The Polynomial

$$p(x)=x^8+x^7-7x^6-6x^5+15x^4+10x^3-10x^2-4x+1$$

Edgar Valdebenito

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Abstract

In this note give some formulas related with the polynomial:

$$p(g)=g^8+g^7-7g^6-6g^5+15g^4+10g^3-10g^2-4g+1$$

1. Introduction: Roots

$$p(g)=0 \Rightarrow g\in\{g_n : n=1,2,3,...,8\}=\{g_1,g_2,g_3,...,g_8\} \tag{1}$$

$$g_1<g_2<g_3<g_4<g_5<g_6<g_7<g_8 \tag{2}$$

$$g_1=-\frac{1}{8}+\frac{\sqrt{17}}{8}-\frac{\sqrt{34-2\sqrt{17}}}{8}-\frac{\sqrt{17+3\sqrt{17}+\sqrt{34-2\sqrt{17}}+2\sqrt{34+2\sqrt{17}}}}{4} \tag{3}$$

$$g_2=-\frac{1}{8}-\frac{\sqrt{17}}{8}-\frac{\sqrt{34+2\sqrt{17}}}{8}-\frac{\sqrt{4\left(17-3\sqrt{17}\right)+\sqrt{34\left(17-\sqrt{17}\right)}-7\sqrt{34-2\sqrt{17}}}}{8} \tag{4}$$

$$g_3=-\frac{1}{8}-\frac{\sqrt{17}}{8}-\frac{\sqrt{34+2\sqrt{17}}}{8}+\frac{\sqrt{4\left(17-3\sqrt{17}\right)+\sqrt{34\left(17-\sqrt{17}\right)}-7\sqrt{34-2\sqrt{17}}}}{8} \tag{5}$$

$$g_4=-\frac{1}{8}-\frac{\sqrt{17}}{8}+\frac{\sqrt{34+2\sqrt{17}}}{8}-\frac{\sqrt{4\left(17-3\sqrt{17}\right)-\sqrt{34\left(17-\sqrt{17}\right)}+7\sqrt{34-2\sqrt{17}}}}{8} \tag{6}$$

$$g_5=-\frac{1}{8}+\frac{\sqrt{17}}{8}+\frac{\sqrt{34-2\sqrt{17}}}{8}-\frac{\sqrt{17+3\sqrt{17}-\sqrt{34-2\sqrt{17}}-2\sqrt{34+2\sqrt{17}}}}{4} \tag{7}$$

$$g_6=-\frac{1}{8}-\frac{\sqrt{17}}{8}+\frac{\sqrt{34+2\sqrt{17}}}{8}+\frac{\sqrt{4\left(17-3\sqrt{17}\right)-\sqrt{34\left(17-\sqrt{17}\right)}+7\sqrt{34-2\sqrt{17}}}}{8} \tag{8}$$

$$g_7 = -\frac{1}{8} + \frac{\sqrt{17}}{8} - \frac{\sqrt{34 - 2\sqrt{17}}}{8} + \frac{\sqrt{17 + 3\sqrt{17} + \sqrt{34 - 2\sqrt{17}} + 2\sqrt{34 + 2\sqrt{17}}}}{4} \tag{9}$$

$$g_8 = -\frac{1}{8} + \frac{\sqrt{17}}{8} + \frac{\sqrt{34 - 2\sqrt{17}}}{8} + \frac{\sqrt{17 + 3\sqrt{17} - \sqrt{34 - 2\sqrt{17}} - 2\sqrt{34 + 2\sqrt{17}}}}{4} \tag{10}$$

2. Trigonometric Relations

$$g_1 = 2\cos\left(\frac{16\pi}{17}\right) = -2\sin\left(\frac{15\pi}{34}\right) \tag{11}$$

$$g_2 = 2\cos\left(\frac{14\pi}{17}\right) = -2\sin\left(\frac{11\pi}{34}\right) \tag{12}$$

$$g_3 = 2\cos\left(\frac{12\pi}{17}\right) = -2\sin\left(\frac{7\pi}{34}\right) \tag{13}$$

$$g_4 = 2\cos\left(\frac{10\pi}{17}\right) = -2\sin\left(\frac{3\pi}{34}\right) \tag{14}$$

$$g_5 = 2\cos\left(\frac{8\pi}{17}\right) = 2\sin\left(\frac{\pi}{34}\right) \tag{15}$$

$$g_6 = 2\cos\left(\frac{6\pi}{17}\right) = 2\sin\left(\frac{5\pi}{34}\right) \tag{16}$$

$$g_7 = 2\cos\left(\frac{4\pi}{17}\right) = 2\sin\left(\frac{9\pi}{34}\right) \tag{17}$$

$$g_8 = 2\cos\left(\frac{2\pi}{17}\right) = 2\sin\left(\frac{13\pi}{34}\right) \tag{18}$$

$$g_n = 2\cos\left(\frac{(18 - 2n)\pi}{17}\right) \quad , n = 1, 2, 3, 4, 5, 6, 7, 8 \tag{19}$$

$$g_{9-n} = 2\cos\left(\frac{2n\pi}{17}\right) \quad , n = 1, 2, 3, 4, 5, 6, 7, 8 \tag{20}$$

$$g_n = -2\cos\left(\frac{(2n - 1)\pi}{17}\right) \quad , n = 1, 2, 3, 4, 5, 6, 7, 8 \tag{21}$$

3. Some Series

$$-\frac{\pi}{g_1}=\frac{17}{15}+15\cdot 34\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(34n)^2-15^2}=2\cdot 17^2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(2n-1)}{(34n-17)^2-4\cdot 16^2} \qquad (22)$$

$$-\frac{\pi}{g_2}=\frac{17}{11}+11\cdot 34\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(34n)^2-11^2}=2\cdot 17^2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(2n-1)}{(34n-17)^2-4\cdot 14^2} \qquad (23)$$

$$-\frac{\pi}{g_3}=\frac{17}{7}+7\cdot 34\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(34n)^2-7^2}=2\cdot 17^2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(2n-1)}{(34n-17)^2-4\cdot 12^2} \qquad (24)$$

$$-\frac{\pi}{g_4}=\frac{17}{3}+3\cdot 34\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(34n)^2-3^2}=2\cdot 17^2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(2n-1)}{(34n-17)^2-4\cdot 10^2} \qquad (25)$$

$$\frac{\pi}{g_5}=\frac{17}{1}+1\cdot 34\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(34n)^2-1^2}=2\cdot 17^2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(2n-1)}{(34n-17)^2-4\cdot 8^2} \qquad (26)$$

$$\frac{\pi}{g_6}=\frac{17}{5}+5\cdot 34\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(34n)^2-5^2}=2\cdot 17^2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(2n-1)}{(34n-17)^2-4\cdot 6^2} \qquad (27)$$

$$\frac{\pi}{g_7}=\frac{17}{9}+9\cdot 34\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(34n)^2-9^2}=2\cdot 17^2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(2n-1)}{(34n-17)^2-4\cdot 4^2} \qquad (28)$$

$$\frac{\pi}{g_8}=\frac{17}{13}+13\cdot 34\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(34n)^2-13^2}=2\cdot 17^2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(2n-1)}{(34n-17)^2-4\cdot 2^2} \qquad (29)$$

4. Infinite Products

$$g_1=2\prod_{n=1}^{\infty}\left(1-\left(\frac{32}{34n-17}\right)^2\right)=-\pi\frac{15}{17}\prod_{n=1}^{\infty}\left(1-\left(\frac{15}{34n}\right)^2\right) \qquad (30)$$

$$g_2=2\prod_{n=1}^{\infty}\left(1-\left(\frac{28}{34n-17}\right)^2\right)=-\pi\frac{11}{17}\prod_{n=1}^{\infty}\left(1-\left(\frac{11}{34n}\right)^2\right) \qquad (31)$$

$$g_3=2\prod_{n=1}^{\infty}\left(1-\left(\frac{24}{34n-17}\right)^2\right)=-\pi\frac{7}{17}\prod_{n=1}^{\infty}\left(1-\left(\frac{7}{34n}\right)^2\right) \qquad (32)$$

$$g_4 = 2\prod_{n=1}^{\infty}\left(1-\left(\frac{20}{34n-17}\right)^2\right) = -\pi\frac{3}{17}\prod_{n=1}^{\infty}\left(1-\left(\frac{3}{34n}\right)^2\right) \tag{33}$$

$$g_5 = 2\prod_{n=1}^{\infty}\left(1-\left(\frac{16}{34n-17}\right)^2\right) = \pi\frac{1}{17}\prod_{n=1}^{\infty}\left(1-\left(\frac{1}{34n}\right)^2\right) \tag{34}$$

$$g_6 = 2\prod_{n=1}^{\infty}\left(1-\left(\frac{12}{34n-17}\right)^2\right) = \pi\frac{5}{17}\prod_{n=1}^{\infty}\left(1-\left(\frac{5}{34n}\right)^2\right) \tag{35}$$

$$g_7 = 2\prod_{n=1}^{\infty}\left(1-\left(\frac{8}{34n-17}\right)^2\right) = \pi\frac{9}{17}\prod_{n=1}^{\infty}\left(1-\left(\frac{9}{34n}\right)^2\right) \tag{36}$$

$$g_8 = 2\prod_{n=1}^{\infty}\left(1-\left(\frac{4}{34n-17}\right)^2\right) = \pi\frac{13}{17}\prod_{n=1}^{\infty}\left(1-\left(\frac{13}{34n}\right)^2\right) \tag{37}$$

5. Roots:Iterative Method

$$x_{k+1} = \frac{1+10x_k^2-20x_k^3-45x_k^4+24x_k^5+35x_k^6-6x_k^7-7x_k^8}{4+20x_k-30x_k^2-60x_k^3+30x_k^4+42x_k^5-7x_k^6-8x_k^7} \quad ,k\in\mathbb{N} \tag{38}$$

$$x_1 = -2 \Rightarrow x_k \to g_1 \tag{39}$$

$$x_1 = -8/5 \Rightarrow x_k \to g_2 \tag{40}$$

$$x_1 = -1 \Rightarrow x_k \to g_3 \tag{41}$$

$$x_1 = -1/2 \Rightarrow x_k \to g_4 \tag{42}$$

$$x_1 = 0 \Rightarrow x_k \to g_5 \tag{43}$$

$$x_1 = 1 \Rightarrow x_k \to g_6 \tag{44}$$

$$x_1 = 3/2 \Rightarrow x_k \to g_7 \tag{45}$$

$$x_1 = 2 \Rightarrow x_k \to g_8 \tag{46}$$

6. Integrals

$$-\frac{\pi}{g_1} = \frac{1}{2}\int_0^1 \sqrt[34]{x^{-19}(1-x)^{-15}}\,dx \tag{47}$$

$$-\frac{\pi}{g_2}=\frac{1}{2}\int_0^1 \sqrt[34]{x^{-23}(1-x)^{-11}}dx \tag{48}$$

$$-\frac{\pi}{g_3}=\frac{1}{2}\int_0^1 \sqrt[34]{x^{-27}(1-x)^{-7}}dx \tag{49}$$

$$-\frac{\pi}{g_4}=\frac{1}{2}\int_0^1 \sqrt[34]{x^{-31}(1-x)^{-3}}dx \tag{50}$$

$$\frac{\pi}{g_5}=\frac{1}{2}\int_0^1 \sqrt[34]{x^{-33}(1-x)^{-1}}dx \tag{51}$$

$$\frac{\pi}{g_6}=\frac{1}{2}\int_0^1 \sqrt[34]{x^{-29}(1-x)^{-5}}dx \tag{52}$$

$$\frac{\pi}{g_7}=\frac{1}{2}\int_0^1 \sqrt[34]{x^{-25}(1-x)^{-9}}dx \tag{53}$$

$$\frac{\pi}{g_8}=\frac{1}{2}\int_0^1 \sqrt[34]{x^{-21}(1-x)^{-13}}dx \tag{54}$$

7. Hypergeometric Relations

$$g_1=2F\left(\frac{48}{17},-\frac{48}{17};\frac{1}{2};\frac{1}{4}\right) \tag{55}$$

$$g_2=2F\left(\frac{42}{17},-\frac{42}{17};\frac{1}{2};\frac{1}{4}\right) \tag{56}$$

$$g_3=2F\left(\frac{36}{17},-\frac{36}{17};\frac{1}{2};\frac{1}{4}\right) \tag{57}$$

$$g_4=2F\left(\frac{30}{17},-\frac{30}{17};\frac{1}{2};\frac{1}{4}\right) \tag{58}$$

$$g_5=2F\left(\frac{24}{17},-\frac{24}{17};\frac{1}{2};\frac{1}{4}\right) \tag{59}$$

$$g_6=2F\left(\frac{18}{17},-\frac{18}{17};\frac{1}{2};\frac{1}{4}\right) \tag{60}$$

$$g_7=2F\left(\frac{12}{17},-\frac{12}{17};\frac{1}{2};\frac{1}{4}\right) \tag{61}$$

$$g_8 = 2F\left(\frac{6}{17}, -\frac{6}{17}; \frac{1}{2}; \frac{1}{4}\right) \qquad (62)$$

8. The Equation: $x^{17} - 1 = 0$

$$x^{17} - 1 = 0 \Rightarrow x = x_k = \cos\left(\frac{2k\pi}{17}\right) + i\sin\left(\frac{2k\pi}{17}\right) \quad , k = 0,1,2,...,16 \qquad (63)$$

$$w = \cos\left(\frac{2\pi}{17}\right) + i\sin\left(\frac{2\pi}{17}\right) \Rightarrow x_k = w^k \ , k = 0,1,2,...,16 \qquad (64)$$

$$w = \frac{1}{2}\left(g_8 + i\sqrt{4 - g_8^2}\right) \qquad (65)$$

9. $g_k^n \ , k = 1,2,...,8; n \in \mathbb{N}$

$$g_k^n = A_n + B_n g_k + C_n g_k^2 + D_n g_k^3 + E_n g_k^4 + F_n g_k^5 + G_n g_k^6 + H_n g_k^7 \qquad (66)$$

$$A_{n+1} = -H_n \qquad (67)$$

$$B_{n+1} = A_n + 4H_n \qquad (68)$$

$$C_{n+1} = B_n + 10H_n \qquad (69)$$

$$D_{n+1} = C_n - 10H_n \qquad (70)$$

$$E_{n+1} = D_n - 15H_n \qquad (71)$$

$$F_{n+1} = E_n + 6H_n \qquad (72)$$

$$G_{n+1} = F_n + 7H_n \qquad (73)$$

$$H_{n+1} = G_n - H_n \qquad (74)$$

$$A_0 = 1, B_0 = C_0 = D_0 = E_0 = F_0 = G_0 = H_0 = 0 \qquad (75)$$

10. The Equation: $x^4 + x^3 - 6x^2 - x + 1 = 0$

$$x^4+x^3-6x^2-x+1=0 \Rightarrow \begin{cases} x_1 = g_1+g_7 \\ x_2 = g_2+g_3 \\ x_3 = g_4+g_6 \\ x_4 = g_5+g_8 \end{cases} \tag{76}$$

11. The Equation: $x^8-17x^6+68x^4-85x^2+17=0$

$$x^8-17x^6+68x^4-85x^2+17=0 \Rightarrow \begin{cases} x_{1,2} = \pm\left(g_1-g_7\right) \\ x_{3,4} = \pm\left(g_2-g_3\right) \\ x_{5,6} = \pm\left(g_4-g_6\right) \\ x_{7,8} = \pm\left(g_5-g_8\right) \end{cases} \tag{77}$$

12. Relations

$$g_1+g_5+g_7+g_8=\frac{\sqrt{17}-1}{2} \tag{78}$$

$$g_2+g_3+g_4+g_6=-\frac{\sqrt{17}+1}{2} \tag{79}$$

$$g_1=g_5^2-2 \tag{80}$$

$$g_3=g_6^2-2 \tag{81}$$

$$g_5=g_7^2-2 \tag{82}$$

$$g_7=g_8^2-2 \tag{83}$$

13. $g_n + g_m = g_k g_r \quad ,1 \le n, m, k, r \le 8$

+	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8
g_1	$2g_1$	$g_1 g_8$	$g_2 g_8$	$g_2 g_7$	$g_3 g_7$	$g_3 g_6$	$g_4 g_6$	$g_4 g_5$
g_2		$2g_2$	$g_1 g_7$	$g_3 g_8$	$g_2 g_6$	$g_4 g_7$	$g_3 g_5$	$g_5 g_6$
g_3			$2g_3$	$g_1 g_6$	$g_4 g_8$	$g_2 g_5$	$g_5 g_7$	$g_3 g_4$
g_4				$2g_4$	$g_1 g_5$	$g_5 g_8$	$g_2 g_4$	$g_6 g_7$
g_5					$2g_5$	$g_1 g_4$	$g_6 g_8$	$g_2 g_3$
g_6						$2g_6$	$g_1 g_3$	$g_7 g_8$
g_7							$2g_7$	$g_1 g_2$
g_8								$2g_8$

$$g_{2n} + g_{2m} = g_{n+m} g_{9-|n-m|} \quad ,1 \le n, m \le 4 \tag{84}$$

$$g_{2n-1} + g_{2m-1} = g_{n+m-1} g_{9-|n-m|} \quad ,1 \le n, m \le 4 \tag{85}$$

$$g_{2n} + g_{2m-1} = g_{|n-m|} g_{9-|n+m-1|} \quad ,1 \le n, m \le 4 \tag{86}$$

14. Plots

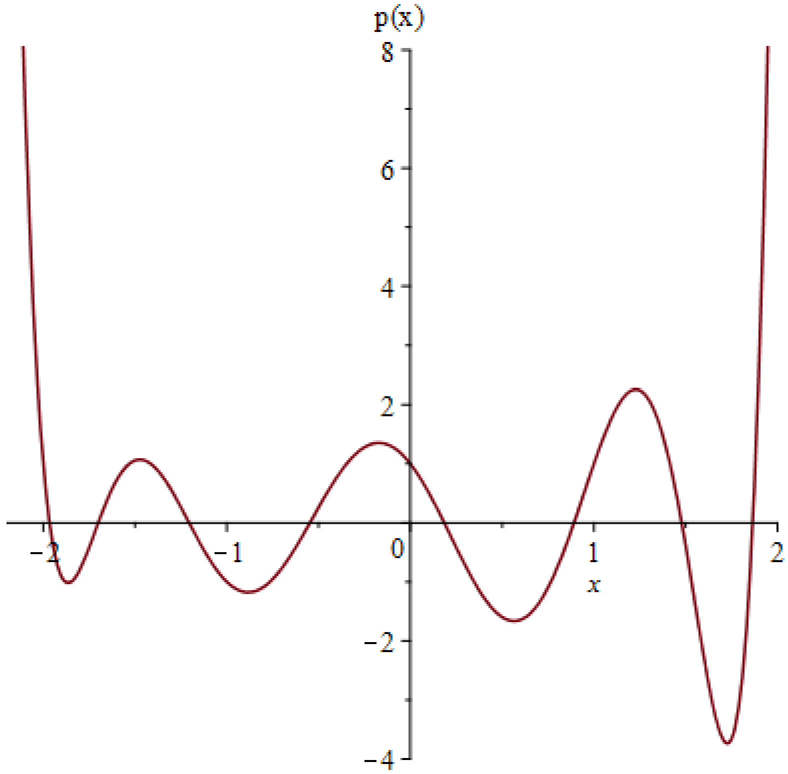

Fig.1. $p(x) = x^8 + x^7 - 7x^6 - 6x^5 + 15x^4 + 10x^3 - 10x^2 - 4x + 1$

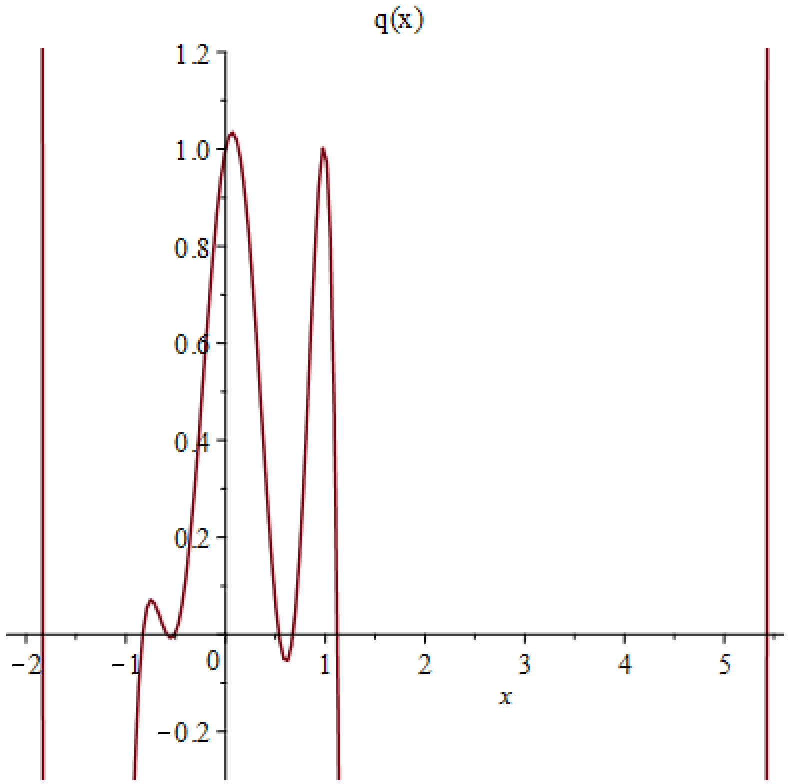

Fig.2. $q(x) = x^8 p\left(\frac{1}{x}\right) = x^8 - 4x^7 - 10x^6 + 10x^5 + 15x^4 - 6x^3 - 7x^2 + x + 1$

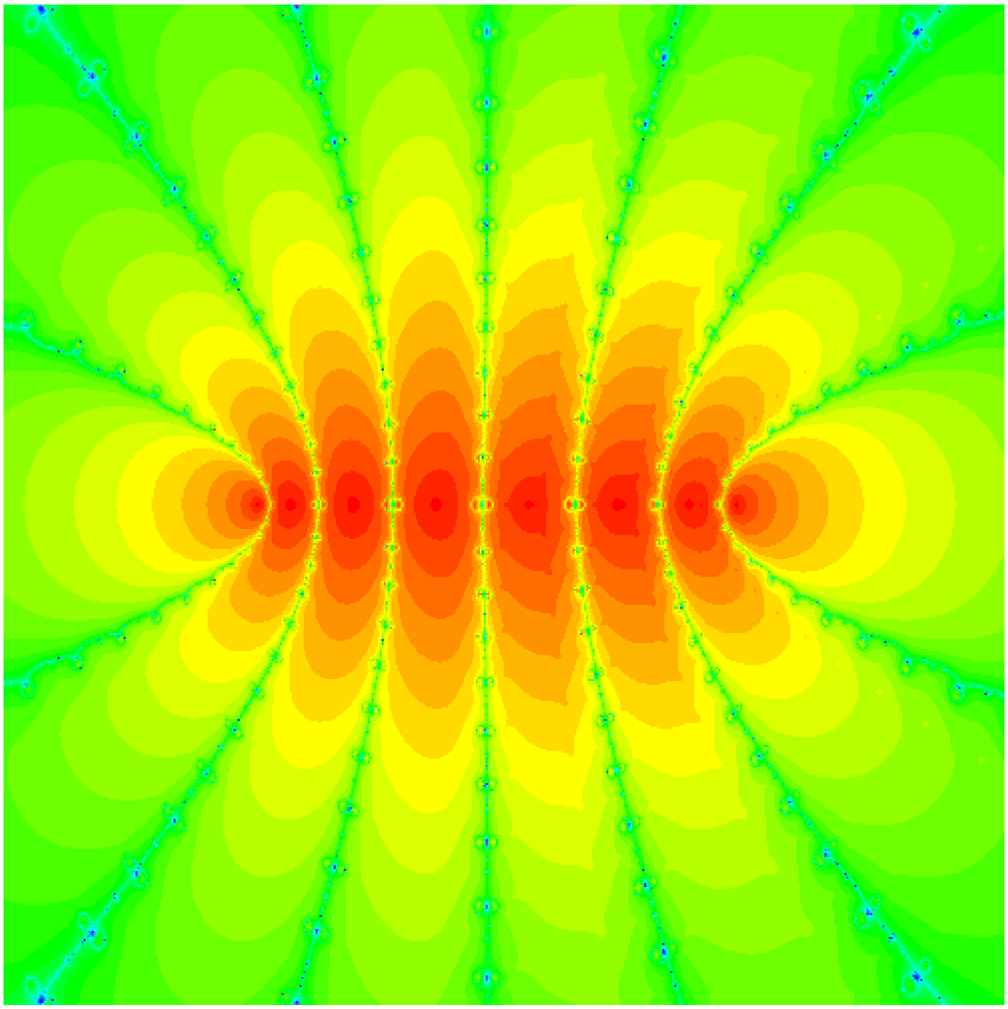

Fig.3. $p(z) = z^8 + z^7 - 7z^6 - 6z^5 + 15z^4 + 10z^3 - 10z^2 - 4z + 1$

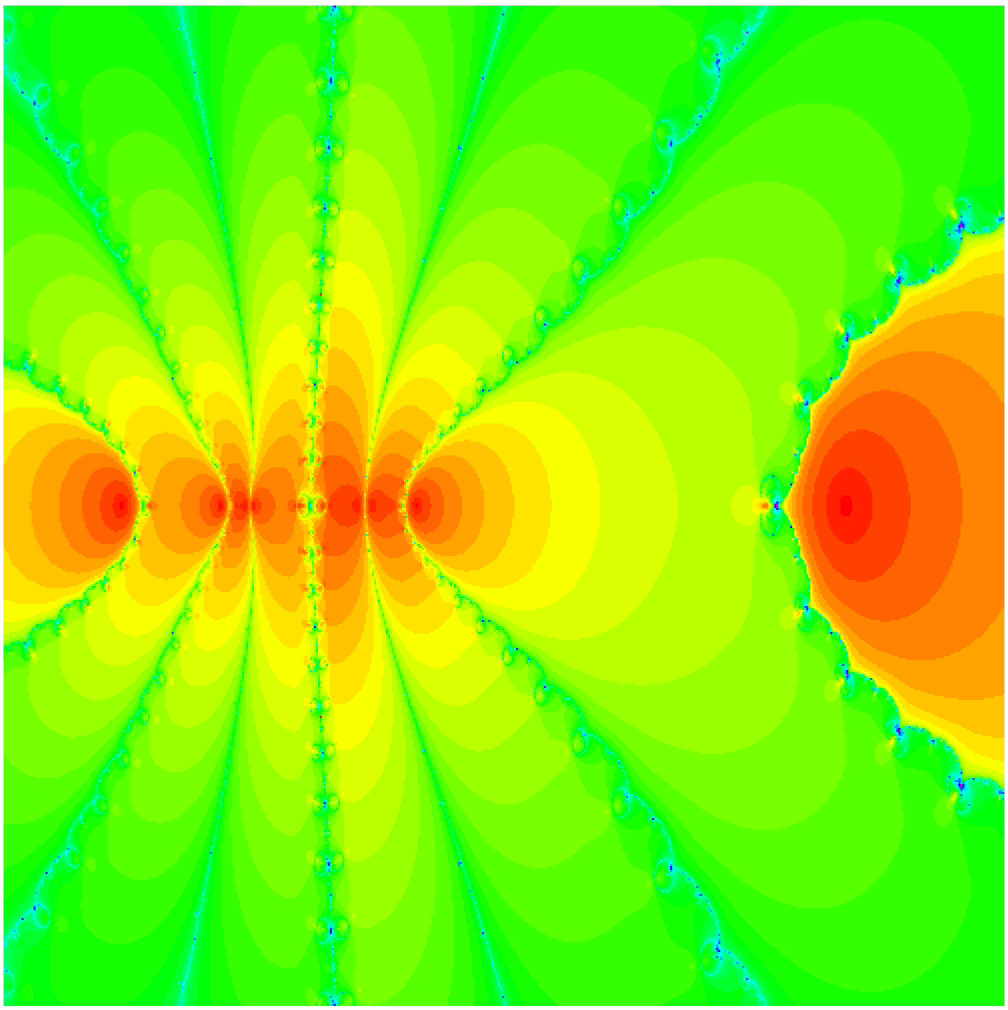

Fig.4. $q(z) = z^8 p\left(\frac{1}{z}\right) = z^8 - 4z^7 - 10z^6 + 10z^5 + 15z^4 - 6z^3 - 7z^2 + z + 1$

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