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ANALYTICAL DESCRIPTION OF THE FLOW OF THE NEWTONIAN LIQUID IN A ROUND TUBE AND ON A HORIZONTAL PLATE

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The object of research is obtaining general integrals and some particular solutions for two common flow conditions of incompressible liquid – laminar and averaged turbulent flow. Mathematical description is based on the system of equations of motion in stresses (Navier) and its special case for the Newtonian liquid. A condition of integrating the equations is the constancy of pressure drop and viscosity along the flow. The block schemes of obtaining the general integrals for flow in a pipe and turbulent flow on a plate are represented.

Are as a result, three new general integrals and four particular solutions, which are compared with the known equations, were found. It was shown that the integrals of the Navier equation describe the distribution of tangential stress for turbulent flow. An analysis of solutions for the distribution of velocity showed that the Poiseuille equation for laminar flow in a pipe and the Blasius curve for laminar flow on a plate are particular solutions of one general integral. An analysis of the particular solutions made it possible to estimate the thickness of the laminar sublayer under turbulent flow condition. The results of the work create prerequisites for a more detailed further analysis of laminar and turbulent flows.

Keywords: laminar and turbulent flows, general integral, particular solutions, distribution of velocity, tangential stresses.

1. Introduction

The problems of hydrodynamics are categorized into internal and external, depending on the position of hard surface relative to the flow. The first problem includes the flows of working media in the equipment for thermal power engineering, hydraulics and hydrotechnology. The second problem is examined with the flow around elements of building structures, aircraft, while solving problems of meteorology, etc. The flow in a round pipe refers to the internal problem, and the flow around a plate – to the external. Both problems have great significance because of their wide spread in technology, which led to a detailed study of these flows for a long period of time. In both problems, they search for the distribution of speed and tangential stresses along the radius of a pipe or near the flat surface of a plate under the laminar and turbulent condition [1–5].

At present, several exact solutions for the flow in a pipe and on a plate with the laminar condition are known. The basis of finding these solutions is the equations of equilibrium, solutions of the equation of boundary layer and others, which are implemented in the form of different calculation methods. At the same time, application of the Nervier-Stokes equation, which has several exact solutions at very small Reynolds numbers and claims for to status of the general equation of motion of the Newtonian liquid, does not lead to the same results [1, 5, 6].

The turbulent flowing condition, for which there are no exact solutions, is of great practical value. In this case, the calculation is conducted on the basis of experimental data and different physical models, implemented in the form of semi-empirical equations and computer programs [2–4, 7]. The procedures, developed on the basis of this approach, are used in the process of designing heat exchange equipment [8] and water pipe-lines [2–4], for the calculation and visualization of processes of convective heat exchange, which are frequently complicated by the additional contributing factors [8–14].

A small number of exact solutions only for one flow condition, obtained without the use of general equations of motion, negatively affect understanding the interrelations between physical processes and their mathematical description.

Thus, relevant appears the search for differential equations and their exact solutions, carried out according to the classical scheme and implemented, for example, in problems of thermal conductivity, theory of elasticity and others

2. Literature review and problem statement

As a result of the conducted research, it is established that with under laminar flow conditions the distribution of speed and tangential stress along the normal to the flow depends only on the Reynolds number, and with the turbulent condition they depend on two parameters – the Reynolds number and relative roughness of the surface of a pipe and a plate. The influence of these factors is manifested in a change in pressure along the flow, which decreases in the course of flowing in a pipe and changes in a more complex way while flowing around a plate [1–4].

In laminar flow, there exists only progressive motion of particles along the axis with the variable speed along the radius. Existence of a small-scale rotation of particles and pulsation of all characteristic thermodynamic magnitudes refers to distinctive properties of turbulent flow.

For calculating the laminar condition of the flow in a pipe, there is an exact solution (Poiseuille), the merit of which is the clear connection between a change in the pressure along the flow (dp_z/dz), viscosity and distribution of speed along the radius. This equation is obtained as a result application of equation of equilibrium to the small element of the flow and corresponds well to experimental data. For calculating turbulent flow in a pipe, averaged by time, two semi-empirical equations are common – exponential and logarithmic, which correspond well to experimental data and are obtained as a result of applications of different hypotheses about the internal structure of turbulent nucleus. With this approach, there is no explicit connection between magnitude dp_z/dz, viscosity and distribution of velocity [3–6].

Calculation of distribution of velocity and tangential stresses on a horizontal plate is carried out with the help of the methods of theory of boundary layer, based on the simplification of the Navier-Stokes equation. The use of theorem of momentum for the element of the flow is another method of this calculation. As a result, it was possible to obtain the Blasius solution in the dimensionless form, which takes the form of a table. This solution connects the distribution of velocity along the normal to the surface of a plate with the coordinate along the flow and viscosity. With the help of this solution it is not possible to establish the influence of magnitude dp_z/dz on the distribution of the unknown values, nevertheless, it agrees well with experimental data and it is considered the exact solution of the equation of the boundary layer [1, 3–5].

As a result of studies of flows in a pipe and on a plate, it was established that turbulent flow in both cases has laminar structure. Near the wall there is a thin laminar sublayer and turbulent flow does not have any mechanical contact with the wall.

During the flow in a pipe, the thickness of laminar sublayer is identical, while during the flow on a plate it is variable. It is considered, that the distribution of velocity in the sublayer is linear; however, there are no calculated equations for finding speed and thickness of the sublayer [2–5].

The noted properties of laminar and turbulent flows are considered classical. At present, these approaches are refined and developed due to the use of numerical modeling and improvement of physical experiment.

Contemporary studies are characterized by a large number of trends, which relate to technology, flows in the environment, behavior of living organisms in the ocean and in the atmosphere etc. [9–15].

A part of the studies are directed toward modeling the processes of the flow, complicated by different additional conditions of flowing. The presence of heat sources, which exist inside the channels of different configuration (internal problem) [9], as well as in the atmosphere (external problem) [10], is one of such conditions. In work [11], it is noted that turbulent flows with the heat sources have a great impact on the properties of geophysical and astrophysical systems, as well as on the interaction of the ocean and the atmosphere. Paper [12] uses numerical modeling for studying the influence of the gradient of velocity and the small-scale rotation of the particles of liquid for explaining the statistical properties of turbulence.

The numerical modeling of these and other flows is based on the use of the Reynolds equations, which include the averaged components of velocity and their pulsating components. In these equations the additional known qualities appear (dual and triple products of pulsations of velocity and others), to obtain which, different semi-empirical models of turbulence are developed. According to numerous authors, this theory, as well as the Reynolds equations, is far from perfection [1, 3, 6, 7].

Paper [13] is dedicated to theoretical description of the slow flow of rarefied gases in thin tubes. For an analysis, the Navier-Stokes equation is used, which gives satisfactory result for small Reynolds numbers (*Re<1*). However, such small Reynolds numbers are not found in the majority of practical problems.

One of the reasons for the problems of calculation methods is the absence in the used equations of the function of velocity rotor, characterizing the key property of the moving liquid – the rotation of particles. This function is absent in the Navier-Stokes and Reynolds equations, in different variants of equations of turbulent flow, it is not considered in special cases of theorem of momentum and others.

Experimental studies frequently use different methods of the flow visualization, which gives a visual idea of the distribution of velocities and temperatures in the flow [15]. The results of such studies are not connected with the problems of calculation methods; however, their performing requires special equipment and is possible for a limited range of problems.

In the present work we used the method of calculation of tangential stresses and velocities, based on the use of the equation of motion in stresses (Navier) and its special case for the Newtonian liquid, in which is the function of velocity rotor is present [16]. The latest system of equations takes the form:

$$X-\frac{1}{\rho}\frac{\partial p_x}{\partial x}+\nu\left[\frac{\partial}{\partial y}\left(\left(\text{rot u}\right)_z+2\frac{\partial u_x}{\partial y}\right)+\frac{\partial}{\partial z}\left(\left(\text{rot u}\right)_y+2\frac{\partial u_z}{\partial x}\right)\right]=\frac{du_x}{dt},$$

$$Z-\frac{1}{\rho}\frac{\partial p_z}{\partial z}+\nu\left[\frac{\partial}{\partial x}\left(\left(\text{rot u}\right)_y+2\frac{\partial u_z}{\partial x}\right)+\frac{\partial}{\partial y}\left(\left(\text{rot u}\right)_x+2\frac{\partial u_y}{\partial z}\right)\right]=\frac{du_z}{dt}. \tag{1}$$

The use of these two systems of equations made it possible to find the distribution of tangential stresses and velocity along the radius with laminar flow in a pipe. Distinctive special feature (1) is considering two forms of motion – progressive, which is characterized by the gradients of linear velocities, and rotatory, which is characterized by the function of rotor. Taking into account the determinations of laminar and averaged turbulent flow, system (1) can be used for describing both flow conditions.

3.The aim and tasks of the study

The purpose of the study is obtaining exact solutions for finding the distribution of velocities and tangential stresses along the normal to the flow in a round pipe and with flowing around a plate with laminar flow and turbulent flow averaged by time.

To achieve the set goals, the following tasks were to be solved:

– to find general integrals for the distribution of velocity and tangential stress along the radius of a pipe and by the thickness of the boundary layer on a flat plate in presence and absence of rotation of particles of the liquid;

– to find some particular solutions of the general integrals, which make it possible to compare them with the known equations;

– to carry out a comparative analysis and to find the distinctive properties of the particular solutions and the known equations.

4. Methods for finding general integrals and their analysis

Solution of the problems in question is carried out by two methods. The first method is based on the general system of differential equations for any continuous medium in stresses (Navier), and the second method uses a special case of the Navier equation, obtained for the Newtonian liquid (1) [16]. Both systems of equations are simplified, integrated when the given problems are explored, then particular solutions are found, which are compared with the known equations. The distribution of tangential stress is found with the help of the first method, and then the distribution of velocity is found with the use of Newton's law for the viscous friction. With the help of the second method, the distribution of velocity is found by integrating the equation of motion of the second order. Both methods supplement each other and must give an identical result. In this case, it is always assumed that incompressible fluid and thermo-physical properties are constant, and the flow in a round pipe and on a flat plate is one-dimensional and steady.

Let us examine the flow in a round pipe and find general integrals for distributing tangential stress and velocity along the radius of a pipe under laminar and turbulent flow conditions (Fig.1).

Fig. 1 presents the calculated scheme of the stabilized flow in a round pipe.

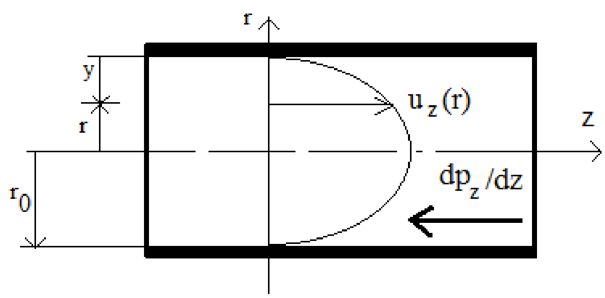

Fig.1. Calculated scheme for the flow in a round pipe

We will use the general equation of motion of continuous medium in stresses in coordinates (r, θ, z). For coordinate z, the general equation takes the form (1):

$$Z-\frac{1}{\rho}\frac{\partial p_z}{\partial z}+\frac{1}{\rho}\left(\frac{\partial \tau_{zr}}{\partial r}+\frac{1}{r}\frac{\partial \tau_{\theta r}}{\partial \theta}+\frac{\tau_{zr}}{r}\right)=\frac{du_z}{dt}, \tag{2}$$

where $p_z = -\sigma_z$ is the pressure along the axis z, which, according to the rule of signs, is opposite to normal stress σ_z.

We will simplify the equation (2), assuming that mass forces and rotation around the axis of a pipe are absent. Then we obtain:

$$\frac{\partial \tau_{zr}}{\partial r}+\frac{\tau_{zr}}{r}=\frac{\partial p_z}{\partial z}. \tag{3}$$

At constant diameter of pipe $dp_z/dz = const$ the solution of equation (3) takes the form:

$$\tau_{zr}=\frac{c_1}{r}+\frac{\frac{dp_z}{dz}\cdot r}{2}. \tag{4}$$

At $c_1 = 0$, we will obtain equation for the distribution of tangential stress in the form:

$$\tau_{zr} = \frac{dp_z}{dz} \cdot \frac{r}{2}, \qquad (5)$$

which coincides with the known formula for laminar flow [1, 2, 4, 5].

Let us find the distribution of velocity along the radius of a pipe for the Newtonian liquid ($\tau = \mu \cdot \mathrm{grad} u$, where μ is the dynamic viscosity).

From (5) we will obtain:

$$\frac{du_z}{dr} = \frac{\tau_{zr}}{\mu} = \frac{1}{2\mu}\frac{dp_z}{dz} r.$$

After the integration

$$u_z(r) = \frac{1}{4\mu}\frac{dp_z}{dz} r^2 + c_2. \qquad (6)$$

Let us find the distribution of velocity $u_z(r)$, using the second method. From (1) in the absence of the rotation of particles (function of rotor it is equal to zero) we will obtain the following equation in the coordinates (r, z):

$$\frac{\partial^2 u_z}{\partial r^2} = \frac{1}{2\mu}\frac{\partial p_z}{\partial z}. \qquad (7)$$

After two-fold integration at $dp_z/\mu \cdot dz = const$ we will find:

$$u_z(r) = \frac{1}{4\mu}\frac{dp_z}{dz} \cdot r^2 + c_1 r + c_2. \qquad (8)$$

Differentiating (8) by r for the Newtonian liquid ($\tau = \mu \cdot du/dr$), we will find the distribution of tangential stresses, which with an accuracy to constant coincides with (5).

Comparison (6) and (8) shows that finding the distribution of velocity is necessary to perform with the aid of the system (1), since this way makes it possible to find a more general equation and to exclude the loss of a constant. At the same time, the distribution of tangential stress can be found by any method.

Let us examine the solution of the same problem at $c_1 \neq 0$ (4).

Then, using the same scheme of calculations, we will find the distribution of velocity. Since

$$\frac{du_z}{dr} = \frac{\tau_{zr}}{\mu} = \frac{1}{2\mu}\frac{dp_z}{dz} r + \frac{c_1}{r}, \qquad (9)$$

after integrating, we will obtain:

$$u_z(r) = \frac{1}{4\mu}\frac{dp_z}{dz} r^2 + c_1 \ln r + c_2. \qquad (10)$$

We will use the second method of solving, for which equation (1) in the cylindrical system of coordinates for the axis z will take the form:

$$
\begin{aligned}
&Z-\frac{1}{\rho}\frac{\partial p_z}{\partial z}+\nu\left(\frac{\partial}{\partial r}\left[\mathrm{rot}(u)_\theta+2\frac{\partial u_z}{\partial r}\right]+\right.\\
&\left.+\frac{1}{r}\frac{\partial}{\partial\theta}\left[(\mathrm{rot}(u)_r+2\frac{\partial u_\theta}{\partial z}\right]+\frac{1}{r}\left[(\mathrm{rot}(u)_\theta+2\frac{\partial u_z}{\partial r}\right]\right)=\frac{du_z}{dt}.
\end{aligned}
\tag{11}
$$

In this equation, in contrast to (7), the rotational effect of particles was taken into account, which requires the calculation of the function of rotor. For this problem the flow rate changes only along the radius and the derivative along axis z is possible to be neglected. We will also consider that the speed does not change depending on angle θ (the flow does not revolve around the axis of a pipe).

Then, calculating the function of rotor and producing reductions, we find:

$$
\frac{\partial^2 u_z}{\partial r^2}+\frac{1}{r}\frac{\partial u_z}{\partial r}=\frac{1}{\mu}\frac{\partial p_z}{\partial z}.
\tag{12}
$$

After the integration at with $dp_z/\mu\cdot dz=const$ we will again obtain the equation (10).

Thus, the general integral for finding the distribution of velocity with the flow without the rotation of particles is equation (8). The general integral for the distribution of velocity with the rotation of particles is equation (10).

In accordance with the physical definition of flow conditions, equations (5) and (8) relate to laminar flow and equations (4) and (10) relate to turbulent flow averaged by time.

Fig. 2 displays the scheme of obtaining general integrals for finding tangential stresses and distribution of velocity for laminar and turbulent flow conditions.

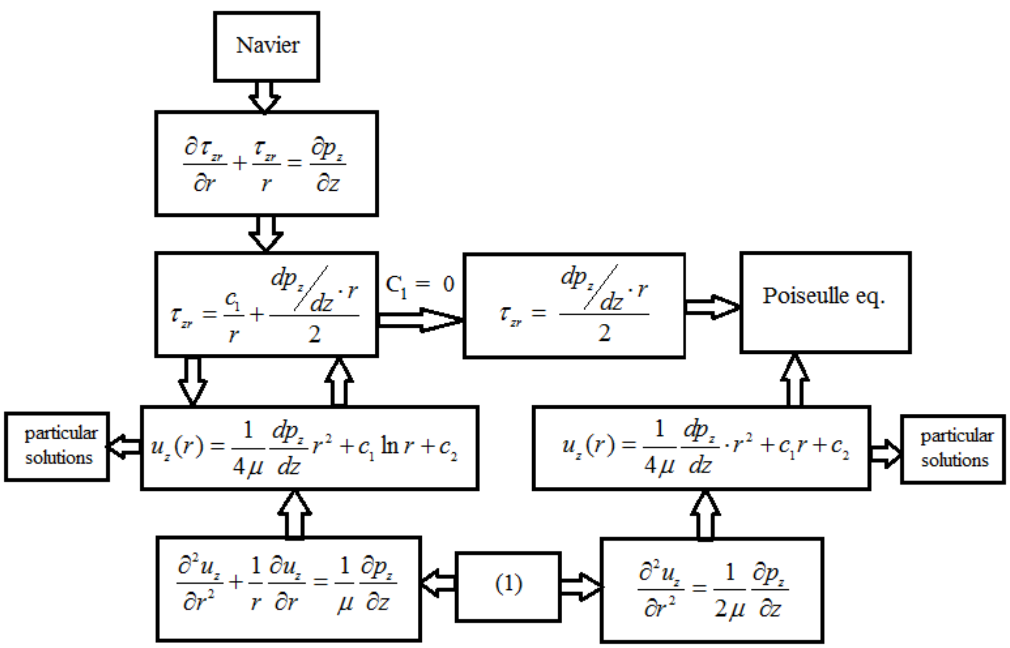

Fig. 2. Scheme of finding general integrals for the flow in a round pipe ($dp_z/\mu\cdot dz=const$)

For finding general integrals with the flow on a horizontal plate we will use the known calculation scheme (Fig. 3) [1–3].

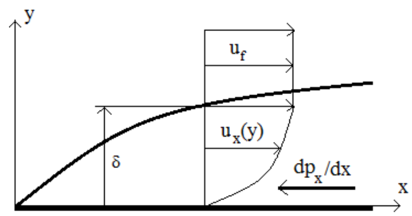

Fig. 3. Scheme of the flow on a plate.

A special case of equation (1) in the Cartesian coordinates in the absence of rotation of particles will take the form:

$$\frac{\partial p_x}{\partial x} = 2\mu \frac{\partial^2 u_x}{\partial y^2},$$

which coincides with (7) at replacement *z* = *x* and *r* = *y*.

Thus, the distribution of velocity during the flow in a pipe and on the surface of a plate without rotation of particles has identical differential equations and, correspondingly, identical general integrals.

Let us use the Navier equation for finding the distribution of tangential stress in the flow near the surface of a plate. For coordinate *x* we will obtain:

$$X - \frac{1}{\rho}\frac{\partial p_x}{\partial x} + \frac{1}{\rho}\left(\frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) = \frac{du_x}{dt}.$$

After simplification in accordance with the previously accepted assumptions

$$\frac{dp_x}{dx} = \frac{d\tau_{yx}}{dy}.$$

After the integration at $dp_x/dx = const$ we will find

$$\tau_x(y) = \frac{dp_x}{dx} y + c_1. \tag{13}$$

The distribution of velocity along the normal to the surface of a plate will be found from equation:

$$\frac{du_x}{dy} = \frac{\tau_{yx}}{\mu} = \frac{1}{\mu}\frac{dp_x}{dx} y + c_1,$$

after integrating

$$u_x(y) = \frac{1}{2\mu}\frac{dp_x}{dx} y^2 + c_1 y + c_2. \tag{14}$$

We will use the second method of finding the distribution of velocity with the help of (1) and consider the rotational effect of particles. Since the velocity changes only along the coordinate y, we will obtain (Fig. 3):

$$\mu\left[\frac{\partial}{\partial y}\left((\text{rot } u)_z + 2\frac{\partial u_x}{\partial y}\right)\right] = \frac{\partial p_x}{\partial x}.$$

Calculating the function (*rot* u_z), we will obtain:

$$\frac{\partial^2 u_x}{\partial y^2} = \frac{1}{\mu}\frac{\partial p_x}{\partial x}. \tag{15}$$

Integrating (15) at $dp_x/\mu \cdot dx = const$, we will find the general integral, which coincides with (14).

Fig. 4 displays the scheme of finding general integrals for the distribution of tangential stress and velocity with the flow on a plate. The use of the Navier equation of and system (1) leads to one and the same result, which is illustrated by vertical pointers.

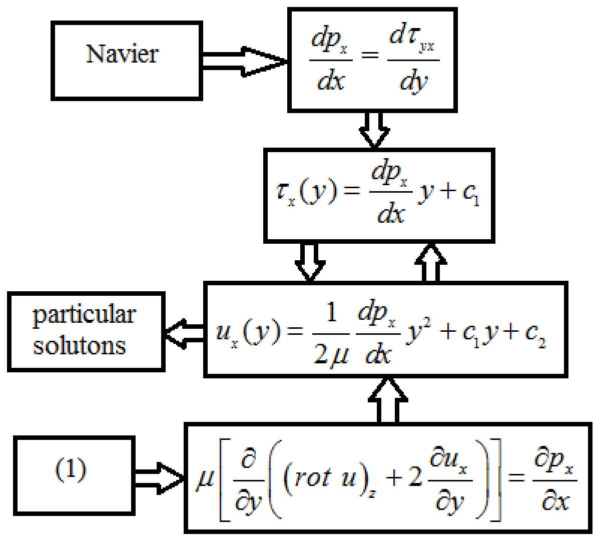

Fig. 4. Scheme of finding general integrals for turbulent flow on a plate

Table 1 presents the results of finding general integrals for laminar and turbulent flows in a round pipe and on the surface of a horizontal plate.

Table 1

General integrals for the flow in a round pipe and on the surface of a plate

Condition	Flow in a pipe	Flow on a plate
Laminar	$u_z(r) = \frac{1}{4\mu}\frac{dp_z}{dz} \cdot r^2 + c_1 r + c_2$	$u_x(y) = \frac{1}{4\mu}\frac{dp_x}{dx} \cdot y^2 + c_1 y + c_2$
Turbulent	$u_z(r) = \frac{1}{4\mu}\frac{dp_z}{dz} r^2 + c_1 \ln r + c_2$	$u_x(y) = \frac{1}{2\mu}\frac{dp_x}{dx} y^2 + c_1 y + c_2$

It follows from Table 1 that for laminar flow in a pipe and on a plate general integrals coincides. The integral for turbulent flow in a pipe differs most significantly from other equations. The existence of function ln(r) requires the formulation of special boundary conditions for taking into account the influence of constant c_1.

5. Particular solutions and comparison with the known equations

Let us examine laminar flow in a pipe under the following boundary conditions: if $r = 0$, then $du_z/dr = 0$, and at $r = r_0$, $u_z = 0$, which we will apply to (8). Then we will obtain the distribution of velocity along the radius, which coincides with the known Poiseuille equation.

$$u_z(r) = \frac{1}{4\mu}\partial p_z / \partial z \cdot \left(r^2{}_0 - r^2\right),$$

at $dp_z/dz > 0$ [3, 4].

A more convenient particular solution is the equation, for which average flow rate is known, since it is easy to find it from the equation of mass balance. For finding this particular solution we will use the following boundary conditions: if $r = 0$, then $du_z/dr = 0$, and at $r = r_0/2$, $u_z(r) = u_{av}$ we will obtain

$$u_z(r) = \frac{1}{4\mu}\frac{dp_z}{dz}\left(r^2 - \frac{r}{r_0}r^2{}_0\right) + 2u_{av}\left(1 - \frac{r}{r_0}\right). \tag{16}$$

Fig. 5 displays the comparison of the distribution of velocity, plotted according to the Poiseuille equation and equation (16) with the help of the Mathcad software package.

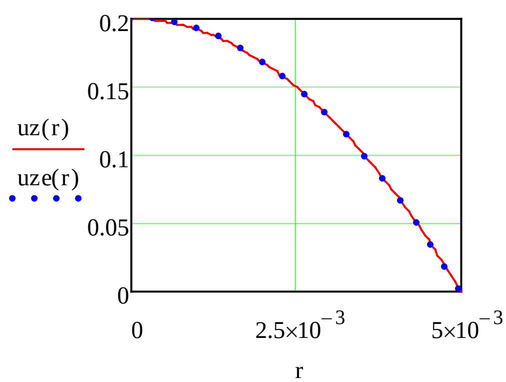

Fig. 5. Distribution of velocity along the radius of a pipe.

Continuous line – according to equation (16), points – according to the Poiseuille equation /water, Re=1000/.

The particular solution for distribution of velocity with turbulent flow in a pipe will be found from the general solutions (4) and (10) with the use of variable y (Fig. 1). Let us apply the boundary conditions to these equations, characteristic for the axis of a pipe: at $y = r_0$, $\tau = 0$ *u* $u_z(y) = u_{max}$.

Then we will obtain:

$$u_z(y) = \frac{1}{4\mu}\frac{dp_z}{dz}\left(y^2 - r_0^2 + 2r_0^2 \ln\frac{r_0}{y}\right) + u_{max}. \tag{17}$$

It follows from this equation that the velocity of turbulent flow on the wall cannot be equal to zero. This property of equation (17) corresponds to physical sense, since near the wall there is a laminar sublayer, which separates the nucleus of turbulent flow from the wall. In the sublayer there is no rotation of particles and equation (17) is not applicable to it.

Fig. 6 displays the comparison of distribution of velocity, plotted according to equation (17) and according to semi-empirical equation $u_z(y) = u_{max}(y/r_0)^{0.16}$ [1, 5].

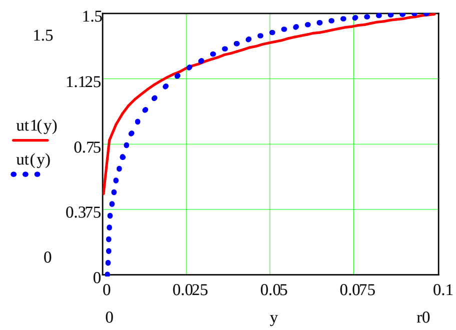

Fig. 6. Comparison of theoretical distribution of velocity in a pipe (designated by the continuous line) and exponential semi-empirical equation (designated by points).

For finding particular solutions with laminar and turbulent flows on a plate we will use varied conditions. For laminar flow condition the boundary layer begins on the wall and the conditions will be the following: at $y = \delta(x)$, $u_x(y) = u_f$, and at $y = 0$, $u_x(y) = 0$. As a result, we will obtain the laminar flow:

$$u_x(y) = \frac{1}{4\mu}\frac{dp_x}{dx}\left[y^2 - y\cdot\delta(x)\right] + \frac{y}{\delta(x)}u_f. \tag{18}$$

Fig. 7 contains the graph, plotted in the dimensionless coordinates

$$u_x(y)/\ u_f = f\left(\eta = \frac{y}{2}\sqrt{\frac{u_f}{\upsilon\cdot x}}\right),$$

accepted in works [1, 5]. For graph construction we used equation (18), numerical data correspond to laminar condition of the flow at distance x from the front edge. The results of comparison of the graph in Fig.7 and the data of the table for the Blasius solution show their satisfactory correspondence [1, 5].

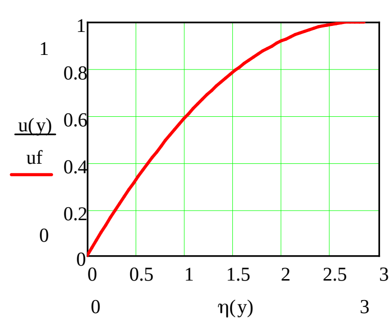

Fig. 7. Theoretical distribution of velocity by the thickness of the boundary layer on a plate (Re_x=5,8·10^3, dp_x/dx=–3,2 Pa/m, air).

For turbulent flow we will assign boundary conditions on the axis of the flow and apply them to the integral (14).

At $y = \delta(x)$, $\tau_x(y) = 0$, and $u_x(y) = u_f$.

Then we obtain:

$$u_x(y) = \frac{1}{2\mu}\frac{dp_x}{dx}\left[y^2 + \delta(x)^2 - 2y \cdot \delta(x)\right] + u_f. \quad (19)$$

Fig. 8 displays a comparison of distribution of velocity, found according to (19) and according to the known semi-empirical equation $ue(y) = u_x(y) = u_f[y/\delta(x)]^{1/7}$.

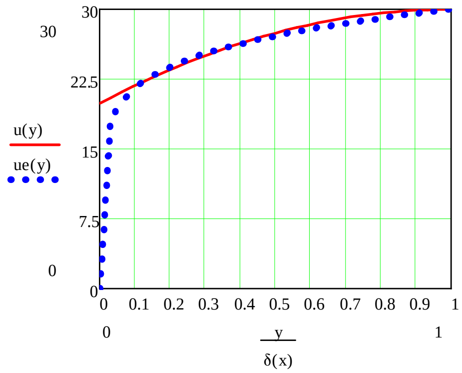

Fig. 8. Comparison of theoretical distribution of velocity ((19) designated by continuous line) and semi-empirical (designated by /$Re_x= 2\cdot10^5$, $dp_x/dx= -35$ Pa/m, air/).

As it follows from the comparison of curves, in the range of $y/\delta(x) = 0\ldots0{,}1$, there occurs a sharp divergence of theoretical and semi-empirical dependence.

It is connected with the existence of the laminar sublayer, in which there is no rotation of particles and the distribution of velocity in this range $y/\delta(x)$ must be calculated from another equation.

Thus, equation (19) can be used only for finding the distribution of velocity in the turbulent part of the boundary layer.

Table 2 displays the particular solutions, which were obtained as a result of the application of boundary conditions to the previously obtained general integrals.

Table 2

Particular solutions for the flows in a pipe and on a plate for different conditions

Condition	Flow in a pipe	Flow on a plate
Laminar	$u_z(r) = \frac{1}{4\mu}\frac{dp_z}{dz}\left(r^2 - r \cdot r_0\right) + 2u_{av}\left(1 - \frac{r}{r_0}\right)$	$u_x(y) = \frac{1}{4\mu}\frac{dp_x}{dx}\left[y^2 - y \cdot \delta(x)\right] + \frac{y}{\delta(x)}u_f$
Turbulent	$u_z(y) = \frac{1}{4\mu}\frac{dp_z}{dz}\left(y^2 - r_0^2 + 2r_0^2 \ln\frac{r_0}{y}\right) + u_{max}$	$u_x(y) = \frac{1}{2\mu}\frac{dp_x}{dx}\left[y^2 + \delta(x)^2 - 2y \cdot \delta(x)\right] + u_f$

Note: $\delta(x)$ – thickness of the boundary layer [1–3]

As it follows from Table 2, all particular solutions consider the influence of viscosity and pressure drop along the flow, which for the flow in a pipe does not change along the axis, but for the flow on a plate, it is necessary to assign or calculate preliminarily.

6. Discussion of results of mathematical description

Mathematical description of laminar and averaged turbulent flow is performed according to the classical scheme, at the basis of which lies the equation of motion for any continuous medium (Navier). In previous works, a particular case of the Navier equation for the Newtonian liquid was found – system (1) [16]. In the present work we used both systems of equations, with the help of which general solutions for one-dimensional steady flows in the cylindrical and Cartesian coordinate system were found. Then the boundary conditions for the flows in a pipe and on a plate were formulated and particular solutions for them were found. These solutions were compared with the experimental results, represented in the form of exact solutions (Poiseuille and Blasius) or semi-empirical equations.

This way of mathematical description made it possible to find precise equations for the distribution of tangential stresses and velocities along the normal to the direction of the flow for all known conditions and Reynolds numbers. In this case, it is assumed that the flow washes a smooth surface, and the influence of roughness over time can be taken into account by introduction of a change in viscosity or dp_z/dz.

An analysis of different ways of finding the calculated equations showed that the integrals of the Navier equation give the distribution of tangential stresses in turbulent flow. It was shown that the Poiseuille and Blasius solutions, relating to different types of problems, obtained in completely different ways, are particular solutions of one general integral.

The positive result of equations is the possibility to calculate the thickness of the near–wall laminar layer, on the outer boundary of which the velocities of laminar and turbulent flows are identical.

In the calculated equations there are no factors, which characterize pulsations of all thermodynamic values or gaps in the flow. This means that these equations are suitable only for describing turbulent flow averaged by time.

7. Conclusions

1. Three new general integrals for two flow conditions of liquid were found, which make it possible to establish the laws of distribution of tangential stresses and velocities along the normal to the flow (8), (10), (14). These laws are obtained as a result of parallel solution of problems with the help of the Navier equation and system (1). This approach allows us to exclude the loss of constants and increases the reliability of mathematical description.

2. For refining the area of application of general integrals, we formulated and used the boundary conditions for two known problems – the flow in a round pipe and on a horizontal plate. An analytical and graphic comparison of the particular solutions, obtained in the work, and the known equations was performed and it was shown that analytical equations describe the structure of flows more accurately.

3. As a result of the analysis, it was shown that the integrals of the Navier equation relate to averaged turbulent flow. This result is proved by integrating system (1) and subsequent use of the Newton law for viscous friction. The comparison of the precise and empirical solutions was carried out, which made it possible to estimate the thickness of the near-wall laminar boundary layer with the existence of the nucleus of turbulent flow. It was shown that the Poiseuille equation for flow in a pipe and the Blasius curve for flow on a plate are particular solutions of one general integral (8).

8.References

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