Global Relativity

Ahmida Bendjoudi
Laboratoire de Physique Mathématique et Subatomique,
Frères Mentouri University, Constantine, Algeria
8ahmida8@gmail.com

A new independent derivation of general theory of relativity using only special relativity principles is shown. Some solutions are derived and discussed.

INTRODUCTION

In general relativity (GR), the curvature of spacetime is directly related to the stress-energy tensor. This provides a rigorous geometric description for the gravitational phenomena. Yet, the theory is geometrically very complicated. Different attempts for alternatives have been made [1–3], but the consistency with special relativity (SR) is confused.

In the present paper [4], we provide an independent derivation of the theory, using only SR principles: we find that the first SR postulate naturally corresponds to the conservation of geometric momenta, and the second to the constancy of the speed of light globally.

The result of the approach is that a new equations for the gravitation theory are discovered, which naturally lead to he usual metrics found through GR.

In fact, we do not find Einstein equations, but new formula (relating the spacetime interval with the stress-energy tensor), which self-sufficiently reproduces the geodesics found through GR: the Schwarzschild and Reissner-Nordstrom metrics. The results simplify the theory of gravitation very much, since the curvature in use is that of lines (not surfaces). Moreover, new insights into the quantitative description of quantum gravity are provided: the approach we present here fundamentally based on the notion of local coordinates, which can lead to new key-concepts about gravitational waves.

RIEMANN GEOMETRY

Riemann manifold is the global space on which Einstein equations solutions are represented. Each point, say p, in it corresponds to the center, say $O_l(p)$, of a local frame. Each local frame has its own local basis, with respect to the gravity center. For a global observer at the gravity center, say O, this basis is the coordinates basis. The location in the global space is defined by curved coordinates, say $\{x^\mu\}$, whereas in the local frames the flat coordinates are used instead, say $\{X^\mu\}$. At a point M, the coordinates basis can be defined using the partial derivative of the global position with respect to the curved coordinates as: $e_\mu = \partial_\mu \mathbf{OM}$. Of course, this basis is tangential to the lines of curved coordinates.

Einstein realized that local frames correspond to the case of SR, whereas the general motion in the global space (which corresponds to a continuous jumping between infinite Minkowski spaces) corresponds to the general case of the theory, he called general relativity. The idea is that: for a local observer in free-fall (moving along a given geodesic), the space with respect to him is Minkowskian. This called Einstein equivalence principle (EEP). Einstein equations are the constraints that define the geodesics, as such, define EEP.

The result of the theory is that:*matter distorts spacetime, and beings living in spacetime follow distorted paths.*

GLOBAL RELATIVITY

SR is the local description of spacetime. For the global observer, the local frame changes at each new point on the geodesic. This leads to the general case of the theory: global relativity.

Let us make the argument more clear. In a local frame $O_l(p)$, the local observer measures the infinitesimal interval as:

$$ds^2 = g_{\mu\nu}(X) dX^\mu dX^\nu \tag{1}$$

where $g_{\mu\nu}(X)$ is the Minkowskian metric. With respect to the global observer, in the general case, each infinitesimal element in the coordinates is split as:

$$dx^\mu = f^\mu_\nu(x) dX^\nu \tag{2}$$

where $f^\mu_\nu(x) = \frac{\partial x^\mu}{\partial X^\nu}$. In the global space, the coordinates are curved; therefore, the relations between the coordinates are not linear. Moreover, each coordinate is parametrized with the parameter of the embedded curve (geodesic) as: $x^\mu(\tau)$. Furthermore, each coordinate may (generally) construct three planes, e.g. for x^1, we have: $\{(x^1, x^2), (x^1, x^3), (x^1, x^4)\}$; therefore, each coordinate may generally construct three curves by eliminating the parameter τ between the couples, e.g. for $x^1(\tau)$, we have: $\{(x^1(\tau), x^2(\tau)), (x^1(\tau), x^3(\tau)), (x^1(\tau), x^4(\tau))\}$; each couple corresponds to a curve. We use the polar coordinates for each couple, with the choice that the coordinate that construct the planes plays the role of the rho-coordinate, e.g. for $\{(x^1, x^2), (x^1, x^3), (x^1, x^4)\}$, the coordinate x^1

plays the role of the rho-coordinate. These planes are used to parametrize the general form of the curve in the global space; hence, they determine the explicit form of the curvature.

Since the coordinates X^μ are flat, in the general case, we write

$$dX^i = \omega^i dT^i, i = \overline{1,3}. \tag{3}$$

That is, the theory (generally) is described with three constants. Therefore

$$dx^\mu = f^\mu_\nu(x) dX^\nu = \left(g^\mu_\nu(X) + g^\mu_4(X)\omega_\rho \delta^\rho_\nu\right) dX^\nu \tag{4}$$

where $\omega_\rho = \frac{1}{\omega^\rho}, \omega_1 = 1$ (Note that $\{x^\nu\}$ locally reduce to $\{X^\nu\}$ and the relations between the coordinates $\{X^\nu\}$ are linear).

An important note to mention is that the differential elements used here are not infinitesimal in the mathematical sense, but in the context of EEP. That is, they correspond (physically) to sufficiently small region of space.

Clearly, the quantities $\{f^\mu_4(x)\}$ correspond to curvatures (of curved lines in 2d spaces), as such, they correspond to accelerations; therefore, multiplying these with the differential element of the time coordinate $d\tau$, we get elements of velocity in the same/opposite direction of the μ-axis.

Of course, $\frac{ds}{d\tau}$ is invariant, as such, the square $g_{\mu\nu}(X)\frac{X^\mu}{d\tau}\frac{dX^\nu}{d\tau} = M$. Since the components of this four-vector are the velocity, thus by adopting constants for these components, we get a geometric stress-energy tensor; its physical interpretation is simple: it represents locally the four-vector of impulsion-energy (caused by gravity) of the test particle under study.

But, how the picture is, for the global observer? The answer is simple; we just apply SR principles. The first SR postulate corresponds to the conservation of momentum, and the second corresponds to the constancy of speed of light (as it is measured by the local and global observers). The result is that we find the generalized geometric stress-energy tensor, say $G_{\mu\nu}$. The gravity theory, therefore, lies in the equation

$$ds^2 = G_{\mu\nu} dx^\mu dx^\nu. \tag{5}$$

It is clear that $G_{\mu\nu}$ reduces to $M_{\mu\nu}$ locally. Considering the general case by adding the stress-energy tensor of ordinary matter gives the final form of the gravity equation:

$$ds^2 = \hat{G}_{\mu\nu} dx^\mu dx^\nu \tag{6}$$

where $\hat{G}_{\mu\nu}$ corresponds to the matter-geometry stress-energy tensor as it is measured by the global observer.

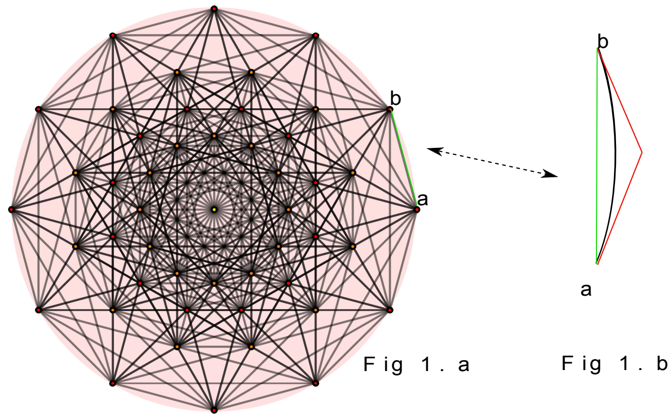

FIG. 1. A circular geodesic; the continuous path (black) corresponds to the global observer; the discrete path (red) corresponds to the local observer; the straight path (green) corresponds to the local R-axis; with its orthogonal axis they construct the local plane. This representation corresponds to the arc $\widehat{ab}$. Each arc in the curve has the same description

Evidently, the last equation defines the metric of spacetime, together with the geodesic equations (Note that the geodesics equations are closely to the local coordinate [5]), the curved paths can be determined explicitly.

THE SPHERICAL SYMMETRIC CASE

For the case of spherical symmetric gravitational field, the orbits are circles. A circular orbit can be represented locally (which depends on the position on the curve; and it takes discrete description as in Regge geometry) and globally (continuously; which we get by taking the lengths of the edges in the discretization to zero; as it looks globally) in a contextual form, see Fig. 1. In this case:

$$f^r_4(x) = \frac{-1}{r} \tag{7}$$

This is the usual curvature of a circular curve in two dimensional space (r, φ). This leads to

$$(dx^r)^2 = \left(1 - \frac{1}{r}\right)(dX^r)^2 = k(r) d(X^r)^2 \tag{8}$$

For dx^t, we use the spacetime diagram (as in SR) together with the last equation, we get the result:

$$f^t_4(x) = \frac{\frac{-1}{r}}{k(r)} \tag{9}$$

which gives

$$(dx^t)^2 = \frac{1}{k(r)}(dX^t)^2 \qquad (10)$$

By replacing these equations in Eq (1), we get the spherical symmetric solution of Einstein equations (Schwarzschild solution):

$$ds^2 = -(dX^t)^2 + (dX^r)^2 + (rd\theta)^2 + (r\sin(\theta)d\varphi)^2 \qquad (11)$$

$$= -k(r)(dx^t)^2 + \frac{1}{k(r)}(dx^r)^2 + (rd\theta)^2 + (r\sin(\theta)d\varphi)^2 \qquad (12)$$

This completes the derivation in question.

Note that the derivation of Eq. (10) can be extracted straightforwardly using the postulate of the constancy of the speed of light of SR, without using the spacetime diagram (i.e. the last derivation fundamentally reflects the speed of light constancy postulate): it is clear that the element of velocity $\frac{dr}{d\tau}$ is tangent to the spherical symmetric geodesic, thus locally this element corresponds to a uniform straight motion (with respect to the observer at this local frame). With respect to the global observer (at the gravity center), this element has an additional contribution (due to the curvature of the curve $f_4^r(x)$, see above); the picture is illustrated in Fig. 1. On the other hand, the distance traveled by light (of course, with the constant of speed c) with respect to the local observer is dX^r, and with respect to the global observer, the distance is dx^r. At the same time, the speed of light is the same with respect to both the local and global observers (the constancy of speed of light postulate). Therefore, the time (with respect to the global observer) must pass slowly when it is compared with its passage with respect to the local observer.

This can be visualized as follows: the light travels through a longer distance (with respect to the local observer) than the distance with respect to the global observer (due to the contraction in length), see Fig 1. At the same time, the starting point of traveling (which is r with respect to the global observer; and $R = 0$ with respect to the local observer; note that the differential in coordinates here is not in the mathematical sense, but in the context of Einstein equivalence principle) and the final one (which is $r + dr$ with respect to the global observer; and dR with respect to the local observer) are the same with respect to both local and global observers (this is obvious; the local frames construct atlas, with are glued together in a consistent way to cover the global Riemann manifold). Therefore, the only way these conditions are satisfied is that the speed of the time-passage (with respect to the two observers) is different.

One may be wondered that: in GR, some solutions naturally involve non-diagonal metrics. Naturally, if the curves corresponding to the curvatures $f_4^\mu(x)$ depend on the corresponding three curved coordinates, then (when we apply the second SR postulate) new constraints are considered, as such, relations between coordinates are resulted, which lead finally (when we compute the square of infinitesimal coordinates elements) to those terms being discussed: if we suppose that the additional infinitesimal length of the curve with respect to the global observer, say dl, depends on more than one variable: $dl = a(x)_\mu dx^\mu$. The metric, therefore, is expected to have the form:

$$ds^2 = b(x)dt^2 + (c(x)_\rho dl^\rho)^2 \qquad (13)$$

where $a(x), b(x), c(x)$ are functions depend on x, Thus, we can see that (in the form of the metric) $(c(x)_\rho dl^\rho)^2$ reproduces the general case.

REISSNER-NORDSTROM SOLUTION

In the case of gravitational field of a charged, non-rotating, spherically symmetric gravity source, the metric is extracted by taking into account the Hamiltonian terms, say $T_{\varphi\varphi}$ or $T_{\theta\theta}$, of the electric charge, say q, and the that of cosmological constant,say Λ. Their Hamiltonian terms are given by [6]: for Λ, $T_{\varphi\varphi} = -r^2\Lambda$; for q, $T_{\varphi\varphi} = \frac{\chi}{r^2}$, where χ is a constant.

First, let us clarify more the expression of the geometry-matter stress-energy tensor $\hat{G}_{\mu\nu}$: (a) the first term corresponds to the contributions of the Minkowski local frame and the induced spacetime (from the curvature), (b) the second corresponds to the usual matter stress-energy tensor. That is

$$\hat{G}_{\mu\nu} = G_{\mu\nu} + T_{\mu\nu} \qquad (14)$$

This leads to the formula (using the above analysis)

$$(dx^r)^2 = (1 - \frac{1}{r} + \frac{\chi}{r^2} - \Lambda r^2)(dX^r)^2 = K(r)(dX^r)^2 \qquad (15)$$

For the element of time, we proceed as above, evidently this gives

$$(dx^t)^2 = \frac{1}{K(r)}(dX^t)^2 \qquad (16)$$

which completes the derivation being discussed:

$$ds^2 = -(dX^t)^2 + (dX^r)^2 + (rd\theta)^2 + (r\sin(\theta)d\varphi)^2 = -K(r)(dx^t)^2 + \frac{1}{K(r)}(dx^r)^2 + (rd\theta)^2 + (r\sin(\theta)d\varphi)^2 \quad (17)$$

THE GEOMETRIC STRESS-ENERGY TENSOR: GENERALIZATION

Here, we derive the Reissner-Nordstrom Solution geometrically, with a nice physical-mathematical realization.

Electric Charge Constant

If the sphere-area (formed by the couple (φ, θ), with radius r) is expanding, then the curvature element $f_4^r(x)$ will naturally have an additional contribution (because of the curvature of the expanding sphere), which leads to the contribution of the electric charge Hamiltonian in the geodesic, exactly as above.

The generalization in this study is that the term $f_4^r(x)$ is replaced with $\hat{f}_4^r(x)$ such that: $\hat{f}_4^r(x) = f_4^r(x) + F_4^{\theta\varphi}(x)$. This is for the this particular case. In the general case, however, the equation Eq (2) becomes:

$$dx^\mu = f_\nu^\mu(x) dX^\nu + F_\nu^\mu(x) dX^\nu \quad (18)$$

where

$$F_\nu^i(x) = \frac{1}{2}\partial_\nu \left(X_j \wedge X_k \epsilon^{ijk}\right), \qquad F_\nu^4(x) = 0, \quad (19)$$

where $i, j, k = \overline{1,3}$, $F_\nu^\mu(x)$ are Gauss (or Riemann) curvatures of the surfaces, and ϵ^{ijk} is Levi-Civita tensor.

Cosmological Constant

If the space is charged with a surface-distribution of energy (hypothetically), then the Hamiltonian will take the form: $G_{22} = 4\pi\Lambda r^2$ where Λ is the constant of distribution. Evidently, this leads to the term of cosmological constant found above. Note that: we have considered the surface-distribution because of the fact that locally we are dealing with a one-form quantity (i.e. infinitesimal charts in the manifold), therefor, globally the case corresponds to a volumetric distribution with a given constant. This is the constant used (in many sources) to interpret the dark energy mysterious.

Note that, if the curvature is negative, the force is attractive; if it is positive, the force is repulsive.

CONCLUSION

Einstein equivalence principle is the geometric realization of special theory of relativity principles.

[1] Misner, Charles W., et al. Freeman and Co. San Francisco, Calif (1973).

[2] Will, Clifford M. Theory and experiment in gravitational physics. Cambridge University Press (1993).

[3] Ni, Wei-Tou. Theoretical Frameworks for Testing Relativistic Gravity. IV. A Compendium of Metric Theories of Gravity and Their Post Newtonian Limits." The Astrophysical Journal 176 (1972).

[4] Bendjoudi, Ahmida. New Approach to General Relativity. DOI: 10.6084/m9.figshare.4810399.v4. viXra:1704.0002 (2017).

[5] Weinberg, Steven. Gravitation and cosmology: principles and applications of the general theory of relativity. Vol. 67. New York: Wiley, 1972.

[6] Hehl, F. W. Obukhov, Foundations of Classical Electrodynamics (2003).