\overline{B'}\chi}\binom{C\overline{B'}\chi}{C\chi\lambda}\chi(x)\lambda(y)$$
$$+ \varepsilon(y)B'(-1)\binom{\overline{A}}{B'\overline{C}}B'\overline{C}(x)\overline{A'}(1-y).$$

Then

$$\sum_{\chi,\lambda}\binom{\overline{A}}{\chi}\binom{\overline{A'}}{\lambda}\binom{\varepsilon}{C\overline{B'}\chi}\binom{C\overline{B'}\chi}{C\chi\lambda}\chi(x)\lambda(y) \tag{4.6}$$
$$= (q-1)^2\left(F_3(A, A'; C\overline{B'}, B'; C; x, y) - \varepsilon(y)B'(-1)\binom{\overline{A}}{B'\overline{C}}B'\overline{C}(x)\overline{A'}(1-y)\right).$$

We see from (1.3), (1.5) and [7, (2.10)] that

$$\sum_{\lambda}\binom{\overline{A'}}{\lambda}\binom{\varepsilon}{B'\lambda}\lambda(y) = -B'(-1)\sum_{\lambda}\binom{A'\lambda}{\lambda}\lambda(y) + (q-1)\binom{\overline{A'}}{\overline{B'}}\overline{B'}(y)$$
$$= -(q-1)B'(-1)\varepsilon(y)\overline{A'}(1-y) + (q-1)\binom{\overline{A'}}{\overline{B'}}\overline{B'}(y).$$

Then, by (1.5),

(4.7)
$$\sum_{\chi,\lambda}\binom{\overline{A}}{\chi}\binom{\overline{A'}}{\lambda}\binom{\varepsilon}{C\overline{B'}\chi}\binom{C\overline{B'}\chi}{C\chi\lambda}\chi(x)\lambda(y)$$
$$= -CB'(-1)\sum_{\chi,\lambda}\binom{\overline{A}}{\chi}\binom{\overline{A'}}{\lambda}\binom{C\overline{B'}\chi}{C\chi\lambda}\chi(-x)\lambda(y) + (q-1)\overline{C}B'(x)\binom{\overline{A}}{\overline{C}B'}\sum_{\lambda}\binom{\overline{A'}}{\lambda}\binom{\varepsilon}{B'\lambda}\lambda(y)$$
$$= -CB'(-1)\sum_{\chi,\lambda}\binom{\overline{A}}{\chi}\binom{\overline{A'}}{\lambda}\binom{C\overline{B'}\chi}{C\chi\lambda}\chi(-x)\lambda(y) - (q-1)^2B'(-1)\overline{C}B'(x)\binom{\overline{A}}{\overline{C}B'}\varepsilon(y)\overline{A'}(1-y)$$
$$+ (q-1)^2\overline{C}B'(x)\binom{\overline{A}}{\overline{C}B'}\binom{\overline{A'}}{\overline{B'}}\overline{B'}(y).$$

From (4.6) and (4.7) we deduce that

$$\sum_{\chi,\lambda}\binom{\overline{A}}{\chi}\binom{\overline{A'}}{\lambda}\binom{C\overline{B'}\chi}{C\chi\lambda}\chi(-x)\lambda(y) \tag{4.8}$$
$$=(q-1)^2\left(\overline{C}B'(-x)\binom{\overline{A}}{\overline{C}B'}\binom{\overline{A'}}{\overline{B'}}\overline{B'}(y)-CB'(-1)F_3(A,A';C\overline{B'},B';C;x,y)\right).$$

It follows from (3.3) that

$$\sum_{\theta}\binom{BB'\overline{C}\theta}{\theta}\binom{B\chi\theta}{BB'\overline{C}\theta}\theta(t)=(q-1){}_2F_1\left(\begin{matrix}BB'\overline{C},B\chi\\ BB'\overline{C}\end{matrix}\middle|\,t\right)$$
$$=(q-1)\binom{B\chi}{BB'\overline{C}}\overline{B}\overline{\chi}(1-t)-(q-1)\overline{B}\overline{B'}C(-t).$$

Then

$$\sum_{\chi,\lambda}\binom{\overline{A}}{\chi}\binom{\overline{A'}}{\lambda}\binom{C\overline{B'}\chi}{C\chi\lambda}\chi(-x)\lambda(y)\sum_{\theta}\binom{BB'\overline{C}\theta}{\theta}\binom{B\chi\theta}{BB'\overline{C}\theta}\theta(t) \tag{4.9}$$
$$=(q-1)CB'(-1)\overline{B}(1-t)\sum_{\chi,\lambda}\binom{\overline{A}}{\chi}\binom{\overline{A'}}{\lambda}\binom{C\overline{B}\overline{B'}}{C\overline{B'}\chi}\binom{C\overline{B'}\chi}{C\chi\lambda}\chi\left(\frac{x}{1-t}\right)\lambda(y)$$
$$-(q-1)\overline{B}\overline{B'}C(-t)\sum_{\chi,\lambda}\binom{\overline{A}}{\chi}\binom{\overline{A'}}{\lambda}\binom{C\overline{B'}\chi}{C\chi\lambda}\chi(-x)\lambda(y).$$

Substituting (4.2) and (4.8) into (4.9), applying (1.2), (1.3), (4.9) and [7, (2.10)] in the left side and cancelling some terms yields

$$\frac{1}{q-1}\sum_{\theta}\binom{BB'\overline{C}\theta}{\theta}F_3(A,A';B\theta,B';C;x,y)\theta(t)$$
$$=\frac{CB'(-1)}{(q-1)^3}\sum_{\chi,\lambda}\binom{\overline{A}}{\chi}\binom{\overline{A'}}{\lambda}\binom{C\overline{B'}\chi}{C\chi\lambda}\chi(-x)\lambda(y)\sum_{\theta}\binom{BB'\overline{C}\theta}{\theta}\binom{B\chi\theta}{BB'\overline{C}\theta}\theta(t)$$
$$+\frac{1}{q-1}\varepsilon(y)B'(-1)\binom{\overline{A}}{B'\overline{C}}B'\overline{C}(x)\overline{A'}(1-y)\sum_{\theta}\binom{BB'\overline{C}\theta}{\theta}\theta(t)$$
$$=\overline{B}(1-t)F_3\left(A,A';B,B';C;\frac{x}{1-t},y\right)$$
$$-\overline{B}(-t)\overline{B'}C(t)\left(\overline{C}B'(-x)\binom{\overline{A}}{\overline{C}B'}\binom{\overline{A'}}{\overline{B'}}\overline{B'}(y)-CB'(-1)F_3(A,A';C\overline{B'},B';C;x,y)\right),$$

which equals the right side. The proof of Theorem 4.3 is completed. □

From Theorem 4.3 and (1.6) we can also deduce another generating function for $F_3(A,A';B,B';C;x,y)$.

Theorem 4.4. *For any* $A, A', B, B', C \in \widehat{\mathbb{F}}_q$ *and* $y, t \in \mathbb{F}_q^* \backslash \{1\}$, $x \in \mathbb{F}_q$, *we have*

$$\frac{1}{q-1}\sum_{\theta}\binom{BB'\overline{C}\theta}{\theta}F_3(A,A';B,B'\theta;C;x,y)\theta(t)$$
$$=\overline{B'}(1-t)F_3\left(A,A';B,B';C;x,\frac{y}{1-t}\right)+\overline{BB'}C(-t)F_3(A,A';B,C\overline{B};C;x,y)$$
$$-\overline{B}(y)\overline{B'}(-t)\overline{C}B\left(-\frac{y}{t}\right)\binom{\overline{A'}}{\overline{C}B}\binom{\overline{A}}{\overline{B}}.$$

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