Single-Source Nets
of Algebraically-Quantized Reflective Liouville Potentials on the Line
II. Use of Krein Determinants for Constructing SUSY Ladders of Rational Potentials Quantized by Multi-Index Gauss-Seed Heine polynomials

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The paper presents the uniform technique for constructing SUSY ladders of rational canonical Sturm-Liouville equations (RCSLEs) conditionally exactly quantized by Gauss-seed (GS) Heine polynomials. Each ladder starts from the RCSLE exactly quantized by classical Jacobi, generalized Laguerre or Romanovski-Routh polynomials. We then use its nodeless almost everywhere holomorphic (AEH) solutions formed by the appropriate set of *non-orthogonal* polynomials to construct multi-step rational SUSY partners of the given Liouville potential on the line. It was proven that eigenfunctions of each RCSLE in the ladder have an AEH form, namely, each eigenfunction can be represented as a weighted polynomial fraction (PFrs), with both numerator and denominator remaining finite at the common singular points of all the RCSLEs in the given ladder. As a result both polynomials satisfy the second-order differential equations of Heine type.

1. Introduction

In our recent study [1] referred to below as Part I we presented the general analysis of SUSY partners of the 1D rational canonical Sturm-Liouville equation (RCSLE) with the second-order poles using almost-everywhere holomorphic (AEH) solutions as factorization functions for the corresponding canonical Liouville-Darboux transformations (CLDTs) defined as the three-step operations including

i) the Liouville transformation of the given RCSLE,

ii) the Darboux transformation of the resultant Liouville potential,

iii) the inverse Liouville transformation back to the original RCSLE.

For the RCSLEs with the singularity at the finite end of the quantization interval we additionally require that CLDTs in question do not change the zero-energy exponent difference for this singular point which assures that the Liouville transformation necessarily converts the given RCSLE into the Schrödinger equation on the line. (SUSY ladders of radial Liouville potentials will be covered in a separate publication [2].)

The purpose of this paper is to apply the aforementioned formalism to the ladders of multi-step SUSU partners of the Gauss-reference (GRef) potentials [3-5] exactly quantized by classical Jacobi, generalized Laguerre or Romanovski [6, 7] ('Romanovski-Routh' in our terms [5]) polynomials and referred to below as the *r*-, *c*-,and *i*-GRef potentials accordingly. Contrary to the *r*- and *c*-GRef potentials broadly cited in the literature [8-27], the *i*-GRef potential, having the Kepler problem in spherical coordinates [28-33] and Gendenshtein [34-36] potential ('trigonometric Rosen-Morse' and 'Scarf II' potentials, respectively, in the Cooper-Khare-Sukhatme [37-39] classification scheme) as their shape-invariant [31, 34] limiting cases, attracted relatively little attention. In this paper we consider only symmetric-tangent-polynomial (sym-TP) reduction of this potential thoroughly studied by Milson [4] and referred to for this reason as 'Milson potential' [5]. The main reason for focusing on this reduction is that (as proven in [5]) each eigenfunction formed by the Romanovski-Routh polynomial of order v is accompanied by an irregular almost-everywhere holomorphic (AEH) solution formed by another Routh polynomial of the same order. If the second Routh polynomial does not have real roots then the latter (irregular at $\pm\infty$) AEH solution can be used as the factorization function (FF) for the single-step CLDT. In particular, one can construct the SUSY ladder of rational SUSY partners of the nearly-symmetric Milson potential which are conditionally exactly quantized by

'Routh seed' (ℜS) Heine polynomials [5, 1]. It is worth emphasizing the nearly-symmetric Milson potential represents the only exception when we consider irregular-at-both-endpoints solutions as seed functions for the DT.

For both *r*- and *c*-GRef potentials we use only seed functions regular at one end which are gathered in our recent survey [40, 41]. It is worth mentioning that the sets of regular Gauss-seed (GS) solutions for these potentials -- referred to as Jacobi-seed' (𝒥S) [40] and 'Laguerre seed' (ℒS) [41], correspondingly -- generally include nodeless solutions formed by polynomials of the order either larger or equal to the number of bound energy levels. For two shape-invariant potentials on the line – the Rosen-Morse potential [42] and Morse oscillator [43] – these are the only GS solutions lying below the ground energy levels, as initially demonstrated by Quesne [44, 45]. This is indeed the common feature of all shape-invariant potentials as it has been pointed to by Odake and Susaki [46] who refer to these solutions as 'overshoot eigenfunctions'.

It has been proven by us [1, 40, 41] that each eigenfunction generally belongs to the quartet of GS solutions of four distinct types formed by polynomials of the same order. Some of these quartets necessarily include nodeless solutions regular at either $-\infty$ or $+\infty$. However all regular solutions disappear in the limit of the constant tangent polynomial (const-TP) corresponding to the mentioned shape-invariant potentials on the line, contrary to 'overshoot eigenfunctions'.

The most important observation made by the author is the CLDT using the so-called 'Jacobi-seed' (**𝒥S**) [40], 'Laguerre seed' (ℒS) [41], or 'Routh seed' (ℜS) [5] solutions as seed functions converts each seed solution into the AEH solution having form of a weighted polynomial fraction (PFr). This remarkable feature of the SUSY ladders of rational Liouville potentials generated by the above CLDTs is reminiscent of the technique utilized by Odake and Sasaki [46, 47] in the particular case of the shape-invariant GRef potentials. However the crucial element of our approach is that we require that power exponents appearing the weight function coincide with one of two characteristic exponents (ChExps) at each finite regular singular point of the resultant RCSLE.

We then take advantage of the fact that each single-step CLDT does not change exponents differences (ExpDiffs) at the intrinsic singularities (finite ends of the quantization interval for *r*- and *c*-GRef potentials or ±i for the Milson potential) for any GRef potential on the line, except [5, 1] the Gendenstein potential which will be covered in a separate publication. Indeed, since the PFr forming the given AEH solution must generally remain finite at all the singular point it may not have zero at any of the intrinsic singularities. Based on the fact that this assertion holds for each single-step CLDT using a regular FF as well as for the seed solutions themselves we can use mathematical induction to prove both numerator and denominator of the PFr in question remain finite at each intrinsic singularity and therefore satisfy the second-order differential equation of Heine type at the origin.

For this proof to be valid, the mentioned FFs must remain nodeless at each step and this why we did not include into our analysis multi-step SUSY partners [47, 48] constructed using pairs of juxtaposed eigenfunctions [49-51]. We also prefer not to use GS solutions irregular at both ends [46, 47] until it is proven that the appropriate FFs are nodeless at each step.

It is also worth mentioning that the CLDTs do change ExpDiffs at the origin for any radial potential and this is the main reason of why we restricted the current analysis solely to multi-step SUSY partners of GRef potentials on the line.

The progress outlined above was to a large extend made due to use of Krein determinants [52], instead of Crum Wroskians [53], in following Bagrov and Samsonov's cutting-edge suggestion [49-51, 54] which (to our knowledge) was mostly ignored in the literature. Though potential use of Krein determinant, instead of Crum Wroskian, was recently mentioned by Grandati [55], that the main advantage of the former representation – the fact that it contains only first derivatives of seed functions which significantly simplifies an analysis of its behavior near singularities -- has not been fully appreciated.

In Appendix A we derive explicit formulas for zero-free energy term in the generic canonical

Sturm-Liouville equation transformed by a multi-step Darboux deformation of the Liouville potential. We express solutions of this equation in terms of both Crum Wroskians and Krein determinants. We traced the representation of solutions in terms of ratios of Crum Wroskians to Schulze-Halberg's paper [56] who uses a different terminology: 'foreign auxiliary equation' (see also [57]) for the Sturm-Liouville equation and 'generalized Darboux transformation' for the CLDT. (Remember that we changed the term 'generalized Darboux transformation'(GDT) introduced in the scattering theory [58-64] for 'Liouville-Darboux transformation' because the latter expression was repeatedly used by various authors in completely different sense [1].)

2. Multi-step SUSY partners of GRef potentials and related GS Bose invariants

Let us consider the CLDT of the GRef potential $V[\xi\,|\,{}_{\iota}\boldsymbol{G}^{K\Im}]$ (ι = 0, 1, or *i*) using the $p \equiv 2j + \ell$ GS solutions $\dagger_K m_K$ as seed functions. (Compared with [1], we dropped the third superscript in the notation of the PFrBs ${}_{\iota}\boldsymbol{G}^{K\Im 0}$ keeping in mind that the present study deals solely with GRef potentials on the line so the omitted superscript is always equal to 0.) It directly follows from the analysis presented in Appendix A leads to the RCSLE

$$\left\{\frac{d^2}{d\xi^2} + I^{o}[\xi;\varepsilon\,|\,{}^{p}_{\iota}\boldsymbol{G}^{K\Im}_{\{\dagger m\}_p}]\right\}\Phi[\xi;\varepsilon\,|\,{}^{p}_{\iota}\boldsymbol{G}^{K\Im}_{\{\dagger m\}_p}] = 0. \tag{2.1}$$

where the reference PFr (RefPFr)

$$I^{o}[\xi\,|\,{}^{p}_{\iota}\boldsymbol{G}^{K\Im}_{\{\dagger m\}_p}] = I^{o}[\xi\,|\,{}_{\iota}\boldsymbol{G}^{K\Im}_{\downarrow\{\dagger m\}_p}] + \Delta I^{o}[\xi\,|\,{}^{p}_{\iota}\boldsymbol{G}^{K\Im}_{\{\dagger m\}_p}] \tag{2.2}$$

in the Bose invariant

$$I^{o}[\xi;\varepsilon\,|\,{}^{p}_{\iota}\boldsymbol{G}^{K\Im}_{\{\dagger m\}_p}] \equiv I^{o}[\xi\,|\,{}^{p}_{\iota}\boldsymbol{G}^{K\Im}_{\{\dagger m\}_p}] + \varepsilon\,{}_{\iota}\wp[\xi;{}_{\iota}T_K] \tag{2.3}$$

is expressed in terms of the Krein determinant

$${}_{\iota}K_p[\xi\,|\,\{\dagger m\}_p] \equiv K\{{}_{\iota}\phi[\xi\,|\,\{\dagger m\}_p];\,{}_{\iota}\varepsilon_{\{\dagger m\}_p}\} \tag{2.4}$$

via (A.9$_0$) and (A.9$_1$) for even and odd p, i.e., for $\ell = 0$ and $\ell = 1$, respectively, namely,

$$\Delta I^{o}[\xi\,|\,{}^{2j+\ell}_{\iota}\boldsymbol{G}^{K\Im}_{\{\dagger m\}_{2j+\ell}}] = \ell\,\Delta I\{\wp[\xi;{}_{\iota}T_K]\}$$
$$+2\,{}_{\iota}\wp^{1/2}[\xi;{}_{\iota}T_K]\frac{d}{d\xi}\left\{{}_{\iota}\wp^{-1/2}[\xi;{}_{\iota}T_K]\,ld\,\left|{}_{\iota}K_{2j+\ell}[\xi\,|\,\{\dagger m\}_{2j+\ell}]\right|\right\}, \tag{2.5}$$

and

$$\Delta I\{{}_{\iota}\wp[\xi]\} = \tfrac{1}{2}\{l\,\dot{d}\;{}_{\iota}\wp[\xi] - \tfrac{1}{2} ld^{\,2}\;{}_{\iota}\wp[\xi]\}. \tag{2.6}$$

As pointed to in Part I, the CLDTs preserve the density function which is chosen (*in this series of publications*) to have the very specific form

$${}_{\iota}\wp[\xi;{}_{\iota}T_K] = \frac{{}_{\iota}a_K\,\Pi_{\Im}^{\delta_T}[\xi;{}_{\iota}\bar{\xi}_T]}{4\prod\limits_{r=0}^{|\iota|}(\xi - {}_{\iota}e_r)^2}, \tag{2.7}$$

where $\delta_T = \Im - K + 1$ for K>0 is the order of the TP zero ${}_{\iota}\xi_{T;1}$ (if any) and $\Pi_n[\xi;\bar{\xi}]$ stands for the monic polynomial with n simple zeros ξ_k (k=1,…,n). For any GRef potential *on the line*, excluding the Gendenshtein (Scarf II) potential [34] discussed in a separate paper, the TP roots ${}_{\iota}\bar{\xi}_T$ differ from the intrinsic singular points ${}_{\iota}e_r$ (${}_1e_r = r$, ${}_0e_0 = 0$, and ${}_ie_0 = -i$, ${}_ie_0 = +i$) which implies that the density function of our interest has the second order pole at each intrinsic singularity. The direct corollary of this observation [1] is that the sequential CLDTs of our interest keep unchanged the energy-dependent characteristic exponents (ChExps) at the intrinsic singular points of the resultant RCSLEs.

Making use of representation (A.27) for the zero-energy free term in CSLE (A.21) in terms of Wroskian ${}_{\iota}W[\xi\,|\,\{\dagger m\}_p]$ formed by GS solutions $\dagger_k m_k$ we can also write (2.5) as

$$\Delta I^{o}[\xi\,|\,{}^{p}_{\iota}G^{K\Im}_{\{\dagger m\}_p}] = -p(p-2)\Delta I\{\wp[\xi;{}_{\iota}T_K]\} \tag{2.8}$$

$$+2\,{}_{\iota}\wp^{1/2}[\xi;{}_{\iota}T_K]\frac{d}{d\xi}\left\{{}_{\iota}\wp^{-1/2}[\xi;{}_{\iota}T_K]\,ld\,\left|{}_{\iota}W[\xi\,|\,\{\dagger m\}_p]\right|\right\}.$$

The Schulze-Halberg [56] representation of the zero-energy free term and similar formula for AEH solutions of RCSLE (2.1) is convenient to relate our results to Odake and Susaki s scheme [45, 46] for constructing multi-step SUSY partners of shape-invariant potentials. We will come back to this analysis in Part III where both shape-invariant potentials on the line: the Rosen-Morse and Morse potentials will be analyzed in great details as the limiting cases of the linear TP (LTP) *r*- and *c*-GRef potentials, respectively.

It is convenient to re-arrange the second term in the right-hand side of (2.5),

$$2\wp^{1/2}[\xi;{}_\iota T_{\mathrm{K}}]\frac{\mathrm{d}}{\mathrm{d}\xi}\left\{{}_\iota\wp^{-1/2}[\xi;{}_\iota T_{\mathrm{K}}]\,ld\,\left|{}_\iota\mathrm{K}_{\mathrm{p}}[\xi\,|\,\mathbf{t}_1\mathrm{m}_1;...;\mathbf{t}_{\mathrm{p}}\mathrm{m}_{\mathrm{p}}]\right|\right\} \tag{2.8}$$

$$=2\,l\dot{d}\,\left|{}_\iota\mathrm{K}_{\mathrm{p}}[\xi\,|\,\mathbf{t}_1\mathrm{m}_1;...,\mathbf{t}_{\mathrm{p}}\mathrm{m}_{\mathrm{p}}]\right|-ld\,\wp[\xi;{}_\iota T_{\mathrm{K}}]\times ld\,\left|{}_\iota\mathrm{K}_{\mathrm{p}}[\xi\,|\,\mathbf{t}_1\mathrm{m}_1;...,\mathbf{t}_{\mathrm{p}}\mathrm{m}_{\mathrm{p}}]\right|,$$

as

$$2\wp^{1/2}[\xi;{}_\iota T_{\mathrm{K}}]\frac{\mathrm{d}}{\mathrm{d}\xi}\left\{\wp^{-1/2}[\xi;{}_\iota T_{\mathrm{K}}]\,ld\,\left|\mathrm{K}_{\mathrm{p}}[\xi\,|\,\mathbf{t}_1\mathrm{m}_1;...;\mathbf{t}_{\mathrm{p}}\mathrm{m}_{\mathrm{p}}]\right|\right\}$$

$$=2\,l\dot{d}\,\left|\mathrm{K}_{\mathrm{p}}[\xi\,|\,\mathbf{t}_1\mathrm{m}_1;...;\mathbf{t}_{\mathrm{p}}\mathrm{m}_{\mathrm{p}}]\right|+2\sum_{r=0}^{|\iota|}\frac{1}{\xi-{}_\iota e_r}\,ld\left|\mathrm{K}_{\mathrm{p}}[\xi\,|\,\mathbf{t}_1\mathrm{m}_1;...;\mathbf{t}_{\mathrm{p}}\mathrm{m}_{\mathrm{p}}]\right| \tag{2.8*}$$

$$-2\sum_{\mathrm{k}'=1}^{\mathfrak{I}}\frac{{}_\iota\rho_{\mathrm{T}}}{\xi-{}_\iota\xi_{\mathrm{T};\mathrm{k}'}}\,ld\,\left|\mathrm{K}_{\mathrm{p}}[\xi\,|\,\mathbf{t}_1\mathrm{m}_1;...;\mathbf{t}_{\mathrm{p}}\mathrm{m}_{\mathrm{p}}]\right|$$

taking into account that

$$ld\,\wp[\xi;{}_\iota T_{\mathrm{K}}]=2\sum_{\mathrm{k}'=1}^{\mathfrak{I}}\frac{{}_\iota\rho_{\mathrm{T}}}{\xi-{}_\iota\xi_{\mathrm{T};\mathrm{k}'}}-2\sum_{r=0}^{|\iota|}\frac{1}{\xi-{}_\iota e_r}, \tag{2.9}$$

where ${}_\iota\rho_{\mathrm{T}}=\frac{1}{2}\delta_{\mathrm{T}}$.   It will be proven in next two sections that the second-order poles appearing in the first and second terms in the right-hand side of (2.8*) compensate each other and that the last term has no singularities at the TP zeros ${}_\iota\xi_{\mathrm{T};\mathrm{k}'}$ for $\ell=0$ as expected from the general analysis presented in Part I. It is worth emphasizing that the latter assertion holds iff density function (2.7) has second-order poles at the intrinsic singular points which is not true either for radial GRef potentials or for two exactly quantized trigonometric potentials: the D/PT potential [64, 65] and its *i*-GRef analogue [29, 33].

Substituting GS solutions

$$
{}_\iota\phi_{\dagger_k m_k}[\xi] \equiv \phi[\xi\,|\,{}_\iota\boldsymbol{G}^{K\Im}_{\downarrow\dagger_k m_k};\dagger_k m_k] = {}_\iota\Theta[\xi;\,{}_\iota\lambda_{0;\dagger_k m_k},{}_\iota\lambda_{1;\dagger_k m_k}]\,\Pi_{m_k}[\xi;{}_\iota\bar{\xi}_{\dagger_k m_k}], \tag{2.10}
$$

where

$$
{}_\iota\Theta[\xi;2\,{}_\iota\rho_0-1,2\,{}_\iota\rho_1-|\iota|] \equiv \begin{cases} (1-i\,\xi)^{i\rho_0}(1+i\,\xi)^{i\rho_1} & \text{for } \iota=i \;\; ({}_i\rho_1={}_i\rho_0^*), \\ \prod\limits_{r=0}^{|\iota|} |\xi-{}_\iota e_r|^{{}_\iota\rho_r} \;\; e^{-\delta_{\iota;0}\,{}_\iota\rho_1\xi} & \text{otherwise,} \end{cases} \tag{2.11}
$$

into Krein discriminant (2.4) one can represent the latter as the weighted polynomial:

$$
\begin{aligned}
&{}_\iota K_{2j+\ell}[\xi\,|\,\dagger_1 m_1;\ldots;\dagger_{2j+\ell}m_{2j+\ell}] \\
&\qquad = {}_\iota\Theta_j[\xi\,|\,\{\dagger m\}_{2j+\ell}]\,{}_\iota P_{{}_\iota\upsilon_{\{\dagger m\}_{2j+\ell}}}[\xi\,|\,\dagger_1 m_1;\ldots;\dagger_{2j+\ell}m_{2j+\ell}],
\end{aligned} \tag{2.12}
$$

where

$$
{}_\iota\Theta_\kappa[\xi\,|\,\{\dagger m\}_p] = \frac{\prod\limits_{k=1}^{p} {}_\iota\Theta[\xi;\,{}_\iota\lambda_{0;\dagger_k m_k},{}_\iota\lambda_{1;\dagger_k m_k}]}{\prod\limits_{r=0}^{|\iota|}(\xi-{}_\iota e_r)^\kappa} \tag{2.13}
$$

and the so-called ‘GS main polynomial determinant’ (GS-MPD) have the form

$$
{}_\iota P_{{}_\iota\upsilon_{\{\dagger m\}_{2j}}}[\xi\,|\,\dagger_1 m_1;\ldots;\dagger_{2j}m_{2j}] \tag{2.14a}
$$

$$
\equiv \begin{vmatrix} \Pi_{m_1}[\xi;{}_\iota\bar{\xi}_{\dagger_1 m_1}] & \ldots & \Pi_{m_{2j}}[\xi;{}_\iota\bar{\xi}_{\dagger_{2j}m_{2j}}] \\ {}_\iota P_{m_1+1}[\xi\,|\,\dagger_1 m_1] & \ldots & {}_\iota P_{m_{2j}+1}[\xi\,|\,\dagger_{2j}m_{2j}] \\ {}_\iota\varepsilon_{\dagger_1 m_1}\Pi_{m_1}[\xi;{}_\iota\bar{\xi}_{\dagger_1 m_1}] & \ldots & {}_\iota\varepsilon_{\dagger_{2j}m_{2j}}\Pi_{m_{2j}}[\xi;{}_\iota\bar{\xi}_{\dagger_{2j}m_{2j}}] \\ & \ldots & \\ {}_\iota\varepsilon^{j-1}_{\dagger_1 m_1}\,{}_\iota P_{m_1+1}[\xi\,|\,\dagger_1,m_1] & \ldots & {}_\iota\varepsilon^{j-1}_{\dagger_{2j}m_{2j}}\,{}_\iota P_{m_{2j}+1}[\xi\,|\,\dagger_{2j},m_{2j}] \end{vmatrix}
$$

or

$${}_{\iota}P_{\iota\upsilon_{\{\dagger m\}_{2j+1}}}[\xi\,|\,\dagger_1 m_1;\ldots;\dagger_{2j+1}m_{2j+1}] \tag{2.14b}$$

$$\equiv(-1)^{j}\begin{vmatrix} \Pi_{m_1}[\xi;{}_{\iota}\bar{\xi}_{\dagger_1 m_1}] & \ldots & \Pi_{m_{2j+1}}[\xi;{}_{\iota}\bar{\xi}_{\dagger_{2j+1}m_{2j+1}}] \\ {}_{\iota}P_{m_1+1}[\xi\,|\,\dagger_1 m_1] & \ldots & {}_{\iota}P_{m_{2j+1}+1}[\xi\,|\,\dagger_{2j+1}m_{2j+1}] \\ & \ldots & \\ {}_{\iota}\varepsilon^{j-1}_{\dagger_1 m_1}\,{}_{\iota}P_{m_1+1}[\xi|\dagger_1 m_1] & \ldots & {}_{\iota}\varepsilon^{j-1}_{\dagger_{2j+1}m_{2j+1}}\,{}_{\iota}P_{m_{2j+1}+1}[\xi|\dagger_{2j+1}m_{2j+1}] \\ {}_{\iota}\varepsilon^{j}_{\dagger_1 m_1}\Pi_{m_1}[\xi;{}_{\iota}\bar{\xi}_{\dagger_1 m_1}] & \ldots & {}_{\iota}\varepsilon^{j}_{\dagger_{2j+1}m_{2j+1}}\Pi_{m_{2j+1}}[\xi;{}_{\iota}\bar{\xi}_{\dagger_{2j+1}m_{2j+1}}] \end{vmatrix}$$

for even ($\ell = 0$) or odd ($\ell = 1$) numbers of steps, respectively. Note that use of the Krein determinant [15], instead of the Crum Wroskian [17], allows one to consider only the first derivatives of the polynomials $\Pi_{m_k}[\xi;{}_{\iota}\bar{\xi}_{\dagger_k m_k}]$ making use of the auxiliary polynomials defined via (7.4′) in [1], with ${}^{*}\bar{\xi}^{(n_{\tilde{\dagger}\tilde{m}})}_{\tilde{\dagger}\tilde{m}} = {}^{*}\bar{\xi}^{(\tilde{m})}_{\tilde{\dagger}\tilde{m}} \equiv \bar{\xi}_{\tilde{\dagger}\tilde{m}}$ 2j+, namely,

$${}_{\iota}P_{m+1}[\xi\,|\,\dagger m] \equiv {}_{\iota}P_{m+1}[\xi;\bar{\iota};\bar{\lambda}_{\dagger m};{}_{\iota}\bar{\xi}_{\dagger m}] \tag{2.15}$$

$$= \widehat{B}_1[\xi;\bar{\iota};\bar{\lambda}_{\dagger m}]\Pi_m[\xi;{}_{\iota}\bar{\xi}_{\dagger m}] + \prod_{r=0}^{|\iota|}(\xi - {}_{\iota}e_r)\ \dot{\Pi}_m[\xi;{}_{\iota}\bar{\xi}_{\dagger m}], \tag{2.15*}$$

where

$$\widehat{B}_1[\xi;\bar{\iota};\bar{\lambda}] \equiv \tfrac{1}{2}\,|\,\iota\,|\,[(\lambda_0+1)(\xi - {}_{\iota}e_1) + (\lambda_1+1)(\xi - {}_{\iota}e_0)] + \tfrac{1}{2}\delta_{\iota,0}(\lambda_0 + 1 - \lambda_1\xi). \tag{2.16}$$

In this series of publications it will be always assumed that PDs (2.14a) and (2.14b) have only simple zeros and therefore can be represented as scaled monomial products

$${}_{\iota}P_{\iota\upsilon_{\{\dagger m\}_p}}[\xi\,|\,\dagger_1 m_1;\ldots;\dagger_p m_p] = {}_{\iota}P_{\iota\upsilon_{\{\dagger m\}_p}}\,|\,\dagger_1 m_1;\ldots;\dagger_p m_p)\,\Pi_{\iota\upsilon_{\{\dagger m\}_p}}[\xi;{}_{\iota}\bar{\xi}_{\{\dagger m\}_p}]. \tag{2.17}$$

It will be proven in next section that polynomials (2.17) remain finite at the intrinsic singular points $_\iota e_r$ (r = 0, |ι|) – the common remarkable feature of the GS Liouville potentials *on the line* (with the Gendenshtein potential as the only exclusion).

Substituting (2.12), coupled with (2.17), into the first two terms in the right-hand side of (2.8*) (while keeping the third term unchanged) we can represent the PFr in question as

$$2\wp^{1/2}[\xi;{}_\iota T_K]\frac{d}{d\xi}\left\{\wp^{-1/2}[\xi;{}_\iota T_K]ld\left|{}_\iota K_{2j+\ell}[\xi\,|\,\dagger_1 m_1;...;\dagger_{2j+\ell}m_{2j+\ell}]\right|\right\}=2Q[\xi;{}_\iota\bar{\xi}_{\{\dagger m\}_{2j+\ell}}]$$

$$+2l\dot{d}\left|{}_\iota\Theta_j[\xi\,|\,\{\dagger m\}_{2j+\ell}]\right|+2ld\left|{}_\iota\Theta_j[\xi\,|\,\{\dagger m\}_p]\right|\sum_{r=0}^{|\iota|}\frac{1}{\xi-{}_\iota e_r} \tag{2.18}$$

$$+2\sum_{r=0}^{|\iota|}\frac{1}{\xi-{}_\iota e_r}\frac{\dot{\Pi}_{\upsilon\{\dagger m\}_{2j+\ell}}[\xi;{}_\iota\bar{\xi}_{\{\dagger m\}_{2j+\ell}}]}{\Pi_{\upsilon\{\dagger m\}_{2j+\ell}}[\xi;{}_\iota\bar{\xi}_{\{\dagger m\}_{2j+\ell}}]}$$

$$-2\sum_{k'=1}^{\Im}\frac{{}_\iota\rho_T}{\xi-{}_\iota\xi_{T;k'}}\times\frac{{}_\iota\dot{K}_{2j+\ell}[\xi\,|\,\dagger_1 m_1;...;\dagger_{2j+\ell}m_{2j+\ell}]}{{}_\iota K_{2j+\ell}[\xi\,|\,\dagger_1 m_1;...;\dagger_{2j+\ell}m_{2j+\ell}]}$$

where the PFr

$$Q[\xi;{}_\iota\bar{\xi}_{\{\dagger m\}_p}]\equiv\ddot{\Pi}[\xi;{}_\iota\bar{\xi}_{\{\dagger m\}_p}]/\Pi[\xi;{}_\iota\bar{\xi}_{\{\dagger m\}_p}]-\dot{\Pi}^2[\xi;{}_\iota\bar{\xi}_{\{\dagger m\}_p}]/\Pi^2[\xi;{}_\iota\bar{\xi}_{\{\dagger m\}_p}]. \tag{2.19}$$

was adopted by us from Quesne's works [44, 45, 66-70] and for this reason is referred to below as 'QPFr'. It will be demonstrated in next section that the second-order poles in the second and third terms in the sum in the right-hand side of (2.18) compensate each other so that the sum in question has only first-order poles at the intrinsic singular points $_\iota e_r$ (r = 0, |ι|).

Let us consider the AEH solution of RCSLE (2.1) obtained by applying the $(2j + \ell)$ -step CLDT in question to the GS solution $\dagger_{2j+\ell+1}m_{2j+\ell+1}$. Substituting (2.12) into generic formula (A.31) for solutions of the CSLE undergone a multi-step CLDT one finds

$$\Phi[\xi;\varepsilon\,|\,{}^{p}_{\iota}\boldsymbol{G}^{K\Im}_{\{\dagger m\}_{2j+\ell}};\dagger_{2j+\ell+1}m_{2j+\ell+1}] \tag{2.20}$$

$$= \frac{{}_{\iota}\Theta_{\ell}[\xi\,|\,\dagger_{2j+\ell+1}m_{2j+\ell+1}]\;{}_{\iota}P_{{}_{\iota}\upsilon_{\{\dagger m\}_{2j+\ell+1}}}[\xi\,|\,\dagger_1 m_1;...;\dagger_{2j+\ell+1}m_{2j+\ell+1}]}{{}_{\iota}\wp^{\frac{1}{2}\ell}[\xi;{}_{\iota}T_K]\;{}_{\iota}P_{{}_{\iota}\upsilon_{\{\dagger m\}_{2j+\ell}}}[\xi\,|\,\dagger_1 m_1;...;\dagger_{2j+\ell}m_{2j+\ell}]}$$

or, making use of (2.7),

$$\Phi[\xi;\varepsilon\,|\,{}^{p}_{\iota}\boldsymbol{G}^{K\Im}_{\{\dagger m\}_{2j+\ell}};\dagger_{2j+\ell+1}m_{2j+\ell+1}] \propto {}_{\iota}\Theta_{0}[\xi\,|\,\dagger_{2j+\ell+1}m_{2j+\ell+1}]\Pi_{\Im}^{-\frac{1}{2}\ell\delta_T}[\xi;{}_{\iota}\bar{\xi}_T]$$

$$\times\frac{{}_{\iota}P_{{}_{\iota}\upsilon_{\{\dagger m\}_{2j+\ell+1}}}[\xi\,|\,\dagger_1 m_1;...;\dagger_{2j+\ell+1}m_{2j+\ell+1}]}{{}_{\iota}P_{{}_{\iota}\upsilon_{\{\dagger m\}_{2j+\ell}}}[\xi\,|\,\dagger_1 m_1;...;\dagger_{2j+\ell}m_{2j+\ell}]}. \tag{2.21}$$

We thus conclude that the PFr in the right-hand side of (2.21) may not vanish at intrinsic singular points. Since the latter statement generally holds for Jacobi, generalized Laguerre, and Routh polynomials, we can prove by induction that the PDs of our current interest remain finite at the mentioned singular points and therefore satisfy the Heine-type second-order differential equations introduced in Part I.

Though representation (2.21) for the AEH solution of RCSLE (2.1) is applicable to the potentials on the line, on the half-line, as well as to two trigonometric potentials mentioned above, the rational potentials on the line represent the very special case when

i) the TP does not vanish at the intrinsic singular points

and as a result

ii) the CLDTs in question do not change the zero-energy ExpDiffs for intrinsic singularities.

This conclusion cannot be extended in general to either radial or trigonometric GRef potentials and this is the main reason of why we restricted our current analysis solely to rational potentials on the line.

Up to now we did not impose any restrictions on sets of GS solutions used as seed functions for the given CLDT. It should be however stressed that we are solely interested in RCSLE conditionally exactly quantized by GS Heine polynomials and for this reason the sets of our current interest include only regular $\mathcal{I}$S or regular $\mathcal{L}$S solutions lying below the ground energy level, i. e.,

$$\{\mathbf{t}m\}_p = \mathbf{a}\{m_{\mathbf{a}}\}_{p_{\mathbf{a}}}; \mathbf{b}\{m_{\mathbf{b}}\}_{p_{\mathbf{b}}} \tag{2.22}$$

assuming that

$$\iota\varepsilon_{\mathbf{t}_k m_k} < \iota\varepsilon_{\mathbf{c}0} \quad \text{for} \;\; k = 1, \ldots, p = p_{\mathbf{a}} + p_{\mathbf{b}} \,. \tag{2.22'}$$

With this choice of GS solutions the AEH solution of type $\mathbf{t}_{2j+\ell+1} = \mathbf{a}$ or $\mathbf{b}$ (lying at the energy $\iota\varepsilon_{\mathbf{t}_{2j+\ell+1} m_{2j+\ell+1}} < \iota\varepsilon_{\mathbf{c}0}$) become nodeless and therefore this should be also true for the PFrs in the right-hand side of (2.21). Since the latter statement generally holds for both Jacobi and generalized Laguerre polynomials to construct GS solutions (2.22), we can prove by induction that the PDs in question may not have nodes inside the quantization interval.

Excluding the shape-invariant Rosen-Morse and Morse potentials, we can also delete $p_{\mathbf{c}}$ lowest bound energy levels by adding $p_{\mathbf{c}}$ eigenfunctions to set (2.22):

$$\{\mathbf{t}m\}_p = \mathbf{a}\{m_{\mathbf{a}}\}_{p_{\mathbf{a}}}; \mathbf{b}\{m_{\mathbf{b}}\}_{p_{\mathbf{b}}}; \mathbf{c}\{v = 0, \ldots, p_{\mathbf{c}} - 1\} \tag{2.24}$$

and also extending constraint (2.23):

$$\iota\varepsilon_{\mathbf{a}m_k} < \iota\varepsilon_{\mathbf{c}p_{\mathbf{c}}} \quad \text{for} \;\; k = 1, \ldots, p_{\mathbf{a}}, \tag{2.24a'}$$

$$\iota\varepsilon_{\mathbf{b}m_{\tilde{k}}} < \iota\varepsilon_{\mathbf{c}p_{\mathbf{c}}} \quad \text{for} \;\; \tilde{k} = 1, \ldots, p_{\mathbf{b}}\,. \tag{2.24b'}$$

However the above arguments are not applicable to CLDTs using pairs of sequential eigenfunctions as seed solutions. Indeed, though the resultant AEH solutions must be also nodeless according the Krein-Adler theorem [52, 71] the denominators of the appropriate PFrs in the right-hand side of (2.21) do have zeros inside the quantization interval and therefore this should be also true for their numerators. This is the main reason of why we did not include ‘juxtaposed’ [49-51] seed solutions into the current discussion.

3. Invariance of ExpDiffs at the intrinsic singular points under multi-step CLDTs with GS functions

According to the arguments presented in Part I CLDTs with AEH FFs keep unchanged the ExpDiffs at the intrinsic singular points ${}_\iota e_r$ (r = 0, |ι|) which implies that PFr (2.5) does not have second-order poles – the characteristic signature of GS Liouville potentials on the line.

To confirm this assertion let us first demonstrate that the second-order poles in the second and third terms in the sum in the right-hand side of (2.18) compensate each other. In fact, differentiating the logarithmic derivative

$$ld\left|{}_\iota\Theta_j[\xi\,|\,\{\dagger m\}_{2j+\ell}]\right| = \sum_{r=0}^{|\iota|}\frac{\sum_{k=1}^{2j+\ell}{}_\iota\rho_{r;\dagger_k m_k}-j}{\xi-{}_\iota e_r} - \tfrac{1}{2}\,\delta_{\iota,0}\sum_{k=1}^{2j+\ell}\nu_{\dagger_k m_k} \qquad (3.1')$$

(with $\nu_{\dagger_k m_k}\equiv {}_0\lambda_{1;\dagger_k m_k}$) gives

$$l\dot{d}\left|{}_\iota\Theta_j[\xi\,|\,\{\dagger m\}_{2j+\ell}]\right| = -\sum_{r=0}^{|\iota|}\frac{\sum_{k=1}^{2j+\ell}{}_\iota\rho_{r;\dagger_k m_k}-j}{(\xi-{}_\iota e_r)^2} \qquad (3.1'')$$

which confirms that the PFr in question

$$2l\dot{d}\left|{}_\iota\Theta_j[\xi\,|\,\{\dagger m\}_{2j+\ell}]\right| + 2ld\left|{}_\iota\Theta_j[\xi\,|\,\{\dagger m\}_{2j+\ell}]\right|\sum_{r=0}^{|\iota|}\frac{1}{\xi-{}_\iota e_r} = \frac{{}_\iota\Theta_{\{\dagger m\}_p}+2|\iota|(j+\ell)}{\prod_{r=0}^{|\iota|}(\xi-{}_\iota e_r)}, \qquad (3.1)$$

has only the first-order pole at any intrinsic singular point $\xi = {}_\iota e_r$, Note that each summand

$$ {}_\iota\Theta({}_\iota\bar{\lambda}_{\dagger m}) \equiv |\iota|({}_\iota\lambda_{0;\dagger m} + {}_\iota\lambda_{1;\dagger m}) - \delta_{\iota,0}\, \nu_{\dagger m}, \tag{3.2} $$

in the sum

$$ {}_\iota\Theta_{\{\dagger m\}_p} \equiv \sum_{k=1}^{p} {}_\iota\Theta({}_\iota\bar{\lambda}_{\dagger m}) \tag{3.2*} $$

appearing in the numerator of PFr (3.1) is an odd function of ${}_\iota\bar{\lambda}_{\dagger m}$:

$$ {}_\iota\Theta({}_\iota\bar{\lambda}) \equiv {}_\iota\Theta(-{}_\iota\bar{\lambda}) . \tag{3.2†} $$

It will be shown in Section 7 that this feature of function (3.2) plays an important role in matching the double-step DT and two-seed Crum representations of the RefPFr $I^{o}[\xi|\,{}^{2}_{\iota}\boldsymbol{G}^{K\Im}_{\dagger_1 m_1;\dagger_2 m_2}]$

Let us now prove that that GS-MPDs (2.17) do not have zeros at the intrinsic singular points ${}_\iota e_r$ (r = 0, $|\iota|$) as far as the TP remains finite at these points. The latter assumption does not hold either for the aforementioned Gendenshtein potential or for radial GS Liouville potentials which will be thereby discussed in separate publications. The proof requires a more detailed analysis of the FF

$$ \begin{aligned} &\phi_{\dagger_{2j+\ell} m_{2j+\ell}}[\xi|\,{}^{2j+\ell-1}_{\iota}\boldsymbol{G}^{K\Im}_{\{\dagger m\}_{2j+\ell-1}\downarrow\dagger_{2j+\ell} m_{2j+\ell}}] \\ &\quad\propto {}_\iota T_K^{\frac{1}{2}(1-\ell)}[\xi]\,{}_\iota\Theta[\xi; 2\,{}_\iota\rho_{0;\dagger_{2j+\ell} m_{2j+\ell}} - 1, 2\,{}_\iota\rho_{1;\dagger_{2j+\ell} m_{2j+\ell}} - |\iota|] \\ &\qquad\times \frac{{}_\iota P_{{}_\iota\upsilon_{\{\dagger m\}_{2j+\ell}}}[\xi|\,\overline{\dagger_1 m_1;\ldots;\dagger_{2j+\ell} m_{2j+\ell}}]}{{}_\iota P_{{}_\iota\upsilon_{\{\dagger m\}_{2j+\ell-1}}}[\xi|\,\overline{\dagger_1 m_1;\ldots;\dagger_{2j+\ell-1} m_{2j+\ell-1}}]} \end{aligned} \tag{3.3} $$

implicitly used for the p$^{\text{th}}$ single-step CLDT which *by definition* acts on the PFrB ${}^{p-1}_{\iota}\boldsymbol{G}^{K\Im}_{\{\dagger m\}_{p-1}\downarrow\dagger_p m_p}$ to generate the PFrB ${}^{p}_{\iota}\boldsymbol{G}^{K\Im}_{\{\dagger m\}_p}$. The cited expression can be directly derived from Krein's formula [52] for the given solution of the Schrödinger equation with the RLP

$V[\xi(x)\,|\,{}^{p-1}_{\iota}\boldsymbol{G}^{K\Im}_{\{\mathbf{t}m\}_{p-1}\downarrow\mathbf{t}_p m_p}]$ by changing seed functions $\psi_{\mathbf{t}_k m_k}[\xi(x)\,|\,{}_{\iota}\boldsymbol{G}^{K\Im}_{\downarrow\{\mathbf{t}m\}_p}]$ for GS solutions

$$\phi_{\mathbf{t}_k m_k}[\xi\,|\,{}_{\iota}\boldsymbol{G}^{K\Im}_{\downarrow\{\mathbf{t}m\}_p}] = {}_{\iota}\wp^{-1/4}[\xi;\,{}_{\iota}T_K]\psi_{\mathbf{t}_k m_k}[\xi\,|\,{}_{\iota}\boldsymbol{G}^{K\Im}_{\downarrow\{\mathbf{t}m\}_p}]. \tag{3.4}$$

Since the power ${}_{\iota}\rho_{r;\mathbf{t}_p m_p}$ coincides with one of the ChExps at the singular point ${}_{\iota}e_{r}$ the PFr in the right-hand side of (3.2) must remain finite at $\xi = {}_{\iota}e_{r}$ for r =0, |ι| (excluding the potentially exceptional case of an integer ExpDiff $-{}_{\iota}\lambda_{r;\mathbf{t}_p m_p} > 0$). This implies that the PFr numerator does not generally vanish at $\xi = {}_{\iota}e_{r}$ unless this is also true for its denominator. Now we can apply mathematical induction to prove that the GS-MPDs do not have zeros at the intrinsic singular points ${}_{\iota}e_{r}$ keeping in mind that this assertion normally holds for the first polynomial in the sequence, i.e., for the Jacobi polynomial if ι=1, for the Laguerre polynomial if ι=0, or for the Routh polynomial if ι=*i*, again putting aside anomalous points along threshold curves (2.38) in [40] or (2.26) in [41] for the regular $\mathscr{J}$S or $\mathscr{L}$S solutions, respectively.

We conclude that PFr (2.8*) does not have a second-order pole at any of the intrinsic singular points ${}_{\iota}e_{r}$. This is also true for universal RI-independent correction (2.5). Indeed, substituting both (2.9) and its derivative with respect to ξ into (2.5) one can represent the latter PFr as

$$\Delta I\{\wp[\xi;\,{}_{\iota}T_K]\} = 2\hat{Q}[\xi;\,{}_{\iota}\bar{\xi}_T;\,{}_{\iota}\rho_T] - \frac{2\,|\,\iota\,|}{(\xi-{}_{\iota}e_0)(\xi-{}_{\iota}e_1)} + 2\sum_{k'=1}^{\Im}\frac{{}_{\iota}\rho_T}{\xi-{}_{\iota}\xi_{T;k'}} \times \sum_{r=0}^{|\iota|}\frac{1}{\xi-{}_{\iota}e_r}, \tag{3.5}$$

where

$$\hat{Q}[\xi;\,{}_{\iota}\bar{\xi}_T;\,{}_{\iota}\rho_T] \equiv {}^1\!/\!{}_2\,{}_{\iota}\rho_T\frac{\ddot{\Pi}_{\Im}[\xi;\,{}_{\iota}\bar{\xi}_T]}{\Pi_{\Im}[\xi;\,{}_{\iota}\bar{\xi}_T]} - {}^1\!/\!{}_2\,{}_{\iota}\rho_T({}_{\iota}\rho_T+1)\frac{\dot{\Pi}^2_{\Im}[\xi;\,{}_{\iota}\bar{\xi}_T]}{\Pi^2_{\Im}[\xi;\,{}_{\iota}\bar{\xi}_T]}. \tag{3.6}$$

An analysis of (3.6) reveals that the universal correction (if appropriate, i.e., iff $\ell = 1$) has second-order poles only at the TP zeros. We thus explicitly corroborated the assertion in [1] that CLDs using $\mathfrak{J}$S, $\mathfrak{L}$S, or $\mathfrak{R}$S solutions as seed functions do not change ExpDiffs at the intrinsic singular points.

4. Erasing singularities at TP zeros by even-step CLDTs

In Part I we suggested the conjecture that the intrinsic singular points ${}_\iota e_r$ are the only common singularities of two sequential RCSLEs in the ladder generated by multi-step CLDTs using GS solutions (2.10) as their seed functions. (In general it is theoretically possible that the given CLDT with an AEH FF turns second-order poles in the initial *arbitrarily chosen* RCSLE into the first-order poles in the resultant partner equation thereby excluding the possibility for a further extension of the SUSY ladder using the technique developed in Part I.) The purpose of this Section is to prove that each TP zero is a regular point of RCSLE (2.1) generated by an even number p = 2j of sequential CLDTs.

Indeed an analysis of universal RI-independent correction (3.1) shows that the coefficients of the second-order poles at the TP zeros ${}_\iota\xi_{T;k'}$ match those in the general expression for the partner RefPFr given by (2.24) in [1] and therefore PFr (2.8*) may have only first-order poles at these points. The immediate corollary from this observation is that the GS-MPD (2.14a) and (2.14b) may not share common zeros with the TP. Otherwise both first and third terms in the right-hand side of (2.8*) would have second-order poles. However, since the coefficients of these poles are negative in both cases they cannot cancel each other.

Let us now explicitly corroborate that the last term in the right-hand side of (2.8*) does not have first-order poles at the TP zeros for $\ell = 0$, in agreement with the arguments presented in Part I. In other words we need to prove that the first derivative of Krein determinant (2.4) vanishes at each point ${}_\iota\xi_{T;k'}$:

$$ {}_\iota\dot{K}_{2j}[{}_\iota\xi_{T;k'} \,|\, \dagger_1 m_1;\ldots,\dagger_{2j} m_{2j}] = 0 \qquad (k' = 1,..,\mathfrak{I}). \tag{4.1}$$

To do it we take advantage of the fact that computation of the first derivative of the Krein determinant only requires differentiation of its last row since all other determinants in the sum has a pair of identical row which gives

$$ {}_{\iota}\dot{K}_{2j}[\xi\,|\,\dagger_1 m_1;\ldots;\dagger_{2j}m_{2j}] = -\,{}_{\iota}\wp[\xi;\,{}_{\iota}T_{K}]\,{}_{\iota}D_{2j}[\xi\,|\,\dagger_1 m_1;\ldots;\dagger_{2j}m_{2j}], \tag{4.2}$$

where

$$ {}_{\iota}D_{2j}[\xi\,|\,\dagger_1 m_1;\ldots,\dagger_{2j}m_{2j}] $$

$$ \equiv \begin{vmatrix} \phi_{\dagger_1 m_1}[\xi|\,{}_{\iota}\boldsymbol{G}^{K\Im 0}_{\downarrow\{\dagger m\}_{2j}}] & \cdots & \phi_{\dagger_{2j} m_{2j}}[\xi|\,{}_{\iota}\boldsymbol{G}^{K\Im 0}_{\downarrow\{\dagger m\}_{2j}}] \\ \dot{\phi}_{\dagger_1 m_1}[\xi|\,{}_{\iota}\boldsymbol{G}^{K\Im 0}_{\downarrow\{\dagger m\}_{2j}}] & \cdots & \dot{\phi}_{\dagger_{2j} m_{2j}}[\xi|\,{}_{\iota}\boldsymbol{G}^{K\Im 0}_{\downarrow\{\dagger m\}_{2j}}] \\ \varepsilon_{\dagger_1 m_1}\phi_{\dagger_1 m_1}[\xi|\,{}_{\iota}\boldsymbol{G}^{K\Im 0}_{\downarrow\{\dagger m\}_{2j}}] & \cdots & \varepsilon_{\dagger_{2j} m_{2j}}\phi_{\dagger_{2j} m_{2j}}[\xi|\,{}_{\iota}\boldsymbol{G}^{K\Im 0}_{\downarrow\{\dagger m\}_{2j}}] \\ & \cdots & \\ \varepsilon^{j-1}_{\dagger_1 m_1}\phi_{\dagger_1 m_1}[\xi|\,{}_{\iota}\boldsymbol{G}^{K\Im 0}_{\downarrow\{\dagger m\}_{2j}}] & \cdots & \varepsilon^{j-1}_{\dagger_{2j} m_{2j}}\phi_{\dagger_{2j} m_{2j}}[\xi|\,{}_{\iota}\boldsymbol{G}^{K\Im 0}_{\downarrow\{\dagger m\}_{2j}}] \\ \varepsilon^{j-1}_{\dagger_1 m_1}\phi_{\dagger_1 m_1}[\xi|\,{}_{\iota}\boldsymbol{G}^{K\Im 0}_{\downarrow\{\dagger m\}_{2j}}] & \cdots & \varepsilon^{j-1}_{\dagger_{2j} m_{2j}}\phi_{\dagger_{2j} m_{2j}}[\xi|\,{}_{\iota}\boldsymbol{G}^{K\Im 0}_{\downarrow\{\dagger m\}_{2j}}] \end{vmatrix}. \tag{4.3}$$

Since the last row in the determinant does not contain the first derivatives of GS solutions the weight of the substituted PD (GS-SPD)

$$ {}_{\iota}\tilde{P}_{{}_{\iota}\tilde{\upsilon}_{\{\dagger m\}_{2j}}}[\xi\,|\,\dagger_1 m_1;\ldots;\dagger_{2j}m_{2j}] \tag{4.4}$$

$$ \equiv \begin{vmatrix} \Pi_{m_1}[\xi;\,{}_{\iota}\bar{\xi}_{\dagger_1 m_1}] & \cdots & \Pi_{m_{2j}}[\xi;\,{}_{\iota}\bar{\xi}_{\dagger_{2j} m_{2j}}] \\ {}_{\iota}P_{m_1+1}[\xi\,|\,\dagger_1 m_1] & \cdots & {}_{\iota}P_{m_{2j}+1}[\xi\,|\,\dagger_{2j} m_{2j}] \\ & \cdots & \\ {}_{\iota}\varepsilon^{j-1}_{\dagger_1 m_1}\Pi_{m_1}[\xi;\,{}_{\iota}\bar{\xi}_{\dagger_1 m_1}] & \cdots & {}_{\iota}\varepsilon^{j-1}_{\dagger_{2j} m_{2j}}\Pi_{m_{2j}}[\xi;\,{}_{\iota}\bar{\xi}_{\dagger_{2j} m_{2j}}] \\ {}_{\iota}\varepsilon^{j}_{\dagger_1 m_1}\Pi_{m_1}[\xi;\,{}_{\iota}\bar{\xi}_{\dagger_1 m_1}] & \cdots & {}_{\iota}\varepsilon^{j}_{\dagger_{2j} m_{2j}}\Pi_{m_{2j}}[\xi;\,{}_{\iota}\bar{\xi}_{\dagger_{2j} m_{2j}}] \end{vmatrix}. $$

in the weighted-polynomial representation of function (4.3),

$$ {}_{\iota}D_{2j}[\xi\,|\,\mathbf{t}_1 m_1;...,\mathbf{t}_{2j} m_{2j}] = -\ {}_{\iota}\Theta_{j-1}[\xi\,|\,\{\mathbf{t}m\}_{2j}]\ {}_{\iota}\tilde{P}_{{}_{\iota}\tilde{\upsilon}_{\{\mathbf{t}m\}_{2j}}}[\xi\,|\,\overline{\mathbf{t}_1 m_1;...;\mathbf{t}_{2j} m_{2j}}], \tag{4.5} $$

differs from that in (2.12) by the monomial product $\prod_{r=0}^{|\iota|}(\xi - {}_{\iota}e_r)$. Substituting (2.7) and (4.5) into (4.2) and dividing the resultant expression by (2.12) we come to the PFr

$$ ld\,|\,{}_{\iota}K_{2j}[\xi\,|\,\mathbf{t}_1 m_1;...;\mathbf{t}_{2j} m_{2j}]\,| $$
$$ = \frac{{}_{\iota}a_K \Pi^{\delta_T}_{\mathfrak{I}}[\xi;{}_{\iota}\bar{\xi}_T]\ {}_{\iota}\tilde{P}_{{}_{\iota}\tilde{\upsilon}_{\{\mathbf{t}m\}_{2j}}}[\xi\,|\,\mathbf{t}_1 m_1;...;\mathbf{t}_{2j} m_{2j}]}{2\prod_{r=0}^{|\iota|}(\xi - {}_{\iota}e_r)\ {}_{\iota}P_{{}_{\iota}\upsilon_{\{\mathbf{t}m\}_{2j}}}[\xi\,|\,\mathbf{t}_1 m_1;...;\mathbf{t}_{2j} m_{2j}]}. \tag{4.6} $$

The last term in the right-hand side of (2.18) can be thus represented as

$$ -2\sum_{k'=1}^{\mathfrak{I}} \frac{{}_{\iota}\rho_T}{\xi - {}_{\iota}\xi_{T;k'}}\, ld\,\left|K_p[\xi\,|\,\mathbf{t}_1 m_1;...;\mathbf{t}_{2j} m_{2j}]\right| $$
$$ = -\frac{{}_{\iota}a_K\ {}_{\iota}\rho_T\ \mathfrak{I}(\xi - {}_{\iota}\xi_T)^{K-1}\ {}_{\iota}\hat{P}_{{}_{\iota}\tilde{\upsilon}_{\{\mathbf{t}m\}_{2j}}}[\xi\,|\,\mathbf{t}_1 m_1;...;\mathbf{t}_{2j} m_{2j}]}{\prod_{r=0}^{|\iota|}(\xi - {}_{\iota}e_r)\,\Pi_{{}_{\iota}\upsilon_{\{\mathbf{t}m\}_{2j}}}[\xi;{}_{\iota}\bar{\xi}_{\{\mathbf{t}m\}_{2j}}]}, \tag{4.7} $$

where we set

$$ {}_{\iota}\hat{P}_{{}_{\iota}\tilde{\upsilon}_{\{\mathbf{t}m\}_{2j}}}[\xi\,|\,\mathbf{t}_1 m_1;...;\mathbf{t}_{2j} m_{2j}] \equiv {}_{\iota}\tilde{P}_{{}_{\iota}\tilde{\upsilon}_{\{\mathbf{t}m\}_{2j}}}[\xi\,|\,\mathbf{t}_1 m_1;...;\mathbf{t}_{2j} m_{2j}]/\ {}_{\iota}P_{{}_{\iota}\upsilon_{\{\mathbf{t}m\}_{2j}}}\,|\,\mathbf{t}_1 m_1;...;\mathbf{t}_p m_p) \tag{4.8} $$

making use of (2.17). This confirms that the term in question and therefore RefPFr (2.2) remain regular at each point ${}_{\iota}\xi_{T;k'}$ if $\ell = 0$ (assuming that zeros of the Krein determinant differ from TP zeros.)

5. The Quesne partial decomposition of multi-step GS RefPFrs

It directly follows from the analysis presented in previous sections that RefPF (2.2) can be represented in the generic form specified by rational formulas (3.25a) and (3.25b) in Part I for $\ell = 0$ and 1, respectively. Namely

$$\mathrm{I}^{\mathrm{o}}[\xi\,|\,{}^{2j}_{\iota}\boldsymbol{G}^{\mathrm{K}\Im}_{\{\dagger m\}_{2j}}] = \mathrm{I}^{\mathrm{o}}[\xi;\,{}_{\iota}\bar{\mathrm{h}}_{\mathrm{o}},\,{}_{\iota}\mathrm{O}^{\mathrm{o}}_{0}] + 2\mathrm{Q}[\xi;\,{}_{\iota}\bar{\xi}_{\{\dagger m\}_{2j}}] + \frac{\Delta\mathrm{O}^{\downarrow}_{\upsilon_{\{\dagger m\}_{2j}}}[\xi\,|\,{}^{2j}_{\iota}\boldsymbol{G}^{\mathrm{K}\Im}_{\{\dagger m\}_{2j}}]}{4\prod\limits_{r=0}^{|\iota|}(\xi - {}_{\iota}e_{r})\Pi_{\upsilon_{\{\dagger m\}_{2j}}}[\xi;\,{}_{\iota}\bar{\xi}_{\{\dagger m\}_{2j}}]} \tag{5.1$_0$}$$

and

$$\mathrm{I}^{\mathrm{o}}[\xi\,|\,{}^{2j+1}_{\iota}\boldsymbol{G}^{\mathrm{K}\Im}_{\{\dagger m\}_{2j+1}}] = \mathrm{I}^{\mathrm{o}}[\xi;\,{}_{\iota}\bar{\mathrm{h}}_{\mathrm{o}},\,{}_{\iota}\mathrm{O}^{\mathrm{o}}_{0}] + 2\mathrm{Q}[\xi;\,{}_{\iota}\bar{\xi}_{\{\dagger m\}_{2j}}] + 2\hat{\mathrm{Q}}[\xi;\,{}_{\iota}\bar{\xi}_{\mathrm{T}};\,{}_{\iota}\rho_{\mathrm{T}}] \tag{5.1$_1$}$$

$$+\frac{\Delta\mathrm{O}^{\downarrow}_{\upsilon_{\{\dagger m\}_{2j+1}}+\Im}[\xi\,|\,{}^{2j}_{\iota}\boldsymbol{G}^{\mathrm{K}\Im}_{\{\dagger m\}_{2j+1}}]}{4\prod\limits_{r=0}^{|\iota|}(\xi - {}_{\iota}e_{r})\Pi_{\upsilon_{\{\dagger m\}_{2j+1}}}[\xi;\,{}_{\iota}\bar{\xi}_{\{\dagger m\}_{2j+1}}]\Pi_{\Im}[\xi;\,{}_{\iota}\bar{\xi}_{\mathrm{T}}]}.$$

It is essential that PFrs (2.19) and (3.5) appearing in the right-hand sides of RefPFrs (5.1$_0$) and (5.1$_1$) include both second- and first-order poles at the singular points with energy-independent ExpDiffs. Since the TP turns into a constant in the extreme cases of the shape-invariant RM and Morse potentials (K=0) there is no need to distinguish between even and odd numbers of steps. As a result the Ref PFrs in question take the form

$$\mathrm{I}^{\mathrm{o}}[\xi\,|\,{}^{\mathrm{p}}_{\iota}\boldsymbol{G}^{00}_{\{\dagger\mathrm{m}\}_{\mathrm{p}}}] = \mathrm{I}^{\mathrm{o}}[\xi\,|\,{}_{\iota}\boldsymbol{G}^{00}_{\downarrow\{\dagger\mathrm{m}\}_{\mathrm{p}}}] + 2\mathrm{Q}[\xi;\,{}_{\iota}\bar{\xi}_{\{\dagger\mathrm{m}\}_{\mathrm{p}}}] + \frac{\Delta\mathrm{O}^{\downarrow}_{\mathrm{n}_{\{\dagger\mathrm{m}\}_{\mathrm{p}}}}[\xi\,|\,{}^{\mathrm{p}}_{\iota}\boldsymbol{G}^{00}_{\{\dagger\mathrm{m}\}_{\mathrm{p}}}]}{4\prod\limits_{r=0}^{|\iota|}(\xi - {}_{\iota}e_{r})\Pi_{\mathrm{n}_{\{\dagger\mathrm{m}\}_{\mathrm{p}}}}[\xi;\,{}_{\iota}\bar{\xi}_{\{\dagger\mathrm{m}\}_{\mathrm{p}}}]}, \tag{5.2}$$

regardless of evenness of p. We [X, 1] thus refer to the given representation as the ‘Quesne partial decomposition’ (QPD) where the term ‘partial’ is used to emphasize that we do not separate explicitly all the second-order poles from the rest of the BI, in contrast with the original definition of BIs of our interest via (2.2) in Part I.

Let us start from evaluating the polynomial numerator of the fraction in the right-hand side of (5.1_0). Substituting (3.1) and (4.7) into (2.18) gives

$$\Delta\mathrm{O}^{\downarrow}_{{}_{\iota}\upsilon_{\{\dagger\mathrm{m}\}_{2\mathrm{j}}}}[\xi\,|\,{}^{2\mathrm{j}}_{\iota}\boldsymbol{G}^{\mathrm{K}\Im}_{\{\dagger\mathrm{m}\}_{2\mathrm{j}}}] = 4\big({}_{\iota}\Theta_{\{\dagger\mathrm{m}\}_{2\mathrm{j}}} + 2\,|\,\iota\,|\,\mathrm{j}\big)\Pi_{\upsilon_{\{\dagger\mathrm{m}\}_{2\mathrm{j}}}}[\xi;\,{}_{\iota}\bar{\xi}_{\{\dagger\mathrm{m}\}_{2\mathrm{j}}}] \tag{5.3}$$

$$+8\sum_{r=0}^{|\iota|}(\xi - {}_{\iota}e_{r})^{|\iota|}\,\dot{\Pi}_{\upsilon_{\{\dagger\mathrm{m}\}_{2\mathrm{j}}}}[\xi;\,{}_{\iota}\bar{\xi}_{\{\dagger\mathrm{m}\}_{2\mathrm{j}}}]$$

$$+2\,{}_{\iota}\rho_{\mathrm{T}}\Im\,{}_{\iota}a_{\mathrm{K}}\,(\xi - {}_{\iota}\xi_{\mathrm{T}})^{\mathrm{K}-1}\,{}_{\iota}\hat{\mathrm{P}}_{{}_{\iota}\tilde{\upsilon}_{\{\dagger\mathrm{m}\}_{2\mathrm{j}}}}[\xi\,|\,\dagger_{1}\mathrm{m}_{1};\ldots;\dagger_{2\mathrm{j}}\mathrm{m}_{2\mathrm{j}}].$$

Differentiating (2.12) with respect to ξ and substituting (2.7) and (4.2) into the left- and right-hand side of the resultant expression, respectively, one can explicitly express polynomial (4.8) in terms of polynomial (2.14a) and its first derivative:

$$-\tfrac{1}{4}\,{}_{\iota}\mathrm{T}_{\mathrm{K}}[\xi]\,{}_{\iota}\hat{\mathrm{P}}_{{}_{\iota}\tilde{\upsilon}_{\{\dagger\mathrm{m}\}_{2\mathrm{j}}}}[\xi\,|\,\dagger_{1}\mathrm{m}_{1};\ldots;\dagger_{2\mathrm{j}}\mathrm{m}_{2\mathrm{j}}]\;/\;{}_{\iota}\mathrm{P}_{{}_{\iota}\upsilon_{\{\dagger\mathrm{m}\}_{2\mathrm{j}}}}\,|\,\dagger_{1}\mathrm{m}_{1};\ldots;\dagger_{\mathrm{p}}\mathrm{m}_{\mathrm{p}}) \tag{5.4}$$

$$=\Big\{\sum_{k=1}^{2\mathrm{j}}\hat{\mathrm{B}}_{1}[\xi;\bar{\iota};\,{}_{\iota}\bar{\lambda}_{\dagger_{k}\mathrm{m}_{k}}] - \mathrm{j}\sum_{r=0}^{|\iota|}(\xi - {}_{\iota}e_{r})^{|\iota|}\Big\}\,{}_{\iota}\mathrm{P}_{{}_{\iota}\upsilon_{\{\dagger\mathrm{m}\}_{2\mathrm{j}}}}[\xi\,|\,\dagger_{1}\mathrm{m}_{1};\ldots;\dagger_{2\mathrm{j}}\mathrm{m}_{2\mathrm{j}}]$$

$$+\prod_{r=0}^{|\iota|}(\xi - {}_{\iota}e_{r})\,{}_{\iota}\dot{\mathrm{P}}_{{}_{\iota}\upsilon_{\{\dagger\mathrm{m}\}_{2\mathrm{j}}}}[\xi\,|\,\dagger_{1}\mathrm{m}_{1};\ldots;\dagger_{2\mathrm{j}}\mathrm{m}_{2\mathrm{j}}],$$

where we took into account [1] that first-order polynomial (2.16) can be alternatively represented as

$$\widehat{B}_1[\xi;\bar{\iota};2\,{}_\iota\bar{\rho}-\bar{1}_2]\equiv B_1[\xi;\bar{\iota};{}_\iota\bar{\rho}] \tag{5.5†}$$

$$\equiv \prod_{r=0}^{|\iota|}(\xi-{}_\iota e_r)\left(\sum_{r'=0}^{|\iota|}\frac{{}_\iota\rho_{r'}}{\xi-{}_\iota e_{r'}}-\delta_{\iota,0}\tfrac{1}{2}\,{}_0\rho_1\right) \tag{5.5}$$

$$=|\iota|[{}_\iota\rho_0(\xi-{}_\iota e_1)+{}_\iota\rho_1(\xi-{}_\iota e_0)]+\delta_{\iota,0}({}_0\rho_0-\tfrac{1}{2}\,{}_0\rho_1\xi), \tag{5.5*}$$

where

$$\bar{\iota}=\begin{cases}({}_\iota e_0,{}_\iota e_1) \text{ for } |\iota|=1\\ \quad\text{or}\\ 0 \quad \text{for } \iota=0.\end{cases} \tag{5.6}$$

The polynomial representing the right-hand side of (5.4) is thus necessarily divisible by the TP so the orders of polynomials (4.4) and (2.14a) differ by K−1:

$${}_\iota\upsilon_{\{\dagger m\}_{2j}}={}_\iota\tilde{\upsilon}_{\{\dagger m\}_{2j}}+K-1. \tag{5.7}$$

For odd numbers of steps (p = 2j+1) we explicitly take advantage of the fact that

$$ld\,|\,{}_\iota K_{2j+\ell}[\xi\,|\,\dagger_1 m_1;...;\dagger_{2j+\ell}m_{2j+\ell}]\,|$$

$$=ld\,{}_\iota\Theta_j[\xi\,|\,\{\dagger m\}_{2j+\ell}]+\frac{\dot{\Pi}_{{}_\iota\upsilon_{\{\dagger m\}_{2j+\ell}}}[\xi;{}_\iota\bar{\xi}_{\{\dagger m\}_{2j+\ell}}]}{\Pi_{{}_\iota\upsilon_{\{\dagger m\}_{2j+\ell}}}[\xi;{}_\iota\bar{\xi}_{\{\dagger m\}_{2j+\ell}}]} \tag{5.8}$$

assuming that the PD has only single zeros ${}_\iota\xi_{\{\dagger m\}_p;k}$ for k = 1,…, $\upsilon_{\{\dagger m\}_p}$:

$${}_\iota P_{\upsilon_{\{\dagger m\}_p}}[\xi;{}_\iota\bar{\xi}_T;\overline{\dagger m}]\propto\Pi_{\upsilon_{\{\dagger m\}_p}}[\xi;{}_\iota\bar{\xi}_{\{\dagger m\}_p}], \tag{5.8*}$$

and then re-group PFrs in the logarithmic derivative (3.14′) in the following fashion

$$ld\,|\,{}_{\iota}\Theta_{j}[\xi\,|\,\{\dagger m\}_{2j+\ell}] = \frac{\sum_{k=1}^{2j+\ell} B_1[\xi;\bar{\iota};\,{}_{\iota}\bar{\rho}_{\dagger_k m_k}]}{\prod_{r=0}^{|\iota|}(\xi - {}_{\iota}e_r)} - j\sum_{r=0}^{|\iota|}\frac{1}{\xi - {}_{\iota}e_r} \tag{5.9}$$

Taking into account that

$$\Pi_{\mathfrak{I}}[\xi;\,{}_{\iota}\bar{\xi}_{T}]\sum_{k'=1}^{\mathfrak{I}}\frac{1}{\xi - {}_{\iota}\xi_{T;k'}} = \dot{\Pi}_{\mathfrak{I}}[\xi;\,{}_{\iota}\bar{\xi}_{T}] = \mathfrak{I}(\xi - {}_{\iota}\xi_{T})^{\mathfrak{I}-1}, \tag{5.10}$$

with

$${}_{\iota}\xi_{T} \equiv \frac{1}{\mathfrak{I}}\sum_{k'=1}^{\mathfrak{I}}{}_{\iota}\xi_{T;k'} \quad (\mathfrak{I} = 1 \text{ or } 2). \tag{5.11}$$

The last term in the right-hand side of (2.18) thus takes the form:

$$-2\sum_{k'=1}^{\mathfrak{I}}\frac{{}_{\iota}\rho_{T}}{\xi - {}_{\iota}\xi_{T;k'}} \times \frac{{}_{\iota}\dot{K}_{2j+1}[\xi\,|\,\overline{\dagger_1 m_1;\ldots;\dagger_{2j+\ell} m_{2j+1}}]}{{}_{\iota}K_{2j+1}[\xi\,|\,\overline{\dagger_1 m_1;\ldots;\dagger_{2j+\ell} m_{2j+1}}]} \tag{5.12}$$

$$= \frac{{}_{\iota}\rho_{T}\,\mathfrak{I}(\xi - {}_{\iota}\xi_{T})^{\mathfrak{I}-1}\, o_{\,{}_{\iota}\upsilon_{\{\dagger m\}_{2j+1}}+1}[\xi\,|\,{}^{2j+1}_{\ \ \iota}\boldsymbol{G}^{K\mathfrak{I}}_{\{\dagger m\}_{2j+1}}]}{4\prod_{r=0}^{|\iota|}(\xi - {}_{\iota}e_r)\,\Pi_{\,{}_{\iota}\upsilon_{\{\dagger m\}_{2j+1}}}[\xi;\,{}_{\iota}\bar{\xi}_{\{\dagger m\}_{2j+1}}]\,\Pi_{\mathfrak{I}}[\xi;\,{}_{\iota}\bar{\xi}_{T}]},$$

where

$$o_{\,{}_{\iota}\upsilon_{\{\dagger m\}_{2j+1}}+1}[\xi\,|\,{}^{2j+1}_{\ \ \iota}\boldsymbol{G}^{K\mathfrak{I}}_{\{\dagger m\}_{2j+1}}] = \hat{o}_{\,{}_{\iota}\upsilon_{\{\dagger m\}_{2j+1}}+1}[\xi\,|\,{}^{2j+1}_{\ \ \iota}\boldsymbol{G}^{K\mathfrak{I}}_{\{\dagger m\}_{2j+1}}] \tag{5.13}$$

$$-8\prod_{r=0}^{|\iota|}(\xi - {}_{\iota}e_r)\,\dot{\Pi}_{\,{}_{\iota}\upsilon_{\{\dagger m\}_{2j+1}}}[\xi;\,{}_{\iota}\bar{\xi}_{\{\dagger m\}_{2j+1}}]$$

and

$$\hat{o}_{\,{}_{\iota}\upsilon_{\{\dagger m\}_{2j+1}}+1}[\xi\,|\,{}^{2j+1}_{\ \ \iota}\boldsymbol{G}^{K\mathfrak{I}}_{\{\dagger m\}_{2j+1}}] = -8\Big\{\sum_{k=1}^{2j+1}B_1[\xi;\bar{\iota};\,{}_{\iota}\bar{\rho}_{\dagger_k m_k}] - j\sum_{r=0}^{|\iota|}(\xi - {}_{\iota}e_r)^{|\iota|}\Big\} \tag{5.13*}$$

$$\times\,\Pi_{\,{}_{\iota}\upsilon_{\{\dagger m\}_{2j+1}}}[\xi;\,{}_{\iota}\bar{\xi}_{\{\dagger m\}_{2j+1}}].$$

After substituting (5.12) and (3.1) into (2.18) and making use of (3.4) the numerator of the last PFr in the right-hand side of (5.11) can be thus represented as

$$\Delta O^{\downarrow}_{\upsilon\{\dagger m\}_{2j+1}+\Im}[\xi\,|\,{}^{2j+1}_{\iota}\boldsymbol{G}^{K\Im}_{\{\dagger m\}_{2j+1}}] = 4\big({}_{\iota}\Theta_{\{\dagger m\}_{2j+1}} + 2\,|\,\iota\,|\,j\big)\Pi_{\Im}[\xi;\,{}_{\iota}\bar{\xi}_{T}]\,\Pi_{\iota\upsilon\{\dagger m\}_{2j+1}}[\xi;\,{}_{\iota}\bar{\xi}_{\{\dagger m\}_{2j+1}}]$$

$$+8\sum_{r=0}^{|\iota|}(\xi-{}_{\iota}e_{r})^{|\iota|}\Pi_{\Im}[\xi;\,{}_{\iota}\bar{\xi}_{T}]\,\dot{\Pi}_{\iota\upsilon\{\dagger m\}_{2j+1}}[\xi;\,{}_{\iota}\bar{\xi}_{\{\dagger m\}_{2j+1}}] \qquad (5.14)$$

$$+\,{}_{\iota}\rho_{T}\,\Im(\xi-{}_{\iota}\xi_{T})^{\Im-1}$$

$$\times\Big\{o_{\iota\upsilon\{\dagger m\}_{2j+1}+1}[\xi\,|\,{}^{2j+1}_{\iota}\boldsymbol{G}^{K\Im}_{\{\dagger m\}_{2j+1}}] + 8\sum_{r=0}^{|\iota|}(\xi-{}_{\iota}e_{r})^{|\iota|}\,\Pi_{\iota\upsilon\{\dagger m\}_{2j+1}}[\xi;\,{}_{\iota}\bar{\xi}_{\{\dagger m\}_{2j+1}}]\Big\}.$$

Note that the coefficient of the second-order pole $(\xi - {}_{\iota}\xi_{T;k'})^{-2}$ in PFr (5.15) matches that in the general expression for RefPFr $I^{o}[\xi\,|\,{}^{\ell}_{\iota}\mathbf{B}]$ given by (2.13) in Part I iff the PD remains finite at $\xi = {}_{\iota}\xi_{T;k'}$.

6. The gauge partial decomposition of multi-step GS RefPFrs

While the QPD provides a compact formula for the RLP the gauge partial decomposition (GPD) originally introduced by us in [72] for shape-invariant potentials (both on the line and half-line and then extended in Part I to the generic GRef potential on the line) is preferable as a starting point for the gauge transformations turning the given RCSLE into the Heine-type differential equations.

The RefPFrs in the GPD formally have the structure

$$I^{o}[\xi\,|\,{}^{2j}_{\iota}\boldsymbol{G}^{K\Im}_{\{\dagger m\}_{2j}}] = I^{o}[\xi;\,{}_{\iota}\bar{h}_{o},\,{}_{\iota}O^{o}_{0}] + 2\hat{Q}^{K\Im}[\xi;\,\bar{\xi}_{\{\dagger m\}_{2j}}] + \frac{\Delta\hat{O}^{\downarrow}_{\iota\upsilon\{\dagger m\}_{2j}}[\xi\,|\,{}^{2j}_{\iota}\boldsymbol{G}^{K\Im}_{\{\dagger m\}_{2j}}]}{4\prod_{r=0}^{|\iota|}(\xi-{}_{\iota}e_{r})\Pi_{\iota\upsilon\{\dagger m\}_{2j}}[\xi;\,{}_{\iota}\bar{\xi}_{\{\dagger m\}_{2j}}]} \qquad (6.1_{0})$$

and

$$I^{o}[\xi|\,{}^{2j+1}_{\iota}\boldsymbol{G}^{K\Im}_{\{\dagger m\}_{2j+1}}] = I^{o}[\xi;\,{}_{\iota}\bar{h}_{o},\,{}_{\iota}O^{o}_{0}] + 2\hat{Q}^{K\Im}[\xi;\bar{\xi}_{\{\dagger m\}_{2j+1}};{}_{\iota}\bar{\xi}_{T}] + \frac{\Delta\hat{O}^{\downarrow}_{\iota\upsilon\{\dagger m\}_{2j+1}} + \Im[\xi|\,{}^{2j+1}_{\iota}\boldsymbol{G}^{K\Im}_{\{\dagger m\}_{2j+1}}]}{4\prod_{r=0}^{|\iota|}(\xi-{}_{\iota}e_{r})\Pi_{\Im}[\xi;{}_{\iota}\bar{\xi}_{T}]\Pi_{\iota\upsilon\{\dagger m\}_{2j+1}}[\xi;{}_{\iota}\bar{\xi}_{\{\dagger m\}_{2j+1}}]}, \qquad \mathbf{(6.1_1)}$$

reminiscent to (5.1_0) and (5.1_1) accordingly, except that the PFr

$$\hat{Q}[\xi;{}_{\iota}\bar{\xi}_{\{\dagger m\}_p}] \equiv \frac{\ddot{\Pi}_{\iota\upsilon\{\dagger m\}p}[\xi;{}_{\iota}\bar{\xi}_{\{\dagger m\}_p}]}{2\Pi_{\iota\upsilon\{\dagger m\}p}[\xi;{}_{\iota}\bar{\xi}_{\{\dagger m\}_p}]} - \frac{\dot{\Pi}^{2}_{\iota\upsilon\{\dagger m\}p}[\xi;{}_{\iota}\bar{\xi}_{\{\dagger m\}_p}]}{\Pi^{2}_{\iota\upsilon\{\dagger m\}p}[\xi;{}_{\iota}\bar{\xi}_{\{\dagger m\}_p}]} \qquad (6.2_0)$$

in (6.1_0) has the twice smaller first term, compared with QPFr (2.19), whereas the PFr

$$\hat{Q}^{K\Im}[\xi;{}_{\iota}\bar{\xi}_{\{\dagger m\}_{2j+1}};{}_{\iota}\bar{\xi}_{T}] \equiv \hat{Q}[\xi;{}_{\iota}\bar{\xi}_{\{\dagger m\}_{2j+1}};1] + \hat{Q}^{K\Im}[\xi;{}_{\iota}\bar{\xi}_{T}] + \Delta\hat{Q}^{K\Im}[\xi;{}_{\iota}\bar{\xi}_{\{\dagger m\}_{2j+1}};{}_{\iota}\bar{\xi}_{T}], \qquad (6.2_1)$$

in (6.1_1) has the mixed term

$$\Delta\hat{Q}^{K\Im}[\xi;{}_{\iota}\bar{\xi}_{\{\dagger m\}_{2j+1}};{}_{\iota}\bar{\xi}_{T}] = -\frac{{}_{\iota}\rho_{T}\Im(\xi-{}_{\iota}\xi_{T})^{J-1}}{\Pi_{\Im}[\xi;{}_{\iota}\bar{\xi}_{T}]} \times \frac{\dot{\Pi}_{\iota\upsilon\{\dagger m\}_{2j+1}}[\xi;{}_{\iota}\bar{\xi}_{\{\dagger m\}_{2j+1}}]}{\Pi_{\iota\upsilon\{\dagger m\}_{2j+1}}[\xi;{}_{\iota}\bar{\xi}_{\{\dagger m\}_{2j+1}}]}. \qquad (6.3)$$

Decomposing the QPFr in (5.1_0) and (5.1_1) as

$$Q[\xi;{}_{\iota}\bar{\xi}_{\{\dagger m\}_p}] = \hat{Q}[\xi;{}_{\iota}\bar{\xi}_{\{\dagger m\}_p}] + \frac{\ddot{\Pi}_{\iota\upsilon\{\dagger m\}p}[\xi;{}_{\iota}\bar{\xi}_{\{\dagger m\}_p}]}{2\Pi_{\iota\upsilon\{\dagger m\}p}[\xi;{}_{\iota}\bar{\xi}_{\{\dagger m\}_p}]}, \qquad (6.4)$$

we come to the following polynomial formulas for the numerators of the fractions in the right-hand sides of (6.1_0) and (6.1_1):

$$\Delta\hat{O}^{\downarrow}_{\iota\upsilon\{\dagger m\}_{2j}}[\xi\,|\,{}^{2j}_{\iota}\boldsymbol{G}^{K\Im}_{\{\dagger m\}_{2j}}] = \Delta O^{\downarrow}_{\iota\upsilon\{\dagger m\}_{2j}}[\xi\,|\,{}^{2j}_{\iota}\boldsymbol{G}^{K\Im 0}_{\{\dagger m\}_{2j}}] \qquad (6.5_0)$$

$$+4\prod_{r=0}^{|\iota|}(\xi-{}_{\iota}e_{r})\ddot{\Pi}_{\iota\upsilon\{\dagger m\}_{2j}}[\xi;{}_{\iota}\bar{\xi}_{\{\dagger m\}_{2j}}]$$

and

$$\Delta\hat{O}^{\downarrow}_{\iota\upsilon\{\dagger m\}_{2j+1}}[\xi\,|\,{}^{2j+1}_{\iota}\boldsymbol{G}^{K\Im}_{\{\dagger m\}_{2j+1}}] = \Delta O^{\downarrow}_{\iota\upsilon\{\dagger m\}_{2j+1}}[\xi\,|\,{}^{2j+1}_{\iota}\boldsymbol{G}^{K\Im}_{\{\dagger m\}_{2j+1}}] \qquad (6.5_1)$$

$$+4\prod_{r=0}^{|\iota|}(\xi-{}_{\iota}e_{r})\ddot{\Pi}_{\iota\upsilon\{\dagger m\}_{2j}}[\xi;{}_{\iota}\bar{\xi}_{\{\dagger m\}_{2j}}]$$

$$+8\,{}_{\iota}\rho_{T}\Im(\xi-{}_{\iota}\xi_{T})^{J-1}\prod_{r=0}^{|\iota|}(\xi-{}_{\iota}e_{r})\dot{\Pi}_{\iota\upsilon\{\dagger m\}_{2j+1}}[\xi;{}_{\iota}\bar{\xi}_{\{\dagger m\}_{2j+1}}].$$

Introducing the auxiliary polynomial

$$\begin{aligned}{}_{\iota}\hat{\Xi}_{\iota\upsilon\{\dagger m\}_{p}}[\xi\,|\,\{\dagger m\}_{p}] \equiv {} & \prod_{r=0}^{|\iota|}(\xi-{}_{\iota}e_{r})\ddot{\Pi}_{\iota\upsilon\{\dagger m\}_{p}}[\xi;{}_{\iota}\bar{\xi}_{\{\dagger m\}_{p}}] \\ & +2\sum_{r=0}^{|\iota|}(\xi-{}_{\iota}e_{r})^{|\iota|}\dot{\Pi}_{\iota\upsilon\{\dagger m\}p}[\xi;{}_{\iota}\bar{\xi}_{\{\dagger m\}p}] \\ & +{}_{\iota}\Theta_{\{\dagger m\}_{p}}\Pi_{\iota\upsilon\{\dagger m\}_{p}}[\xi;{}_{\iota}\bar{\xi}_{\{\dagger m\}_{p}}]\end{aligned} \qquad (6.6)$$

and making use of (5.13) and (5.15) for p=2j+1 we can then represent (6.5_0) and (6.5_1)

$$\Delta\hat{O}^{\downarrow}_{\iota\upsilon\{\dagger m\}_{2j}}[\xi\,|\,{}^{2j}_{\iota}\boldsymbol{G}^{K\Im}_{\{\dagger m\}_{2j}}] = 4\,{}_{\iota}\hat{\Xi}_{\iota\upsilon\{\dagger m\}_{2j}}[\xi\,|\,\{\dagger m\}_{2j}] \qquad (6.7_0)$$

$$+2\,{}_{\iota}\rho_{T}\Im\,{}_{\iota}a_{K}\,(\xi-{}_{\iota}\xi_{T})^{K-1}\,{}_{\iota}\hat{P}_{\iota\tilde{\upsilon}\{\dagger m\}_{2j}}[\xi\,|\,\dagger_1 m_1;...;\dagger_{2j}m_{2j}]$$

and

$$\Delta\hat{O}^{\downarrow}_{\iota\upsilon\{\dagger m\}_{2j+1}+\Im}[\xi\,|\,{}^{2j+1}_{\iota}\boldsymbol{G}^{K\Im}_{\{\dagger m\}_{2j+1}}]$$

$$=4\Pi_{\Im}[\xi;{}_{\iota}\bar{\xi}_{T}]\{{}_{\iota}\hat{\Xi}_{\iota\upsilon\{\dagger m\}_{2j+1}}[\xi\,|\,\{\dagger m\}_{2j+1}]+(1-|\iota|)\,\Pi_{\iota\upsilon\{\dagger m\}_{2j+1}}[\xi;{}_{\iota}\bar{\xi}_{\{\dagger m\}_{2j+1}}]\}$$

$$+{}_{\iota}\rho_{T}\,\Im(\xi-{}_{\iota}\xi_{T})^{\Im-1}\hat{o}_{\iota\upsilon\{\dagger m\}_{2j+1}+1}[\xi\,|\,{}^{2j+1}_{\iota}\boldsymbol{G}^{K\Im}_{\{\dagger m\}_{2j+1}}] \qquad (6.7_1)$$

accordingly.

For the single-step (p=1) SUSY partners of the GRef potentials one can exclude the second derivative of the monomial product $\Pi_{m}[\xi;{}_{\iota}\bar{\xi}_{\dagger m}]$ from auxiliary polynomial ${}_{\iota}\hat{\Xi}_{m}[\xi\,|\,\dagger m]$ taking into account that the latter satisfies second-order differential equation (3.35) in [1]:

$$\prod_{r=0}^{|\iota|}(\xi-{}_{\iota}e_{r})\ddot{\Pi}_{m}[\xi;{}_{\iota}\bar{\xi}_{\dagger m}]+2B_{1}[\xi;\bar{\iota};{}_{\iota}\bar{\rho}_{\dagger m}]\dot{\Pi}_{m}[\xi;{}_{\iota}\bar{\xi}_{\dagger m}]$$
$$+C_{0}({}_{\iota}\varepsilon_{\dagger m}\,|\,{}_{\iota}\boldsymbol{G}_{\downarrow\dagger m};\bar{\sigma}_{\dagger})\Pi_{m}[\xi;{}_{\iota}\bar{\xi}_{\dagger m}]=0, \qquad (6.8)$$

where

$$C_{0}({}_{\iota}\varepsilon_{\dagger m}\,|\,{}_{\iota}\boldsymbol{G}_{\downarrow\dagger m};\bar{\sigma}_{\dagger})={}_{\iota}C^{0}_{0}({}_{\iota}\lambda_{0;\dagger m},{}_{\iota}\lambda_{1;\dagger m})+\tfrac{1}{4}({}_{\iota}O^{o}_{0}+{}_{\iota}d\,{}_{\iota}\varepsilon_{\dagger m}), \qquad (6.9)$$

and

$${}_{\iota}C^{0}_{0}({}_{\iota}\lambda_{0;\dagger m},{}_{\iota}\lambda_{1;\dagger m})=\tfrac{1}{2}|\iota|({}_{\iota}\lambda_{0;\dagger m}+1)({}_{\iota}\lambda_{1;\dagger m}+1)-\tfrac{1}{2}({}_{0}\lambda_{0;\dagger m}+1){}_{0}\lambda_{1;\dagger m}. \qquad (6.9^*)$$

This gives

$$ {}_{\iota}\hat{\Xi}_{m}[\xi\,|\,\dagger m] = 2\hat{B}_1[\xi;\bar{\iota};-\lambda_0,-\lambda_1]\dot{\Pi}_{m}[\xi;{}_{\iota}\bar{\xi}_{\dagger m}] $$
$$ -[{}_{\iota}C_0^0(-{}_{\iota}\lambda_{0;\dagger m},-{}_{\iota}\lambda_{1;\dagger m}) + {}^1\!/_4({}_{\iota}O_0^o + {}_{\iota}d\,{}_{\iota}\varepsilon_{\dagger m})]\Pi_m[\xi;{}_{\iota}\bar{\xi}_{\dagger m}], \tag{6.10} $$

where we also took advantage of the symmetry relations

$$ \hat{B}_1[\xi;\bar{\iota};\lambda_0,\lambda_1] + \hat{B}_1[\xi;\bar{\iota};-\lambda_0,-\lambda_1] = \sum_{r=0}^{|\iota|}(\xi - {}_{\iota}e_r)^{|\iota|} \tag{6.11†} $$

and

$$ {}_{\iota}C_0^0({}_{\iota}\lambda_{0;\dagger m},{}_{\iota}\lambda_{1;\dagger m}) - {}_{\iota}C_0^0(-{}_{\iota}\lambda_{0;\dagger m},-{}_{\iota}\lambda_{1;\dagger m}) = {}_{\iota}\Theta_{\dagger m} \tag{6.11*} $$

for first-order polynomials ($5.5^\dagger$) and functions (6.9*) accordingly, with the right-hand side of (6.11^*) defined via (3.1*). Substituting (6.10) together with the simplified expression

$$ \hat{o}_{m+1}[\xi\,|\,{}^{1}_{\iota}\boldsymbol{G}^{K\mathfrak{I}}_{\dagger m}] = -8\hat{B}_1[\xi;\bar{\iota};{}_{\iota}\bar{\lambda}_{\dagger m}]\Pi_m[\xi;{}_{\iota}\bar{\xi}_{\dagger m}] \tag{6.12} $$

for polynomial (5.13*) into (6.7_1), with j=0, we come to single-step formula (6.30a') in Part I for the polynomial

$$ \hat{O}^{\downarrow}_{{}_{\iota}\upsilon_{\{\dagger m\}_p}+\mathfrak{I}}[\xi\,|\,{}^{p}_{\iota}\boldsymbol{G}^{K\mathfrak{I}}_{\{\dagger m\}_p}] = {}_{\iota}O_0^o\,\Pi_{\mathfrak{I}}[\xi;{}_{\iota}\bar{\xi}_T]\Pi_{{}_{\iota}\upsilon_{\{\dagger m\}_p}}[\xi;{}_{\iota}\bar{\xi}_{\{\dagger m\}_p}] $$
$$ + \Delta\hat{O}^{\downarrow}_{{}_{\iota}\upsilon_{\{\dagger m\}_p}+\mathfrak{I}}[\xi\,|\,{}^{p}_{\iota}\boldsymbol{G}^{K\mathfrak{I}}_{\{\dagger m\}_p}] \tag{6.13} $$

in the GPD of the RefPFr

$$I^{o}[\xi\,|\,{}^{p}_{\iota}\boldsymbol{G}^{K\Im}_{\{\dagger m\}_p}] = -\frac{{}_{\iota}h_{o;0}}{4(\xi - {}_{\iota}e_0)^2} - |\iota|\frac{{}_{\iota}h_{o;1}}{4(\xi - {}_{\iota}e_1)^2} - \tfrac{1}{4}\delta_{\iota,0}\,\nu_o^2 + 2\hat{Q}[\xi;\,{}_{\iota}\bar{\xi}_{\{\dagger m\}_p},\,{}_{\iota}\bar{\xi}_T]$$

$$+\frac{\hat{O}^{\downarrow}_{\iota\upsilon\{\dagger m\}_p} + \Im[\xi\,|\,{}^{p}_{\iota}\boldsymbol{G}^{K\Im}_{\{\dagger m\}_p}]}{4\prod\limits_{r=0}^{|\iota|}(\xi - {}_{\iota}e_r)\,\Pi_{\Im}[\xi;\,{}_{\iota}\bar{\xi}_T]\,\Pi_{\iota\upsilon\{\dagger m\}_p}[\xi;\,{}_{\iota}\bar{\xi}_{\{\dagger m\}_p}]}, \quad (6.14)$$

with p = 1.

The main advantage of GPD (6.14) is that one can easily convert the original RCSLE to the Heine-type equations

$$\hat{\mathbb{D}}\{{}^{p}_{\iota}\boldsymbol{G}^{K\Im}_{\{\dagger m\}_p};\dagger' m'\}\mathrm{Hi}_{\iota\upsilon\{\dagger m\}_{p+1}}[\xi\,|\,{}^{p}_{\iota}\boldsymbol{G}^{K\Im}_{\{\dagger m\}_p};\dagger' m'] = 0 \quad (6.15)$$

using the appropriate gauge transformations [ ]. The latter can be directly obtained from (3.15) in Part I by setting $\varepsilon = {}_{\iota}\varepsilon_{\dagger' m'}$ which gives

$$\hat{\mathbb{D}}\{{}^{2j+\ell}_{\iota}\boldsymbol{G}^{K\Im}_{\{\dagger m\}_{2j+\ell}};\dagger' m'\} \equiv \hat{\mathbb{D}}\{{}_{\iota}\varepsilon_{\dagger' m'}\,|\,{}^{2j+\ell}_{\iota}\boldsymbol{G}^{K\Im}_{\{\dagger m\}_{2j+\ell}};\bar{\sigma}\} \quad (6.16^*)$$

$$= {}^{\ell}_{\iota}\hat{D}\{{}_{\iota}\bar{\rho}_{\dagger' m'};\,{}_{\iota}\bar{\xi}_{\{\dagger m\}_{2j+\ell}};\,{}_{\iota}\bar{\xi}_T\} \quad (6.16)$$

$$+ C_{\iota\upsilon\{\dagger m\}_{2j+\ell}} + \ell\Im[\xi;\,{}_{\iota}\varepsilon_{\dagger' m'}\,|\,{}^{2j+\ell}_{\iota}\boldsymbol{G}^{K\Im}_{\{\dagger m\}_{2j+\ell}};\bar{\sigma}_{\dagger'}],$$

where we set the energy-dependent second-order differential operator $\hat{\mathbb{D}}\{\varepsilon\,|\,{}^{\ell}_{\iota}\mathbf{B}^{K\Im};\bar{\sigma}\}$ in therightr hand of (6.16*) is defined as follows:

$$\hat{\mathbb{D}}\{\varepsilon\,|\,{}^{2j+\ell}_{\iota}\boldsymbol{G}^{K\Im}_{\{\dagger m\}_{2j+\ell}};\bar{\sigma}\} \equiv {}^{\ell}_{\iota}\hat{D}\{\bar{\rho}(\varepsilon\,|\,{}_{\iota}\boldsymbol{G}^{K\Im}_{\downarrow\{\dagger m\}_{2j+\ell}};\bar{\sigma});\,{}_{\iota}\bar{\xi}_{\{\dagger m\}_p};\,{}_{\iota}\bar{\xi}_T\}$$
$$+ C_{\iota\upsilon\{\dagger m\}_{2j+\ell}} + \ell\Im[\xi;\varepsilon\,|\,\boldsymbol{G}^{K\Im}_{\downarrow\{\dagger m\}_{2j+\ell}};\bar{\sigma}] \quad (6.17)$$

The second-order differential operator and the free term in the right-hand side of (6.16) are defined via relations (3.16) and (3.29) in [1], respectively. It directly follows from the form of AEH solutions that equation (6.15) has polynomial solutions

$$\mathrm{Hi}_{\iota\upsilon_{\{\dagger m\}_{p+1}}}[\xi \,|\, {}^{p}_{\iota}\boldsymbol{G}^{K\Im}_{\{\dagger m\}_{p}}; \dagger_{p+1} m_{p+1}] \equiv \Pi_{\iota\upsilon_{\{\dagger m\}_{p+1}}}[\xi; {}_{\iota}\bar{\xi}_{\{\dagger m\}_{p+1}}] \tag{6.18}$$

referred to by us as GS-Heine polynomials. By setting $\varepsilon = {}_{\iota}\varepsilon_{\dagger' m'}$ in the aforementioned relations one finds

$$\begin{aligned} &{}^{\ell}_{\iota}\hat{D}\{{}_{\iota}\bar{\rho}_{\dagger' m'}; {}_{\iota}\bar{\xi}_{\{\dagger m\}_{2j+\ell}}; {}_{\iota}\bar{\xi}_{T}\} \\ &\quad = \prod_{r=0}^{|\iota|}(\xi - {}_{\iota}e_{r})\,\Pi_{\iota\upsilon_{\{\dagger m\}_{2j+\ell+1}}}[\xi; {}_{\iota}\bar{\xi}_{\{\dagger m\}_{2j+\ell}}]\,\Pi^{\ell}_{\Im}[\xi; {}_{\iota}\bar{\xi}_{T}]\frac{d^{2}}{d\xi^{2}} \\ &\qquad + 2\,{}^{\ell}_{\iota}B_{\iota\upsilon_{\{\dagger m\}_{2j+\ell}}+\ell\Im+|\iota|}[\xi; {}_{\iota}\bar{\rho}_{\dagger' m'}; {}_{\iota}\bar{\xi}_{\{\dagger m\}_{p}}; {}_{\iota}\bar{\xi}_{T}]\frac{d}{d\xi} \end{aligned} \tag{6.19}$$

and

$$\begin{aligned} &C_{\iota\upsilon_{\{\dagger m\}_{2j+\ell}}+\ell\Im}[\xi; {}_{\iota}\varepsilon_{\dagger' m'} \,|\, {}^{2j+\ell}_{\iota}\boldsymbol{G}^{K\Im}_{\{\dagger m\}_{2j+\ell}}; \bar{\sigma}_{\dagger'}] = \tfrac{1}{4}\hat{O}^{\downarrow}_{\iota\upsilon_{\{\dagger m\}_{2j+\ell}}}[\xi \,|\, {}^{2j+\ell}_{\iota}\boldsymbol{G}^{K\Im}_{\{\dagger m\}_{2j+\ell}}] \\ &\qquad + \{\tfrac{1}{4}\,{}_{\iota}d\,{}_{\iota}\varepsilon_{\dagger' m'} + 2\,{}_{\iota}\rho_{0;\dagger' m'}[|\iota|\,{}_{\iota}\rho_{1;\dagger' m'} - \tfrac{1}{2}\delta_{\iota,0}\,\nu_{\dagger' m'}]\} \\ &\qquad\qquad \times \Pi_{\iota\upsilon_{\{\dagger m\}_{2j+\ell+1}}}[\xi; {}_{\iota}\bar{\xi}_{\{\dagger m\}_{2j+\ell}}]\,\Pi^{\ell}_{\Im}[\xi; {}_{\iota}\bar{\xi}_{T}] \\ &\qquad - 2\,B_{1}[\xi; \bar{\tau}; {}_{\iota}\bar{\rho}_{\dagger' m'}]\,\dot{\Pi}_{\iota\upsilon_{\{\dagger m\}_{2j+\ell+1}}}[\xi; {}_{\iota}\bar{\xi}_{\{\dagger m\}_{2j+\ell}}]\,\Pi^{\ell}_{\Im}[\xi; {}_{\iota}\bar{\xi}_{T}] \\ &\qquad - 2\ell\,{}_{\iota}\rho_{T}\,B_{1}[\xi; \bar{\tau}; {}_{\iota}\bar{\rho}_{\dagger' m'}]\,\Pi_{\iota\upsilon_{\{\dagger m\}_{2j+\ell+1}}}[\xi; {}_{\iota}\bar{\xi}_{\{\dagger m\}_{2j+\ell}}]\,\dot{\Pi}^{\ell}_{\Im}[\xi; {}_{\iota}\bar{\xi}_{T}]\}, \end{aligned} \tag{6.20}$$

where the first-order polynomial $B_1[\xi;\bar{\iota};{}_\iota\bar{\rho}]$ is defined via (5.5*) in previous section. The polynomial coefficient of the first derivative in the right-hand side of (6.18) can be thus represented as

$$
\begin{aligned}
&{}^{\ell}_{\iota}B_{\iota\upsilon_{\{\dagger m\}_p}+\ell\Im+|\iota|}[\xi;{}_\iota\bar{\rho}_{\dagger'm'};{}_\iota\bar{\xi}_{\{\dagger m\}_p};{}_\iota\bar{\xi}_T] \\
&\qquad = \Pi_{\iota\upsilon_{\{\dagger m\}_p}}[\xi;{}_\iota\bar{\xi}_{\{\dagger m\}_p}]\Pi^{\ell}_{\Im}[\xi;{}_\iota\bar{\xi}_T]B_1[\xi;\bar{\iota};{}_\iota\bar{\rho}_{\dagger'm'}] \\
&\qquad - \Pi_{\iota\upsilon_{\{\dagger m\}_p}}[\xi;{}_\iota\bar{\xi}_{\{\dagger m\}_p}]\Pi^{\ell}_{\Im}[\xi;{}_\iota\bar{\xi}_T]\prod_{r=0}^{|\iota|}(\xi-{}_\iota e_r) \\
&\qquad\qquad \times\left(\sum_{k=1}^{\iota\upsilon_{\{\dagger m\}_p}}\frac{1}{\xi-{}_\iota\xi_{\{\dagger m\}_p;k}}+\frac{1}{\Im}\sum_{k'=1}^{\ell\Im}\frac{1}{\xi-{}_\iota\xi_{T;k'}}\right),
\end{aligned}
\tag{6.21}
$$

where

$$
{}_\iota\xi_T \equiv \frac{1}{\Im}\sum_{k'=1}^{\Im}{}_\iota\xi_{T;k'}. \tag{6.22}
$$

As outlined in Part I, we can then make use of the appropriate gauge transformation to convert the given RCSLE to the second-order differential equation solved by GS Heine polynomials which is the main result of this paper.

8. Conclusions and further developments

The very specific common feature of the RCSLEs associated with multi-step rational SUSY partners of the *r*-GRef potential on the line is that they have only regular singularities, including infinity. In the generic case of the second-order tangent polynomial [1] discussed here the exponent differences (ExpDiffs) for singularities at the ends of the quantization interval as well as the ExpDiff for infinity is unaffected by CLDTs. This implies that polynomial determinants formed by Jacobi-seed ($\mathscr{J}$S) solutions [40] satisfy the Fuschian equation and thereby are referred to us as $\mathscr{J}$S Heine polynomials.

On other hand, the CLDTs of the linear-TP(LTP) *r*-GRef potential analyzed in Part III do change the ExpDiff at infinity. It turns out that the PD constructed in the aforementioned way has as a rule zero of order k $\geq$ 1 at the origin. To obtain the appropriate 𝒥S Heine polynomials one thus need to divide the PD by z^k. We moved a study on SUSY ladders of the LTP potentials $V[\xi\,|\,{}_{\iota}\boldsymbol{G}^{1\Im}]$ into Part III because they have some specific features which do not fit the general pattern for the generic *r*- and *c*-GRef potentials $V[\xi\,|\,{}_{\iota}\boldsymbol{G}^{2\Im}]$ generated using second-order TPs. We also postponed any study on its shape-invariant limiting cases (already addressed in the literature [73, 74, 44-47] in this context.

An analysis of the LTP *r*- and c-GRef potentials is significantly simplified by the fact that energies of AEH solutions are determined by roots of quadratic (instead of quartic) equations so that we can directly formulate constraints selecting regular AEH solutions below the ground-energy level. When the TPs turn into constants the derived constraints become equivalent to the parameter ranges obtained by Quesne [44, 45] for the RM (ι=1) and Morse (ι=0) potentials. This would present a convenient opportunity to more precisely relate Quesne's works to our general approach. Without going into details let us only mention two other astounding attributes of the LTP *r*- and c-GRef potentials:

i) their single-step SUSY partners constructed by means of CLDTs with one of four basic FFs exactly quantized in terms of Heun or *c*-Heun polynomials for ι=1 or 0, respectively;

ii) both *r*- and c-GRef potentials preserve their form under some double-step CLDTs with basic GS solutions.

In Part IV we will present a more thorough analysis of the double-step SUSY partners of GRef potentials. A special attention will be given to CLDTs using the basic 𝒥S and ℒS solutions as seed functions keeping in mind that the resultant RLPs are conditionally exactly quantized by Heun and *c*-Heun polynomials, respectively.

The only exception from this rule is the Gendenshtein (Scarf II) potential [3] which is constructed using the TP with zeros at the singular points $-i$ and $+i$ of the given RCSLE. The remarkable feature of this exceptional family of rational potentials on the line is that ChExps at the singular points $-i$ and $+i$ of the RCSLE are energy-independent and as result each of the mentioned SUSY partners is quantized by a finite set of orthogonal polynomials.

As pointed to in [5] the symmetric curves [3, 14, 4] form the intersection between *r*- and *i*-GRef potentials. As a result it can be alternatively quantized via both ultraspherical [75] and Masjedjamei [76] (symmetric Romanovski-Routh) polynomials. The most important consequence from this observation is that multi-step RCSLEs constructed using irregular Gegenbauer-seed ($\mathfrak{G}$S) or alternatively symmetric Routh-seed (sym-$\mathfrak{R}$S) solutions [5] allow the dual quantization scheme via both $\mathfrak{G}$S and sym-$\mathfrak{R}$S Heine polynomials. The main advantage of sym-$\mathfrak{R}$S Heine polynomials is that they form orthogonal sets.

In particular this implies that the symmetric Rosen-Morse (sym-RM) potential – the "soliton" potential in terms of [46, 47] – also. Quantization of the Schrödinger equation with multi-step symmetric 'algebraically-deformed' [77, 78] soliton potentials via Gegenbauer-seed ($\mathfrak{G}$S) Heine polynomials (in our classification scheme) was discussed in detail in [79-83]. The appropriate finite orthogonal sets of sym-$\mathfrak{R}$S Heine polynomials will be analyzed in detail in [84].

As mentioned above the benchmark feature of the GRef potential on the line is that line the density function of the appropriate RCSLE has the second-order pole at the origin and as a result the CLDTs of our interest do not change the ExpDiff at this singular point. On the contrary, for the Liouville transformation to convert the given RCSLE to the Schrödinger equation on the half-line the density function must have the first-order pole at the origin (if any). As a result, the Darboux transformations do change the exponent differences (ExpDiffs) for the zero singular point, contrary to the ladders formed rational SUSY partners of GRef potentials on the line. The direct consequence of this change is that the polynomial determinants used to define the Heine polynomials in question generally vanish at the above singularity so one first needs to determine the order of this zero root bearing in mind that each Heine polynomial must remain finite at each singular point by definition. The explicit expression for the order of the zero root in terms of the number of regular-at-origin GS solutions used to construct the given rational SUSY partner of the radial GRef potential will be given in [85].

Appendix A

SUSY ladders of canonical Sturm-Liouville equations

The purpose of this appendix is to study transformation properties of the canonical Sturm-Liouville equation (CSLE)

$$\left\{ \frac{d^2}{d\xi^2} + I^o[\xi; Q^o] + \varepsilon \wp[\xi] \right\} \Phi[\xi; \varepsilon; Q^o] = 0 \tag{A.1}$$

under multi-step Darboux transformations (DTs) of the corresponding Liouville potential expressed in terms of the variable ξ. For the reasons explained in Introduction the operators generated by these transformations in the space of the energy-dependent solutions $\Phi[\xi; \varepsilon; Q^o]$ are referred to us as CLDTs. In principle the derivation presented below can be done with no relation to the Schrödinger equation obtained from (A.1) via the appropriate Liouville transformation, as it has been independently done in [56] for the multi-step GDTs. However applying Krein's conventional formalism [52, 49-51, 54] to the resultant Schrödinger equation (instead of deriving all the results from scratch) allow us significantly simplify the arguments.

Let $\psi_{\tau_{k=1,\dots,p}}[\xi(x); Q^o_{\downarrow\{\tau\}_p}]$ be seed solutions of the Schrödinger equation with the Liouville potential $V[\xi(x); Q^o]$ associated with CSLE (A.1). (Here and below subscript $\downarrow\{\tau\}_p$ indicates that the potential parameters are restricted to the region where all the p seed solutions co-exist with each other.) The Crum Wroskian $W\{\psi_{\{\tau\}p}[\xi(x)]\}$ formed by these solutions can be converted into the Krein determinant

$$K_\psi[\xi; \{\tau\}_p] \equiv K\{\psi_{\{\tau\}p}[\xi]; \varepsilon_{\{\tau\}p}\} \tag{A.2}$$

formed by seed functions $\psi_{\tau_k}[\xi]$ of ξ using the easily verified relation

$$W\{\psi_{\tau_{k=1,\ldots,2j+\ell}}[\xi(x)]\} = \wp^{-\frac{1}{2}j}[\xi(x)]K_{\psi}[\xi(x);\{\tau\}_{2j+\ell}\}, \tag{A.3}$$

where we have to distinguish between even ($\ell = 0$) and odd ($\ell = 1$) numbers of steps, namely,

$$K\{\psi_{\tau_{k=1,\ldots,2j}}[\xi];\varepsilon_{\{\tau\}_{2j}}\} = \begin{vmatrix} \psi_{\tau_1}[\xi] & \ldots & \psi_{\tau_{2j}}[\xi] \\ \dot{\psi}_{\tau_1}[\xi] & \ldots & \dot{\psi}_{\tau_{2j}}[\xi] \\ \varepsilon_{\tau_1}\psi_{\tau_1}[\xi] & \ldots & \varepsilon_{\tau_{2j}}\psi_{\tau_{2j}}[\xi] \\ & & \\ \varepsilon_{\tau_1}^{j-1}\dot{\psi}_{\tau_1}[\xi] & \ldots & \varepsilon_{\tau_{2j}}^{j-1}\dot{\psi}_{\tau_{2j}}[\xi] \end{vmatrix} \tag{A.4$_0$}$$

and

$$K_{\xi}\{\psi_{\tau_{k=1,\ldots,2j+1}}[\xi];\varepsilon_{\{\tau\}_{2j+1}}\} = \begin{vmatrix} \psi_{\tau_1}[\xi] & \ldots & \psi_{\tau_{2j+1}}[\xi] \\ \dot{\psi}_{\tau_1}[\xi] & \ldots & \dot{\psi}_{\tau_{2j+1}}[\xi] \\ & \ldots & \\ \varepsilon_{\tau_1}^{j-1}\dot{\psi}_{\tau_1}[\xi] & \ldots & \varepsilon_{\tau_{2j}}^{j-1}\dot{\psi}_{\tau_{2j}}[\xi] \\ \varepsilon_{\tau_1}^{j}\psi_{\tau_1}[\xi] & \ldots & \varepsilon_{\tau_{2j+1}}^{j}\psi_{\tau_{2j+1}}[\xi] \end{vmatrix}, \tag{A.4$_1$}$$

with dot standing for the derivative with respect to ξ we can represent the Liouville potential obtained from $V[\xi(x);Q^{o}]$ via the p-step DT with the seed solutions $\psi_{\tau_{k=1,\ldots,p}}[\xi(x);Q^{o}_{\downarrow\{\tau\}_p}]$ as follows

$$V_{\{\tau\}_{2j+\ell}}[\xi;Q^{o}_{\downarrow\{\tau\}_{2j+\ell}}] = V[\xi;Q^{o}_{\downarrow\{\tau\}_{2j+\ell}}] + 2j\Delta V\{\wp[\xi]\} \tag{A.5}$$

$$-2\wp^{-\frac{1}{2}}[\xi]\frac{d}{d\xi}\left\{\wp^{-\frac{1}{2}}[\xi]\, ld\,|K_{\xi}\{\psi_{\tau_{k=1,\ldots,2j+\ell}}[\xi];\varepsilon_{\{\tau\}_{2j+\ell}}|\right\}$$

where

$$\Delta V\{\wp[\xi]\} \equiv -\tfrac{1}{2}\wp^{-\frac{1}{2}}[\xi]\frac{d}{d\xi}\left(\wp^{-\frac{1}{2}}[\xi]\ ld\,\wp[\xi]\right) \tag{A.6}$$

$$= {}^1\!/\!_4\,\wp^{-3}[\xi]\{3\dot{\wp}^2[\xi] - 2\wp[\xi]\ddot{\wp}[\xi]\}. \tag{A.6′}$$

Our next step is to express Liouville potential (A.5) in terms of the solutions

$$\phi_{\tau_k}[\xi;Q^o_{\downarrow\{\tau\}_p}] = \wp^{-1/4}[\xi]\,\psi_{\tau_k}[\xi;Q^o_{\downarrow\{\tau\}_p}] \tag{A.7}$$

of CSLE (A.1). Keeping in mind that the Krein discriminant transforms under multiplication of each seed solution by the same function in the same way as the Crum Wroskian:

$$K\{f[\xi]\psi_{\tau_{k=1,\dots,p}}[\xi];\varepsilon_{\{\tau\}_p}\} = f^p[\xi]K\{\psi_{\tau_{k=1,\dots,2j+\ell}}[\xi];\varepsilon_{\{\tau\}_p}\} \tag{A.8}$$

one finds

$$K\{\psi_{\{\tau\}_p}[\xi];\varepsilon_{\{\tau\}_p}\} = \xi^{1/4p}\,K\{\phi_{\{\tau\}_p}[\xi];\varepsilon_{\{\tau\}_p}\}, \tag{A.9}$$

where

$$K\{\phi_{\{\tau\}_{2j}};\varepsilon_{\{\tau\}_{2j}}\} = \begin{vmatrix} \phi_{\tau_1}[\xi;Q^o_{\downarrow\{\tau\}_{2j}}] & \dots & \phi_{\tau_{2j}}[\xi;Q^o_{\downarrow\{\tau\}_{2j}}] \\ \dot{\phi}_{\tau_1}[\xi;Q^o_{\downarrow\{\tau\}_{2j}}] & \dots & \dot{\phi}_{\tau_{2j}}[\xi;Q^o_{\downarrow\{\tau\}_{2j}}] \\ \varepsilon_{\tau_1}\phi_{\tau_1}[\xi;Q^o_{\downarrow\{\tau\}_{2j}}] & \dots & \varepsilon_{\tau_{2j}}\phi_{\tau_{2j}}[\xi;Q^o_{\downarrow\{\tau\}_{2j}}] \\ & \dots & \\ \varepsilon^{j-1}_{\tau_1}\phi_{\tau_1}[\xi;Q^o_{\downarrow\{\tau\}_{2j}}] & \dots & \varepsilon^{j-1}_{\tau_{2j}}\phi_{\tau_{2j}}[\xi;Q^o_{\downarrow\{\tau\}_{2j}}] \\ \varepsilon^{j-1}_{\tau_1}\dot{\phi}_{\tau_1}[\xi;Q^o_{\downarrow\{\tau\}_{2j}}] & \dots & \varepsilon^{j-1}_{\tau_{2j}}\dot{\phi}_{\tau_{2j}}[\xi;Q^o_{\downarrow\{\tau\}_{2j}}] \end{vmatrix} \tag{A.10$_0$}$$

and

$$K\left\{\phi_{\{\tau\}_{2j+1}}[\xi;Q^{o}_{\downarrow\{\tau\}_{2j+1}}];\varepsilon_{\{\tau\}_{2j+1}}\right\} \tag{A.10$_1$}$$

$$= \begin{vmatrix} \phi_{\tau_1}[\xi;Q^{o}_{\downarrow\{\tau\}_{2j+1}}] & \ldots & \phi_{\tau_{2j+1}}[\xi;Q^{o}_{\downarrow\{\tau\}_{2j+1}}] \\ \dot{\phi}_{\tau_1}[\xi;Q^{o}_{\downarrow\{\tau\}_{2j+1}}] & \ldots & \dot{\phi}_{\tau_{2j+1}}[\xi;Q^{o}_{\downarrow\{\tau\}_{2j+1}}] \\ \varepsilon_{\tau_1}\phi_{\tau_1}[\xi;Q^{o}_{\downarrow\{\tau\}_{2j+1}}] & \ldots & \varepsilon_{2j+1}\phi_{2j+1}[\xi;Q^{o}_{\downarrow\{\tau\}_{2j+1}}] \\ & \ldots & \\ \varepsilon^{j-2}_{\tau_1}\dot{\phi}_{\tau_1}[\xi;Q^{o}_{\downarrow\{\tau\}_{2j+1}}] & \ldots & \varepsilon^{j-2}_{\tau_{2j+1}}\dot{\phi}_{2j+1}[\xi;Q^{o}_{\downarrow\{\tau\}_{2j+1}}] \\ \varepsilon^{j-1}_{\tau_1}\phi_{\tau_1}[\xi;Q^{o}_{\downarrow\{\tau\}_{2j+1}}] & \ldots & \varepsilon^{j-1}_{\tau_{2j+1}}\phi_{\tau_{2j+1}}[\xi;Q^{o}_{\downarrow\{\tau\}_{2j+1}}] \end{vmatrix}.$$

In particular, combining (A.3) and (A.9) we come to the following general relation

$$W\left\{\psi_{\tau_{k=1,\ldots,2j+\ell}}[\xi(x)]\right\} = \wp^{1/4\,\ell}[\xi(x)]K_{\phi}[\xi(x);\{\tau\}_{2j+\ell}\}, \tag{A.11}$$

between the Krein determinant

$$K_{\phi}[\xi;\{\tau\}_{p}] \equiv K\left\{\phi_{\{\tau\}_{p}}[\xi];\varepsilon_{\{\tau\}_{p}}\right\} \tag{A.12}$$

formed by seed solutions of an arbitrary RCSLE and the Crum Wroskian formed by seed solutions of the corresponding Schrödinger equation after the former function is converted from ξ to x.

Substituting (A.9) into the right-hand side of (A.5) thus gives

$$V_{\{\tau\}_{2j+\ell}}[\xi;Q^{o}_{\downarrow\{\tau\}_{2j+\ell}}] = V[\xi;Q^{o}_{\downarrow\{\tau\}_{2j+\ell}}] - \ell\Delta V\{\wp[\xi]\} \tag{A.13}$$

$$-2\wp^{-1/2}[\xi]\frac{d}{d\xi}\left\{\wp^{-1/2}[\xi]\,ld\,|K_{\phi}[\xi;\{\tau\}_{p}]|\right\}.$$

It seems useful to present an alternative representation for Liouville potential (A.14), in following Quesne's prescription [67] originally implemented in her pioneering study for multi-step rational SUSY partners of the isotonic oscillator. Namely, to find an explicit expression for the 'algebraically deformed' [77, 78] isotonic oscillator in terms of generalized Laguerre polynomials and their derivatives she converted Wroskian (A.11) from x to $\xi(x)$ via the conventional formula [86]

$$W\{\psi_{\{\tau\}_p}[\xi(x)]\} = \wp^{-1/4\,p(p-1)}[\xi(x)]\,W_\psi[\xi(x);\{\tau\}_{2j+\ell}], \tag{A.14}$$

where we set

$$W_\psi[\xi\,|\,\{\tau\}_p] \equiv W\{\psi_{\{\tau\}_p}[\xi]\} \tag{A.15}$$

and also took into account that the derivative of $\xi(x)$ with respect to x is related to the density function $\wp[\xi]$ via the conventional formula

$$\xi'(x) = \wp^{-1/2}[\xi(x)]\ . \tag{A.16}$$

This implies that Liouville potential (A.14) can be alternatively represented as

$$V_{\{\tau\}_{2j+\ell}}[\xi;Q^{o}_{\downarrow\{\tau\}_{2j+\ell}}] = V[\xi;Q^{o}_{\downarrow\{\tau\}_{2j+\ell}}] + p(p-1)\Delta V\{\wp[\xi]\}$$
$$-2\wp^{-1/2}[\xi]\frac{d}{d\xi}\{\wp^{-1/2}[\xi]\,ld\,|W_\psi[\xi\,|\,\{\tau\}_p]|\}. \tag{A.17}$$

By converting Wroskian (A.15) to the Wroskian $W_\phi[\xi\,|\,\{\tau\}_p]$ formed by seed solutions $\phi_{\tau_k}[\xi;Q^{o}_{\downarrow\{\tau\}_p}]$ of CSLE (A.1):

$$W\{\phi_{\{\tau\}_p}[\xi]\} = \wp^{-1/4\,p}[\xi]\,W_\psi[\xi;\{\tau\}_{2j+\ell}] \tag{A.18}$$

and bearing in mind that

$$W\{\phi_{\{\tau\}_{2j+\ell}}[\xi;Q^{o}]\} = \wp^{j(j-1+\ell)}[\xi]K\{\phi_{\{\tau\}p}[\xi;Q^{o}];\varepsilon_{\{\tau\}p}\}, \tag{A.19}$$

or, which is equivalent,

$$j(j-1+\ell) = \tfrac{1}{4}p(p-2) + \tfrac{1}{4}\ell, \tag{A.20}$$

we come back to (A.11) which confirms that representations (A.13) and (A.17) are equivalent.

Let us now remind the reader that our final goal is to find the zero-energy free term $I^{o}_{\{\tau\}_p}[\xi;Q^{o}_{\downarrow\{\tau\}_p}]$ in the CSLE

$$\left\{\frac{d^2}{d\xi^2} + I^{o}_{\{\tau\}_p}[\xi;Q^{o}_{\downarrow\{\tau\}_p}] + \varepsilon\wp[\xi]\right\}\Phi_{\{\tau\}_p}[\xi;\varepsilon;Q^{o}_{\downarrow\{\tau\}_{p+1}}\,|\,\tau_{p+1}] = 0 \tag{A.21}$$

obtained from (A.1) via the p-step CLDT in question. By excluding the Schwarzian derivative from the standard relation between this function and the corresponding Liouville potential:

$$V[\xi;Q^{o}_{\downarrow\{\tau\}_{2j+\ell}}] = -\wp^{-1}[\xi]I^{o}_{\{\tau\}_p}[\xi;Q^{o}_{\downarrow\{\tau\}_p}] - \tfrac{1}{2}\{\xi,x\} \tag{A.22}$$

one finds

$$I^{o}_{\{\tau\}_p}[\xi;Q^{o}_{\downarrow\{\tau\}_p}] - I^{o}[\xi;Q^{o}_{\downarrow\{\tau\}_p}] = \wp[\xi]\{V[\xi;Q^{o}_{\downarrow\{\tau\}_{2j+\ell}}] - V_{\{\tau\}_{2j+\ell}}[\xi;Q^{o}_{\downarrow\{\tau\}_{2j+\ell}}]\}. \tag{A.23}$$

Substituting (A.13) into (A.23) brings us to the expression sought for:

$$\begin{aligned} I^{o}[\xi;Q^{o}_{\downarrow\{\tau\}_{2j+\ell}}\,|\,\{\tau\}_{2j+\ell}] &= I^{o}[\xi;Q^{o}_{\downarrow\{\tau\}_{2j+\ell}}] + \ell\,\Delta I^{o}\{\wp[\xi]\} \\ &\quad + 2\wp^{1/2}[\xi]\frac{d}{d\xi}\left\{\wp^{-1/2}[\xi]\,\mathit{ld}\,|\,K_{\phi}[\xi;\{\tau\}_p]\,|\right\}, \end{aligned} \tag{A.24}$$

where

$$\Delta I^{o}\{\wp[\xi]\} \equiv \wp^{1/2}[\xi]\frac{d}{d\xi}\left(\wp^{-1/2}[\xi]\;\; ld\,\wp[\xi]\right). \qquad (A.25)$$

Representing Liouville potential (A.17)

$$V_{\{\tau\}_{2j+\ell}}[\xi;Q^{o}_{\downarrow\{\tau\}_{2j+\ell}}] = V[\xi;Q^{o}_{\downarrow\{\tau\}_{2j+\ell}}] + p(p-2)\Delta V\{\wp[\xi]\} \qquad (A.26)$$

$$-2\wp^{-1/2}[\xi]\frac{d}{d\xi}\left\{\wp^{-1/2}[\xi]\; ld\,|\,W_{\phi}[\xi\,|\,\{\tau\}_p]\,|\right\}.$$

one can re-write zero-energy free term (A.24) as

$$I^{o}[\xi;Q^{o}_{\downarrow\{\tau\}_{2j+\ell}}\,|\,\{\tau\}_{2j+\ell}] = I^{o}[\xi;Q^{o}_{\downarrow\{\tau\}_{2j+\ell}}] - p(p-2)\,\Delta I^{o}\{\wp[\xi]\} \qquad (A.27)$$

$$+2\wp^{1/2}[\xi]\frac{d}{d\xi}\left\{\wp^{-1/2}[\xi]\; ld\,|\,W_{\phi}[\xi\,|\,\{\tau\}_p]\,|\right\}.$$

One can verify that Sturm-Liouville equation (1) in Schulze-Halberg's paper [56] turns into CSLE (A.21) with zero-energy free term (A.27) if we put g = 0, f = $\wp^{-1}[\xi]$, and then make trivial substitutions

$$V_1 = -\wp^{-1}[\xi]I^{o}[\xi;Q^{o}_{\downarrow\{\tau\}_p}],\; V_2 = -\wp^{-1}[\xi]I^{o}[\xi;Q^{o}_{\downarrow\{\tau\}_p}\,|\,\{\tau\}_p] \qquad (A.28)$$

and

$$\Delta I^{o}\{f^{-1}\} = -f^{-1}\ddot{f} + {}^{1}\!/_{2}\, ld\, f$$

in his formula (5) for the potential V_2.

In the particular case p =1 (j = 0, ℓ =1) one finds

$$I^{o}[\xi;Q^{o}_{\downarrow\tau_1};\wp[\xi]\,|\,\tau_1] = I^{o}[\xi;Q^{o}_{\downarrow\tau_1}] + \Delta I^{o}\{\wp[\xi]\} + 2\wp^{1/2}[\xi]\frac{d}{d\xi}\left\{\wp^{-1/2}[\xi]\; l\dot{d}\,|\,\phi_{\tau_1}[\xi;Q^{o}_{\downarrow\tau_1}]\,|\right\}$$

(A.29)

in agreement with (B.20) in [1] or (7) in [56].

Since any solution of RCSLE (A.21) is related to the appropriate solution of the Schrödinger equation (converted to the variable ξ) via the conventional formula

$$\Phi_{\{\tau\}_p}[\xi;\varepsilon;Q^{o}_{\downarrow\{\tau\}_{p+1}}\,|\,\tau_{p+1}] = \wp^{-1/4}[\xi]\Psi_{\{\tau\}_p}[\xi;\varepsilon;Q^{o}_{\downarrow\{\tau\}_p}\,|\,\tau_{p+1}] \tag{A.30}$$

it can be represented as

$$\Phi_{\{\tau\}_p}[\xi;\varepsilon;Q^{o}_{\downarrow\{\tau\}_{2j+\ell+1}}\,|\,\tau_{2j+\ell+1}] = \frac{\wp^{-1/2\ell}[\xi]K_{\phi}[\xi;\{\tau\}_{2j+\ell+1}]\}}{K_{\phi}[\xi;\{\tau\}_{2j+\ell}]}. \tag{A.31}$$

Alternatively, making use of (A.19) and (A.20), we can write the latter formula as

$$\Phi_{\{\tau\}_p}[\xi;\varepsilon;Q^{o}_{\downarrow\{\tau\}_{p+1}}\,|\,\tau_{2p+1}] = \frac{\wp^{1/4p}[\xi]W_{\phi}[\xi\,|\,\{\tau\}_{p+1}]}{W_{\phi}[\xi\,|\,\{\tau\}_p]}, \tag{A.32}$$

in agreement with (3) in [56].

It is crucial that both expressions (A.27) and (A.31) were derived in [56] with no reference to the Schrödinger equation with the associated Liouville potential. One can thus start from these expressions and then express both zero-energy free term and solutions of CSLE (A.21) in terms of Krein determinants using (A.19). By deriving (A.24) and (A.31) in such a way we would stay within the framework of the Sturm-Liouville theory, with no need in the Liouville transformation to the Schrödinger equation.

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