Identification of the Algebra of Consciousness

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Abstract : After an articulated exposition of the basic features of the Clifford algebra we give evidence that the basic elements of this algebra may represent the basic entities of the mind. According also to the previous basic results of V.A. Lefebvre on conscience , we delineate also some peculiarities of the consciousness and we give proof that they may be correctly represented by this algebra.

Keywords : consciousness, Clifford algebra, quantum cognition, logical origin of quantum mechanics, computational neuroscience, algebra of consciousness.

Consciousness is an abstract identity marked from several and unique features but mainly is marked from two basic salient and peculiar properties.

(-) It is an entity that has self-awareness and this is to say that it has in its inner the image of itself. In most cases we speak of self-image to represent such peculiar feature.

(-) The other marking property is that it has awareness of an external space -time located abstract entity.

Every one is convinced that it is extremely difficult to conceive and to represent a system existing in Nature and having such self-referential properties and this is the reason because we mark the problem to represent the consciousness from several and several years of activity as the basic " hard problem ".

Constantly the hard problem involves the interest of biological, medical and, in general, Life Sciences, it is a basic problem in neurology as well as in science of mind here including the tentative to approach this problem under the field of the philosophy and of physics. Physicists usually approach the problem under a particular perspective: may we use the basic foundations of physics to explain the two previous mentioned peculiar features of consciousness ? In detail, have we the mathematical instruments that pertaining to physical formulation, are able to approach the hard problem giving explanation of the previous mentioned basic features?

The list of physicists who have engaged in this hard problem is of course endless but we acknowledge, in particular, one scholar who in years of activity has conducted studies and has given us some fundamental indications and results. This scientist is V.A. Lefebvre who, in fact, has culminated his activity with a celebrated book entitled "The Algebra of the Conscience ". Here we consider the algebra of consciousness: a tentative to indicate us that consciousness may be described by an algebra and thus by a mathematical tool.

This is precisely the question that we attempt to develop in this paper, to describe for the first time this algebra, able to delineate the two basic peculiar features of consciousness that we have previously indicated.

Let us start to present this algebra.

Let us start with a proper definition of the 3-D space Clifford (geometric) algebra Cl_3.

It is an associative algebra generated by three vectors e_1, e_2, and e_3 that satisfy the orthonormality relation

$$e_j e_k + e_k e_j = 2\delta_{jk} \qquad \text{for } j,k,\lambda \in [1,2,3] \qquad (1.1)$$

That is,

$e_\lambda^2 = 1$ and $e_j e_k = -e_k e_j$ for $j \neq k$

Let **a** and **b** be two vectors spanned by the three unit spatial vectors in $Cl_{3,0}$. By the orthonormality relation the product of these two vectors is given by the well known identity: $ab = a \cdot b + i(a \times b)$ where $i = e_1 e_2 e_3$ is a Clifford algebraic representation of the imaginary unity that commutes with vectors.

The (1.1) are well known in quantum mechanics. Here we give proof under an algebraic profile. Let us follow the approach that, starting with 1981, was developed by Y. Ilamed and N. Salingaros [1]. Imaeda and Edmonds also used extensively this algebra in the past [2,3].

Let us admit that the three abstract basic elements, e_i, with $i = 1,2,3$ admit the following two postulates:

a) it exists the scalar square for each basic element:
$$e_1 e_1 = k_1,\ e_2 e_2 = k_2,\ e_3 e_3 = k_3 \text{ with } k_i \in \Re. \qquad (1.2)$$

In particular we have also the unit element, e_0, such that that

$e_0 e_0 = 1$, and $e_0 e_i = e_i e_0$

b) The basic elements e_i are anticommuting elements, that is to say:
$$e_1 e_2 = -e_2 e_1,\qquad e_2 e_3 = -e_3 e_2,\ e_3 e_1 = -e_1 e_3. \qquad (1.3)$$

Theorem n.1.

Assuming the two postulates given in (a) and (b) with $k_i = 1$, the following commutation relations hold for such algebra :

$$e_1 e_2 = -e_2 e_1 = ie_3;\ e_2 e_3 = -e_3 e_2 = ie_1;\ e_3 e_1 = -e_1 e_3 = ie_2;\ i = e_1 e_2 e_3,\ (e_1^2 = e_2^2 = e_3^2 = 1) \qquad (1.4)$$

They characterize the Clifford Si algebra. We will call it the algebra A(Si).

Proof.

Consider the general multiplication of the three basic elements e_1, e_2, e_3, using scalar coefficients $\omega_k, \lambda_k, \gamma_k$ pertaining to some field:

$$e_1 e_2 = \omega_1 e_1 + \omega_2 e_2 + \omega_3 e_3 \quad ;\ e_2 e_3 = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3\ ;$$

$$e_3 e_1 = \gamma_1 e_1 + \gamma_2 e_2 + \gamma_3 e_3 . \tag{1.4a}$$

Let us introduce left and right alternation: for any (i,j), associativity exists $e_i e_i e_j = (e_i e_i) e_j$ and $e_i e_j e_j = e_i (e_j e_j)$ that is to say

$$e_1 e_1 e_2 = (e_1 e_1) e_2 ; \quad e_1 e_2 e_2 = e_1 (e_2 e_2) ; \quad e_2 e_2 e_3 = (e_2 e_2) e_3 ; \quad e_2 e_3 e_3 = e_2 (e_3 e_3) ; \quad e_3 e_3 e_1 = (e_3 e_3) e_1 ;$$
$$e_3 e_1 e_1 = e_3 (e_1 e_1) . \tag{1.5}$$

Using the (1.4) in the (1.5) it is obtained that

$$k_1 e_2 = \omega_1 k_1 + \omega_2 e_1 e_2 + \omega_3 e_1 e_3 ; \qquad k_2 e_1 = \omega_1 e_1 e_2 + \omega_2 k_2 + \omega_3 e_3 e_2 ;$$

$$k_2 e_3 = \lambda_1 e_2 e_1 + \lambda_2 k_2 + \lambda_3 e_2 e_3 ; \qquad k_3 e_2 = \lambda_1 e_1 e_3 + \lambda_2 e_2 e_3 + \lambda_3 k_3 ;$$

$$k_3 e_1 = \gamma_1 e_3 e_1 + \gamma_2 e_3 e_2 + \gamma_3 k_3 ; \qquad k_1 e_3 = \gamma_1 k_1 + \gamma_2 e_2 e_1 + \gamma_3 e_3 e_1 . \tag{1.6}$$

From the (1.6), using the assumption (b), we obtain that

$$\frac{\omega_1}{k_2} e_1 e_2 + \omega_2 - \frac{\omega_3}{k_2} e_2 e_3 = \frac{\gamma_1}{k_3} e_3 e_1 - \frac{\gamma_2}{k_3} e_2 e_3 + \gamma_3 ;$$

$$\omega_1 + \frac{\omega_2}{k_1} e_1 e_2 - \frac{\omega_3}{k_1} e_3 e_1 = -\frac{\lambda_1}{k_3} e_3 e_1 + \frac{\lambda_2}{k_3} e_2 e_3 + \lambda_3 ;$$

$$\gamma_1 - \frac{\gamma_2}{k_1} e_1 e_2 + \frac{\gamma_3}{k_1} e_3 e_1 = -\frac{\lambda_1}{k_2} e_1 e_2 + \lambda_2 + \frac{\lambda_3}{k_2} e_2 e_3 \tag{1.7}$$

We have that it must be

$$\omega_1 = \omega_2 = \lambda_2 = \lambda_3 = \gamma_1 = \gamma_3 = 0 \tag{1.8}$$

and

$$-\lambda_1 k_1 + \gamma_2 k_2 = 0 \qquad \gamma_2 k_2 - \omega_3 k_3 = 0 \qquad \lambda_1 k_1 - \omega_3 k_3 = 0 \tag{1.9}$$

The following set of solutions is given:

$$k_1 = -\gamma_2 \omega_3 , \; k_2 = -\lambda_1 \omega_3 \; , k_3 = -\lambda_1 \gamma_2 \tag{1.10}$$

that is to say

$$\omega_3 = \lambda_1 = \gamma_2 = i \tag{1.11}$$

In this manner, as a theorem, the existence of such algebra is proven. The basic features of this algebra are given in the following manner

$$e_1^2 = e_2^2 = e_3^2 = 1 ; e_1 e_2 = -e_2 e_1 = i e_3 ; e_2 e_3 = -e_3 e_2 = i e_1 ; e_3 e_1 = -e_1 e_3 = i e_2 ; i = e_1 e_2 e_3 \tag{1.12}$$

The content of the theorem n.1 is thus established: given three abstract basic elements as defined in (a) and (b) ($k_i = 1$) , an algebraic structure is given with four generators (e_0, e_1, e_2, e_3).

Note that in the algebra A(Si) the $e_i \,(i = 1,2,3)$ have an intrinsic potentiality that is to say an ontic potentiality or equivalently an irreducible intrinsic indetermination. Since $e_i^2 = 1 \,(i = 1,2,3)$, we may think to attribute them or the numerical value +1 or the numerical value –1 . Both such alternatives (+1 and -1) both coexist ontologically.

A generic member of our algebra A(Si) is given by

$$x = \sum_{i=0}^{4} x_i e_i = x_0 + \mathbf{x} \tag{1.13}$$

with x_i pertaining to some field $\Re$ or C .

We may define [2] the hyperconjugate $\hat{x}$

$$\hat{x} = x_0 - \mathbf{x}$$

the complex conjugate

$$x^* = x_0^* + \mathbf{x}^\circ$$

and the conjugate

$$\bar{x} = x_0^* - \mathbf{x}^*$$

The *Norm* of x is defined as

$$\textit{Norm}\,(\mathrm{x}) = x\,\hat{x} = \hat{x}\,x = x_0^2 - x_1^2 - x_2^2 - x_3^2 \tag{1.14}$$

with

Norm (xy) = Norm (x) Norm (y)

The proper inverses of the basic elements $e_i \,(i = 1,2,3)$ are themselves. Given the member x, its inverse x^{-1}

is $\hat{x}$ / *Norm* (x) with *Norm* (x) $\neq 0$

We may transform Clifford members according to Linear Transformations

$$x' = AxB + C \tag{1.14a}$$

with unitary norms for the employed Clifford members A, B and $C = 0$ for linear homogeneous transformation.

Let us now take a step on.

As previously said, in the algebra A (Si) the e_i ($i=1,2,3$) have an intrinsic potentiality that is to say an ontic potentiality or equivalently an irreducible intrinsic indetermination. Since $e_i^2=1$ ($i=1,2,3$), we may think to attribute them or the numerical value +1 or the numerical value –1. Let us give proof of such our basic assumption.

Since the e_i are abstract entities, having the potentiality that we may think to attribute them the numerical values, ± 1, they have an intrinsic and irreducible indetermination. Therefore, we may admit to be $p_1(+1)$ the probability that e_1 assumes the value $(+1)$ and $p_1(-1)$ the probability that it assumes the value -1. We may represent the mean value that is given by

$$< e_1 >= (+1)p_1(+1)+(-1)p_1(-1) \quad (1.15)$$

Considering the same corresponding notation for the two remaining basic elements, we may introduce the other two mean values:

$$< e_2 >= (+1)p_2(+1)+(-1)p_2(-1), \quad (1.16)$$

$$< e_3 >= (+1)p_3(+1)+(-1)p_3(-1).$$

We have

$$-1 \le < e_i > \le +1 \quad i=(1,2,3) \quad (1.17)$$

Selected the following generic element of the algebra A(Si):

$$x=\sum_{i=1}^{3} x_i e_i \qquad x_i \in \Re \quad (1.18)$$

Note that

$$x^2 = x_1^2 + x_2^2 + x_3^2 \quad (1.19)$$

Its mean value results to be

$$< x >= x_1 < e_1 > + x_2 < e_2 > + x_3 < e_3 > \quad (1.20)$$

Let us call

$$b=(x_1^2+x_2^2+x_3^2)^{1/2} \quad (1.21)$$

so that we may attribute to x the value $+b$ or $-b$

We have that

$$-b \le x_1 < e_1 > + x_2 < e_2 > + x_3 < e_3 > \le b \quad (1.22)$$

The (1.22) must hold for any real number x_i, and, in particular, for

$$x_i =< e_i >$$

so that we have that

$$x_1^2 + x_2^2 + x_3^2 \leq b$$

that is to say

$$b^2 \leq b \quad \rightarrow b \leq 1$$

so that we have the fundamental relation

$$< e_1 >^2 + < e_2 >^2 + < e_3 >^2 \leq 1 \qquad (1.23)$$

These results were also previously obtained by Jordan but not using Clifford algebra [4]. Our results are contained in [5-34] where we mention also the contributions of also authors that inspired our work as in particular Altafini , Orlov , Cini.

This is a basic relation of irreducible indetermination that we are writing in our Clifford algebraic elaboration.

Let us observe some important features:

(a) In absence of numerical attribution to the e_i (and in analogy with physics this means ….in absence of a measurement, that is to say in absence of direct observation of one quantum observable), the (1.23) holds. If we attribute instead a definite numerical value to one of the three entities, as example we attribute to e_3 the numerical value +1, we have $< e_3 >= 1$, and the (1.23)) is reduced to

$$< e_1 >^2 + < e_2 >^2 = 0 , < e_1 >=< e_2 >= 0 , \qquad (1.24)$$

and we have complete, irreducible, indetermination for e_1 and for e_2 .

(b) Finally, the (1.23) affirms that we never can attribute simultaneously definite numerical values to two basic non commutative elements e_i .

We may now summarize the obtained results. First, we retain that the first axiom of the $A(Si)$ algebra, the (1.2) with $k_i = 1$, indicates that the abstract basic elements e_i have an ontic potentiality, that is to say that they have an irreducible indeterminism as supported finally from the (1.23). In order to characterize such features we have introduced the concept of mean value for such algebraic entities and, consequently, that one of potentiality. When we attempt to attribute a numerical values to an abstract element, as it happens as example, in the (1.24), we perform an operation that in physics has a counterpart that is called an act of measurement. For us, any measurement is a semantic act. No matter if the measurement is performed by a technical instrument or by an human observer. In any case it is realized having at its basic arrangement, a semantic act. If we remain in the restricted domain of the $A(Si)$, we are in some sense in a condition that, on the general plane, may be assimilated to that one in which we have human or technical systems that are in some manner forced to answer to questions (the attribution of numerical values to the basic elements) which they cannot understand in line of principle. As consequence, the probabilities that we have used in the (1.15) and in the (1.16) are fundamentally different from classical probabilities under a basic conceptual profile. In classical probability theory, as it is known, probabilities represent a lack of information about preexisting and pre-established properties of systems .In the present case we have instead a situation in which we have not an algorithm in $A(Si)$ to execute a semantic act devoted to identify the meaning of a statement in terms of truth values and in relation to another statement. So we need to introduce probabilities that pertain now not to a missing our knowledge but to basic intrinisc foundation of irreducible indetermination in the inner structure of our reality.

Let us evidence another important feature of Clifford algebra $A(Si)$.

In Clifford algebra $A(Si)$ we have idempotents (as counterpart we have projection operators in quantum mechanics). In von Neumann language projection operators can be interpreted as logical statements.

Let us give some example of idempotents in Clifford algebra.

It is well known the central role of density matrix in traditional quantum mechanics . In the Clifford algebraic scheme, we have a corresponding algebraic member that is given in the following manner

$$\rho = a + be_1 + ce_2 + de_3 \tag{2.1}$$

with

$$a = \frac{|c_1|^2}{2} + \frac{|c_2|^2}{2},\ b = \frac{c_1^* c_2 + c_1 c_2^*}{2},\ c = \frac{i(c_1 c_2^* - c_1^* c_2)}{2}, \quad d = \frac{|c_1|^2 - |c_2|^2}{2} \tag{2.2}$$

where the e_i are the basic elements in our algebraic Clifford scheme while in matrix notation, e_1, e_2, and e_3 in standard quantum mechanics are the well known Pauli matrices. The complex coefficients c_i $(i = 1,2)$ are the well known probability amplitudes for the considered quantum state

$$\psi = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \quad \text{and} \quad |c_1|^2 + |c_2|^2 = 1 \tag{2.3}$$

For a pure state in quantum mechanics it is $\rho^2 = \rho$. In our scheme a theorem may be demonstrated in Clifford algebra. It is that

$$\rho^2 = \rho \leftrightarrow a = \frac{1}{2} \text{ and } a^2 = b^2 + c^2 + d^2 \tag{2.4}$$

The details of this our theorem are given in references . We have also $Tr(\rho) = 2a = 1$. In this manner we have the necessary and sufficient conditions for ρ to represent a Clifford member whose counterpart in standard quantum mechanics represents a potential state or, equivalently, a superposition of states.

Let us consider still other two of such idempotents in $A(Si)$

$$\psi_1 = \frac{1 + e_3}{2} \quad \text{and} \quad \psi_2 = \frac{1 - e_3}{2} \tag{2.5}$$

It is easy to verify that $\psi_1^2 = \psi_1$ and $\psi_2^2 = \psi_2$.

Let us examine now the following algebraic relations:

$$e_3\psi_1 = \psi_1 e_3 = \psi_1 \quad (2.6)$$

$$e_3\psi_2 = \psi_2 e_3 = -\psi_2 \quad (2.7)$$

Similar relations hold in the case of e_1 or e_2.

Here is one central aspect of the present paper. By a pure semantic act, looking at the (2.6) and (2.7), we reach only a conclusion. With reference to the idempotent ψ_1, the algebra A(Si) (see the (2.6)), attributes to e_3 the numerical value of $+1$ while, with reference to the idempotent ψ_2, the algebra A(Si) attributes to e_3 (see the (2.7)), the numerical value of -1 .

The basic point is that at the basis we have a semantic act .

However, assuming the attribution $e_3 \to$ +1, from the (1.4) we have that new commutation relations should hold in a new Clifford algebra , given in the following manner:

$$e_1^2 = e_2^2 = 1, \quad i^2 = -1; \qquad e_1e_2 = i, \qquad e_2e_1 = -i, \qquad e_2i = -e_1,$$

$$ie_2 = e_1, \qquad e_1i = e_2, \qquad ie_1 = -e_2 \quad (2.8)$$

with three new basic elements (e_1, e_2, i) instead of (e_1, e_2, e_3) .

We totally agree with the possible criticism that such our argument to express the (2.8) on the basis of a rough attribution to e_3 with our mind may be in itself very rough, and, in any case, only pertaining, still again, a pure semantic operation . This is what we intend to evidence with the greatest emphasis. We have performed only a SEMANTIC ACT pertaining to cognition of our mind. I have realid a logic statemnet as it corresponds roughly in our mind . Actually we are admitting that in the case in which we attribute to e_3 the numerical value +1, roughly , this is to say : considering e_3 as a pure symbol a new algebraic structure should arise with new generators whose rules should be given in (2.8) instead of the (1.4). Therefore, the arising central problem is that we should be able to proof the real existence of such new algebraic structure with rules given in the (2.8). We repeat: in the case of the starting algebraic structure, the algebra A(Si), we showed by theorem that it exists with its proper rules:

$$e_1^2 = e_2^2 = e_3^2 = 1;$$

$$e_1e_2 = -e_2e_1 = ie_3\ ;\ e_2e_3 = -e_3e_2 = ie_1;\ e_3e_1 = -e_1e_3 = ie_2\ ; i = e_1e_2e_3 \quad (2.9)$$

In the present case in which we attribute to e_3 the numerical value +1, and we do this operation using only our mind and in particular our cognition, we should demonstrate that really it exists a new algebra given in the following manner

$$e_1^2 = e_2^2 = 1;\ i^2 = -1;$$

$$e_1e_2 = i\ ,\ e_2e_1 = -i\ ,\ e_2i = -e_1, ie_2 = e_1, e_1i = e_2\ ,\ ie_1 = -e_2 \quad (2.10)$$

If we arrive to demonstrate that such algebraic structure certainly exists in the field of the Clifford algebra , , we have given for the first time demonstration and confirmation that we may have a representation of our mental activity whose counterpart is an universal theorem. In brief , the important result is that for the first time we have obtained a representation of our mental operations and we have shown that its counterpart is characterize by an universal theorem. We have an algebraic representation of our mental activity. Obviously such theorem must hold also in the other case in which we attribute to e_3 the numerical value -1 in our mind. If such theorem exists, we will call it the theorem n.2

Let us go to give proof of the existing theorem n.2.

First of all we have to emphasize once again that we are attributing to the previous Clifford basic element e_3 a numerical value only on the basis of a semantic act. Consequently we are reasoning only of basic abstract entities of our mind not of material objects . Let us go to demonstrate the real existence of the theorem 2 .

Theorem n.2 .

Assuming the postulates given in (a) and (b) with $k_1 = 1$, $k_2 = 1$, $k_3 = -1$, the following commutation rules hold for such new algebra:

$e_1^2 = e_2^2 = 1$; $i^2 = -1$;

$e_1 e_2 = i$, $e_2 e_1 = -i$, $e_2 i = -e_1$, $ie_2 = e_1$, $e_1 i = e_2$, $ie_1 = -e_2$ **(**2.11)

They characterize the Clifford Ni algebra. We will call it the algebra $N_{i,+1}$

Proof

To give proof, rewrite the (1.4a) in our case, and performing step by step the same calculations of the previous proof, we arrive to the solutions of the corresponding homogeneous algebraic system that in this new case are given in the following manner:

$k_1 = -\gamma_2 \omega_3$; $k_2 = -\lambda_1 \omega_3$; $k_3 = -\lambda_1 \gamma_2$ (2.12)

where this time it must be $k_1 = k_2 = +1$ and $k_3 = -1$. It results

$\lambda_1 = -1$; $\gamma_2 = -1$; $\omega_3 = +1$ (2.13)

and the proof is given.

The content of the theorem n.2 is thus established. **When we attribute to e_3 the numerical value +1 as a semantic act of our mind, we pass from the Clifford algebra $A(Si)$ to a new Clifford algebra $N_{i,+1}$ whose algebraic structure is no more given from the (2.9) of the algebra $A(Si)$ but from the following new basic rules:**

$e_1^2 = e_2^2 = 1$; $i^2 = -1$;

$e_1 e_2 = i$, $e_2 e_1 = -i$, $e_2 i = -e_1$, $ie_2 = e_1$, $e_1 i = e_2$, $ie_1 = -e_2$ (2.14)

The theorem n.2 also holds in the case in which we attribute to e_3 the numerical value of -1 .

Assuming the postulates given in (a) and (b) with $k_1 = 1$, $k_2 = 1$, $k_3 = -1$, the following commutation rules hold for such new algebra

$$e_1^2 = e_2^2 = 1\,; \; i^2 = -1\,;$$

$$e_1 e_2 = -i \;,\; e_2 e_1 = i\,,\; e_2 i = e_1\,, i e_2 = -e_1\,, e_1 i = -e_2\,,\; i e_1 = e_2 \tag{2.15}$$

They characterize the Clifford Ni algebra. We will call it the algebra $N_{i,-1}$

To give proof , consider the solutions of the (2.12) that are given in this new case by

$$\lambda_1 = +1\,; \gamma_2 = +1;\; \omega_3 = -1 \tag{2.16}$$

and the proof is given.

The content of the theorem n.2 is thus established. When we attribute to e_3 the numerical value –1, we pass from the Clifford algebra $A(Si)$ to a new Clifford algebra $N_{i,-1}$ whose algebraic structure is not given from the (2.9) of the algebra $A(Si)$ and not even from the (2.14) but from the following new basic rules:

$$e_1^2 = e_2^2 = 1\,; \; i^2 = -1\,;$$

$$e_1 e_2 = -i \;,\; e_2 e_1 = i\,,\; e_2 i = e_1\,, i e_2 = -e_1\,, e_1 i = -e_2\,,\; i e_1 = e_2 \tag{2.17}$$

In a similar way, proofs may be obtained when we consider the cases attributing numerical values (± 1) to e_1 or to e_2 .

The Clifford algebra, $N_{1,\pm 1}$, given in the (2.15) and in the (2.17) are the dihedral Clifford algebra N_i **.**

In conclusion, we have shown two basic theorems, the theorem n.1 and the theorem n.2. As any mathematical theorem they have maximum rigour, and an aseptic mathematical content that cannot be questioned. The basic statement that we reach by the proof of such two theorems is that in Clifford algebraic framework, we have the Clifford algebra A(Si) and inter-related Clifford algebras $N_{i,\pm 1}$. When we consider (e_1, e_2, e_3) as the three abstract elements with rules given in (2.9) , we are in the Clifford algebra A(Si) .When we attribute with our mind to e_3 the numerical value +1, we pass from the algebra $A(Si)$ to the Clifford algebra $N_{i,+1}$. Instead, when we pass from the Clifford algebra $A(Si)$ to the Clifford algebra $N_{i,-1}$, we are attributing to e_3 the numerical value –1 .

The same conceptual facts hold when we reason for Clifford basic elements e_1 or to e_2 , attributing in this case a possible numerical value (± 1) or to e_1 or to e_2 , respectively.

The basic conclusion is the following: for the first time we have considered three basic abstract Clifford elements . We have verified that using such abstract elements we may perform what we have called semantic acts. Semantic acts relate cognition. We cannot escape the conclusion that we are considering mind entities. We have identified that mental entities may be represented by a proper algebraic structure.

The first great objective of the present paper has been reached.

Obviously the implications of such shown theorems for the measurement problem in quantum mechanics are of relevant interest.

If one looks at the algebraic rules and commutation relations given in the (2.9), the algebra $A(Si)$ immediately remembers that they are universally valid in quantum mechanics. It links the Pauli matrices that are sovereign in quantum theory. Still the isomorphism between Pauli matrices and Clifford algebra $A(Si)$ is well established at any order.

Passing from the algebra A(Si) to $N_{i,\pm 1}$ it happens an interesting feature. Consider the case, as example, of e_3. While in $A(Si)$ e_3 is an abstract algebraic element that has the potentiality to assume or the value +1 or the value –1 (in correspondence, in quantum mechanics it is an operator with possible eigenvalues ± 1), when we pass in the algebra $N_{i,\pm 1}$, e_3 is no more an abstract element in this algebra, it becomes a parameter to which we may attribute the numerical value +1, and we have $N_{i,+1}$ whose three abstract element now are (e_1, e_2, i) with commutation rules given in the (2.14). If we attribute to e_3 the numerical value -1, we are in $N_{i,-1}$ whose three abstract elements are still (e_1, e_2, i), and the commutation rules are given in (2.17). Reading this statement in the language and in the framework of a quantum mechanical measurement, it means that if we are measuring the given quantum system S with a measuring apparatus and, as result of the actualized and performed measurement, we read the result +1, we are in the corresponding algebraic case, in the algebra $N_{i,+1}$. If instead, performing the measurement, we read the result –1, in this case we are in the algebra $N_{i,-1}$. In each of the two cases this means that a collapse of the wave function has happened.

During a process of quantum measurement, speaking in terms of Clifford algebraic framework, we could have the passage from the Clifford algebra A(Si), in the case in which the result of the measurement of e_3 is +1 (read on the instrument), and instead we could have the passage to the new $N_{i,-1}$ Clifford algebra, in the case in which the result of the quantum measurement of e_3 gives value –1 (read on the instrument).

In such way it seems that a reformulation of von Neumann's projection postulate may be suggested. The reformulation is that, during a quantum measurement (wave-function collapse), we have the passage from the Clifford algebra A(Si), to the new Clifford algebra $N_{i,\pm 1}$. In brief :

Quantum Measurement at a cognitive level (wave-function collapse) = passage from algebra $A(Si)$ to $N_{i,\pm 1}$.

In conclusion we think that the two previously shown theorems in Clifford algebraic framework give justification of the von Neumann's projection postulate and they seem to suggest, in addition, that we may use the passage from the algebra $A(Si)$ to $N_{i,\pm 1}$ to describe actually performed quantum measurements.

A detailed exposition of such results has been discussed by us in papers given in references but we may discuss still here some illustrative examples.

Let us start discussing a preliminary application.

Assume a two –level microscopic quantum system S with two states u_+, u_- corresponding to energy eigenvalues ε_+, ε_-. The Hamiltonian operator H_S can be written

$$H_S = \frac{1}{2}\varepsilon_+(1+e_3) + \frac{1}{2}\varepsilon_-(1-e_3) = \frac{1}{2}(\varepsilon_+ + \varepsilon_-) + \frac{1}{2}(\varepsilon_+ - \varepsilon_-)e_3 \qquad (2.18)$$

In the standard quantum methodological approach we have that

$$u_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \quad u_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad , \text{ and } \quad H_S u_i = \varepsilon_i u_i \, . \qquad (2.19)$$

We may also choose $\varepsilon_+ = \varepsilon$ and $\varepsilon_- = 0$ simplifying the (2.18) to

$$H_S = \frac{1}{2}(1+e_3)\varepsilon \qquad (2.20)$$

Indicate an arbitrary state of such quantum microsystem as

$$\psi_S = c_+ u_+ + c_- u_- \qquad (2.21)$$

where, according to Born's rule, we have

$$c_+ = \sqrt{p_+}e^{i\delta_1} \quad , \quad c_- = \sqrt{p_-}e^{i\delta_2} \quad (2.22)$$

$$p_j \ (j = +,-) \qquad (2.23)$$

corresponding probabilities with $p_+ + p_- = 1$.

This is the standard quantum mechanical formulation of the system.

Let us admit now that we want to measure the energy of S using a proper apparatus . The rules of quantum mechanics tell us that we will obtain the value ε with probability p_+, and the value zero with probability p_-. After the measurement the state of S will be either u_+ or u_- according to the measured value of the energy. The experiment will enable us also to estimate p_+ as well as p_-.

In such simple quantum mechanical example we have, as known, the (2.18)), e_3, the (2.20) that are linear Hermitean operators with quantum states acting on the proper Hilbert space.

Let us see instead the question from our Clifford algebraic point of view.

The e_3, and H_S given in the (2.18) or in the (2.20) are members of the $A(Si)$ Clifford algebra with basic rules $e_1^2 = e_2^2 = e_3^2 = 1$

$$e_1e_2 = -e_2e_1 = ie_3 \ ; \ e_2e_3 = -e_3e_2 = ie_1 \ ; \ e_2e_3 = -e_3e_2 = ie_1 \ ; \ i = e_1e_2e_3 \qquad (2.24)$$

However, on the basis of theorems n.1 and n.2 shown in the previous sections, starting with the Clifford algebra A(Si), we must use the existing Clifford, dihedral algebra, $N_{i,\pm 1}$ when we arrive to attribute (by a measurement) as example to e_3 in one case the numerical value +1 and, in the other case, the numerical value –1.

In the first case we have a dihedral Clifford N_i algebra that is given in the following manner:

$$e_1^2 = e_2^2 = 1 \; i^2 = -1$$

$$e_1 e_2 = i \;\;,\;\; e_2 e_1 = -i \;,\;\; e_2 i = -e_1 \;,\;\; e_2 i = -e_1 \;,\;\; ie_2 = e_1 \;,\;\; e_1 i = e_2 \;,\;\; ie_1 = -e_2 \qquad (2.25)$$

attributing to e_3 the numerical value +1 (in analogy with quantum mechanics: the quantum measurement process has given as result +1). In the second case, we have instead that

$$e_1^2 = e_2^2 = 1 \,; \;\; i^2 = -1 \,;$$

$$e_1 e_2 = -i \;,\; e_2 e_1 = i \,,\; e_2 i = e_1 \,, ie_2 = -e_1 \,, e_1 i = -e_2 \,,\; ie_1 = e_2 \qquad (2.26)$$

that holds when we have arrived to attribute to e_3 the numerical value –1 by a direct measurement.

Reasoning in terms of a Clifford algebraic framework, we are authorized to apply the passage from algebra A(Si) to algebra $N_{i,\pm 1}$ in the (2.18). From it , we obtain:

$$H_{S(Clifford-element)} = \varepsilon_+ \qquad (2.27)$$

if the instrument has given as result of the measurement, the value +1 to e_3 (Clifford algebraic parameter of dihedral $N_{i,+1}$ algebra), and

$$H_{S(Clifford-element)} = \varepsilon_- \qquad (2.28)$$

In the first case, we have

$$H_{S(Clifford-element)} = \varepsilon \qquad (2.29)$$

and in the second case, we have

$$H_{S(Clifford-element)} = 0 \qquad (2.30)$$

Consider now the second application .

Let us introduce a two state quantum system S with connected quantum observable $\sigma_3(e_3)$. We have

$$\left|\psi\right\rangle = c_1\left|\varphi_1\right\rangle + c_2\left|\varphi_2\right\rangle \;\;,\;\; \varphi_1 = \begin{pmatrix}1\\0\end{pmatrix} \;\;,\;\; \varphi_2 = \begin{pmatrix}0\\1\end{pmatrix} \qquad (2.31)$$

and

$$\left|c_1\right|^2 + \left|c_2\right|^2 = 1$$

As we know, the density matrix of such system is easily written

$$\rho = a + be_1 + ce_2 + de_3 \qquad (2.32)$$

with

$$a = \frac{|c_1|^2 + |c_2|^2}{2}\ ,\ b = \frac{c_1^* c_2 + c_1 c_2^*}{2}\ ,\ c = \frac{i(c_1 c_2^* - c_1^* c_2)}{2}\ ,\ d = \frac{|c_1|^2 - |c_2|^2}{2} \tag{2.33}$$

where in matrix notation, e_1 , e_2, and e_3 are the well known Pauli matrices

$$e_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ e_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ e_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.34}$$

Of course, the analogy still holds. The (2.32) is still an element of the $A(Si)$ Clifford algebra. As Clifford algebraic member, the (2.32) satisfies the requirement to be $\rho^2 = \rho$ and Tr($\rho) = 1$ under the conditions $a = 1/2$ and $a^2 - b^2 - c^2 - d^2 = 0$ as we evidenced in the (2.4) . In the algebraic framework, let us admit that we attribute to e_3 the value +1 (that is to say … the quantum observable σ_3 assumes the value +1 during quantum measurement) or to e_3 the numerical value –1 (that is to say… the quantum observable σ_3 assumes the value –1 during the quantum measurement). As previously shown, in such two cases the algebra A, (Si) no more holds, and it will be replaced from the Clifford $N_{i,\pm 1}$. To examine the consequences, starting with the algebraic element (2.32), write it in the two equivalent algebraic forms that are obviously still in the algebra A(Si).

$$\rho = \frac{1}{2}(|c_1|^2 + |c_2|^2) + \frac{1}{2}(c_1 c_2^*)(e_1 + e_2 i) + \frac{1}{2}(c_1^* c_2)(e_1 - ie_2) + \frac{1}{2}(|c_1|^2 - |c_2|^2)e_3 \tag{2.35}$$

and

$$\rho = \frac{1}{2}(|c_1|^2 + |c_2|^2) + \frac{1}{2}(c_1 c_2^*)(e_1 + ie_2) + \frac{1}{2}(c_1^* c_2)(e_1 - e_2 i) + \frac{1}{2}(|c_1|^2 - |c_2|^2)e_3 \tag{2.36}$$

Both such expressions contain the following interference terms.

$$\frac{1}{2}(c_1 c_2^*)(e_1 + e_2 i) + \frac{1}{2}(c_1^* c_2)(e_1 - ie_2) \tag{2.37}$$

and

$$\frac{1}{2}(c_1 c_2^*)(e_1 + ie_2) + \frac{1}{2}(c_1^* c_2)(e_1 - e_2 i) \tag{2.38}$$

Let us consider now that the quantum measurement gives as result +1 for e_3. In this case there are the (2.35) and the (2.37) that we must take in consideration. On the basis of our principle, we know that the previous Clifford algebra A(Si) no more holds, but instead it is valid the $N_{1,+1}$ that has the following new commutation rules:

$$e_1 e_2 = i\ ,\ e_2 e_1 = -i\ ,\ e_2 i = -e_1, ie_2 = e_1, e_1 i = e_2\ ,\ ie_1 = -e_2 \tag{2.39}$$

Inserting such new commutation rules in the (2.35) and in the (2.36), the interference terms are erased and the density matrix, given in the (2.35), now becomes

$$\rho \to \rho_M = |c_1|^2 \tag{2.40}$$

The collapse has happened.

In the same manner let us consider instead that the quantum measurement gives as result -1 for e_3. In this case there are the (2.36) and the (2.38) that we take in consideration The Clifford algebra $A(Si)$ no more holds, but instead it is valid the $N_{1,-1}$ that has the following new commutation rules

$$e_1 e_2 = -i \text{ , } e_2 e_1 = i \text{, } e_2 i = e_1 \text{, } ie_2 = -e_1 \text{, } e_1 i = -e_2 \text{, } ie_1 = e_2 \tag{2.41}$$

Inserting such new commutation rules in the (2.36) and (2.38), remembering that the parameter e_3 now assumes value –1, one sees that the interference terms are erased and the density matrix now becomes

$$\rho \to \rho_M = |c_2|^2 \tag{2.42}$$

The collapse has happened.

By using the Clifford bare bone skeleton , we conclude that quantum mechanics now becomes a self-consistent theory since by the $A(Si)$ and $N_{i,\pm 1}$ algebras, the formulation becomes able to describe the collapse of the wave function without recovering an outside ad hoc postulate on quantum measurement as initially formulated by von Neumann.

Let us examine in detail von Neumann results [36].

Consider the spinor basis given in (2.31).

According to such *projection postulate* the complete phase-damping way for a two state system may be written

$$D(\rho) = |0><0|\rho|0><0| \quad + \quad |1><1|\rho|1><1|$$

where the effect of this mapping is to zero-out the off-diagonal entries of a density matrix:

$$D\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}$$

If we have a set of mutually orthogonal projection operators ($P_1, P_2, \ldots.., P_m$) which complete to identity, i.e., $P_i P_j = \delta_{ij} P_j$ and $\sum_i P_i = 1$ when a measurement is carried out on a system with state $|\psi>$ then

(1) The result *i* is obtained with probability $p_i = <\psi|P_i|\psi>$

(2) The state collapses to $\frac{1}{\sqrt{p_i}} P_i |\psi>$

The projection operators are the idempotents in the $A(Si)$ Clifford algebra.

We have that

$|0><0|$ and $|1><1|$ (2.43)

are respectively the idempotents

$\frac{1+e_3}{2}$ and $\frac{1-e_3}{2}$ (2.44)

We have that

$(\frac{1+e_3}{2})\rho(\frac{1+e_3}{2})$ (2.45)

that gives

$(\frac{1+e_3}{2})\rho(\frac{1+e_3}{2}) = \alpha\ (\frac{1+e_3}{2})$ (2.46)

and explicitly

$$\begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix} \quad (2.47)$$

In the case of

$\frac{1-e_3}{2}$ (2.48)

one obtains as result

$\beta(\frac{1-e_3}{2})$ (2.49)

and explicitly

$$\begin{pmatrix} 0 & 0 \\ 0 & \delta \end{pmatrix}$$

The sum gives

$$\begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}$$

Generally speaking, given an observable with connected linear Hermitean operator O having eigenvalues $O_1, O_2, \ldots\ldots\ldots$

we have

Prob.$(O_n) = Tr(P_n\rho)$ (2.49a)

that obviously is fully justified by our $N_{i,\pm1}$ theorem.

In conclusion we have given a full Clifford algebraic justification of von Neumann's projection postulate.

Note that we have involved idempotents in the $A(Si)$ Clifford algebraic quantum scheme, and they have projectors as counterpart in standard quantum physics. We cannot ignore a fundamental step: according to J. von Neumann projection operators represent logical statements .We have verified that they assume the same meaning in our algebraic scheme. Consequently we cannot escape to the conclusion previously introduced. Measurements must be intended as semantic acts, and conceptual entities are represented in our scheme as a motor device as well as objects and matter dynamics.

Another question : demonstrated that mental entities are their algebraic structure represented from the Clifford algebra , may we identify a time dynamics of such our Clifford admitted basic entities ? The answer is again positive .

Consider the quantum system S and indicate by ψ_0 the state at the initial time 0. The state at any time t will be given by

$$\psi(t) = U(t)\psi_0 \text{ and } \psi_0 = \psi(t=0) \qquad (2.50)$$

An Hamiltonian H must be constructed such that the evolution operator U(t), that must be unitary, gives $U(t) = e^{-iHt}$.

It is well known that, given a finite N-level quantum system described by the state ψ, its evolution is regulated according to the time dependent Schrödinger equation

$$i\hbar \frac{d\psi(t)}{dt} = H(t)\psi(t) \quad \text{with } \psi(0) = \psi_0 . \qquad (2.51)$$

Let us introduce a model for the hamiltonian H(t). We indicate by H_0 the hamiltonian of the system S, and we add to H_0 an external time varying hamiltonian, $H_1(t)$, representing the perturbation to which the system S is subjected by action of the measuring apparatus. We write the total hamiltonian as

$$H(t) = H_0 + H_1(t) \qquad (2.52)$$

so that the time evolution will be given by the following Schrödinger equation

$$i\hbar \frac{d\psi(t)}{dt} = [H_0 + H_1(t)]\psi(t) \qquad (2.52a)$$

We have that

$$i\hbar \frac{dU(t)}{dt} = H(t)U(t) = [H_0 + H_1(t)]U(t) \quad \text{and } U(0)=I \qquad (2.53)$$

where U(t) pertains to the special group SU(N).

This is an argument that holds in quantum mechanics . The basic question that we have to solve is : have we the possibility to represent such Hamiltonian formalism only using abstract Clifford entities ? In other terms , have we the possibility to give here an Hamiltonian representation of our mental entities and of basic interaction with the outside? The answer is positive .

Let $A_1, A_2, \ldots, A_n$, ($n=N^2-1$), are skew-hermitean matrices forming a basis of Lie algebra SU(N). In this manner one arrives to write the explicit expression of the Hamiltonian H(t). It is given in the following manner

$$-iH(t) = -i[H_0 + H_1(t)] = \sum_{j=1}^{n} a_j A_j + \sum_{j=1}^{n} b_j A_j \qquad (2.54)$$

where a_j and $b_j = b_j(t)$ are respectively the constant components of the free hamiltonian and the time-varying control parameters characterizing the action of the measuring apparatus , just the semantic act. If we introduce T, the time ordering parameter (for details on this elaboration consider first of all the development that is due to Altafini, ref.20), we have

$$U(t) = T\exp(-i\int_0^t H(\tau)d\tau) = T\exp(-i\int_0^t (a_j + b_j(\tau))A_j d\tau) \qquad (2.55)$$

that is the well known Magnus expansion. Locally U(t) may be expressed by exponential terms as it follows

$$U(t) = \exp(\gamma_1 A_1 + \gamma_2 A_2 + \ldots\ldots + \gamma_n A_n) \qquad (2.56)$$

on the basis of the Wein-Norman formula

$$\Xi(\gamma_1, \gamma_2, \ldots\ldots, \gamma_n)\begin{pmatrix} \dot{\gamma}_1 \\ \dot{\gamma}_2 \\ \ldots \\ \dot{\gamma}_n \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \ldots \\ a_n + b_n \end{pmatrix} \qquad (2.57)$$

with Ξ n x n matrix, analytic in the variables γ_i. We have $\gamma_i(0) = 0$ and $\Xi(0) = I$, and thus it is invertible. We obtain

$$\begin{pmatrix} \dot{\gamma}_1 \\ \dot{\gamma}_2 \\ \ldots \\ \dot{\gamma}_n \end{pmatrix} = \Xi^{-1}\begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \ldots \\ a_n + b_n \end{pmatrix} \qquad (2.58)$$

Consider a simple case based on the superposition of only two states. We have

$$\psi = [y_1, y_2]^T \quad \text{and} \quad |y_1|^2 + |y_2|^2 = 1 \qquad (2.59)$$

We have here an SU(2) unitary transformation, selecting the skew symmetric basis for SU(2). We will have that

$$e_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad (2.60)$$

The following matrices are given

$$A_j = \frac{i}{2} e_j \quad , \text{j = 1,2,3} \tag{2.61}$$

The reader may now ascertain that the previously developed formalism is moving in direct correspondence with our Clifford algebra A(Si).

We are now in the condition to express H(t) and U(t) in our case of interest. The most simple situation we may examine is that one of fixed and constant control parameters b_j. The Hamiltonian H will become fully linear time invariant and its exponential solution will take the following form

$$e^{t(\sum_{j=1}^{3}(a_j+b_j)A_j)} = \cos(\frac{k}{2}t)I + \frac{2}{k} sen(\frac{k}{2}t)\left(\sum_{j=1}^{3}(a_j+b_j)A_j\right) \tag{2.62}$$

with $k = \sqrt{(a_1+b_1)^2 + (a_2+b_2)^2 + (a_3+b_3)^2}$. In matrix form it will result

$$U(t) = \begin{pmatrix} \cos\frac{k}{2}t + \frac{i}{k} sen\frac{k}{2}t(a_3+b_3) & \frac{1}{k} sen\frac{k}{2}t[a_2+b_2+i(a_1+b_1)] \\ \frac{1}{k} sen\frac{k}{2}t[-a_2-b_2+i(a_1+b_1)] & \cos\frac{k}{2}t - \frac{i}{k} sen\frac{k}{2}t(a_3+b_3) \end{pmatrix} \tag{2.63}$$

and, obviously, it will result to be unimodular as required.

Starting with this matrix representation of time evolution operator U(t), we may deduce promptly the dynamic time evolution of quantum state at any time t writing

$$\psi(t) = U(t)\psi_0 \tag{2.64}$$

assuming that we have for ψ_0 the following expression

$$\psi_0 = \begin{pmatrix} c_{true} \\ c_{false} \end{pmatrix} \tag{2.65}$$

having adopted for the true and false states (or yes-not states, +1 and –1 corresponding eigenvalues of such states) the following matrix expressions

$\varphi_{true} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\varphi_{false} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Finally, one obtains the expression of the state $\psi(t)$ at any time

$$\psi(t)=\left[c_{true}\left[\cos\frac{k}{2}t+\frac{i}{k}sen\frac{k}{2}t(a_3+b_3)\right]+c_{false}\left[\frac{1}{k}sen\frac{k}{2}t\left[(a_2+b_2)+i(a_1+b_1)\right]\right]\right]\varphi_{true}$$

$$+ \tag{2.66}$$

$$\left[c_{true}\left[\frac{1}{k}sen\frac{k}{2}t\left[i(a_1+b_1)-(a_2+b_2)\right]\right]+c_{false}\left[\cos\frac{k}{2}t-\frac{i}{k}sen\frac{k}{2}t(a_3+b_3)\right]\right]\varphi_{false}$$

As consequence, the two probabilities $P_{true}(t)$ and $P_{false}(t)$, will be given at any time t by the following expressions

$$P_{true}(t)=(A^2+B^2)\cos^2\frac{k}{2}t+\frac{1}{k^2}sen^2\frac{k}{2}t(P^2+Q^2)+\frac{senkt}{k}(AP+BQ) \tag{2.67}$$

and

$$P_{false}(t)=(C^2+D^2)\cos^2\frac{k}{2}t+\frac{1}{k^2}sen^2\frac{k}{2}t(S^2+R^2)+\frac{senkt}{k}(RC+DS) \tag{2.68}$$

where

A= Re c_{true} , B=Im c_{true}, C=Re c_{false} , D=Im c_{false} ,

$P=-D(a_1+b_1)+C(a_2+b_2)-B(a_3+b_3)$,

$Q=C(a_1+b_1)+D(a_2+b_2)+A(a_3+b_3)$,

$R=-B(a_1+b_1)-A(a_2+b_2)+D(a_3+b_3)$,

$S=A(a_1+b_1)-B(a_2+b_2)-C(a_3+b_3)$

Until here we have developed only standard elaborations that of course we performed in previous our publications. The reason to have developed here such formalism has been to evidence that at each step it has its corresponding counterpart in Clifford algebraic framework A(Si), and thus we may apply to it the two theorems previously demonstrated, passing from the algebra $A(Si)$ to $N_{i,\pm1}$. In fact, to this purpose, it is sufficient to multiply the (2.63) by the (2.65) to obtain the final forms of $c_{true}(t)$ and $c_{false}(t)$

In the final state we have that

$$\psi_t=\begin{pmatrix}c_{true}(t)\\ c_{false}(t)\end{pmatrix} \tag{2.69}$$

We may now write the density matrix that will result to have the same structure of the previously case given in the (2.32) but obviously with explicit evidence of time dependence. In the Clifford algebraic framework it will pertain still to the Clifford algebra A(Si). In order to describe the wave-function collapse we have to repeat the same procedure that we developed previously from the (2.32) to the (2.42), considering that, in accord to our criterium, we have to pass from the algebra A(Si) to $N_{i,\pm1}$, and obtaining

$$\rho\to\rho_M=\left|c_{true}(t)\right|^2 \tag{2.70}$$

in the case $N_{i,+1}$

and

$$\rho \rightarrow \rho_M = \left| c_{false}(t) \right|^2 \qquad (2.71)$$

in the case $N_{i,-1}$, as required in the collapse.

Note that, using Clifford algebra, we have given now a complete theoretical elaboration of the problem of wave function reduction in quantum mechanics also considering the process under the profile of the time dynamics.

Evidences of such elaboration have been given by us at cognitive level using introducing also experimental verifications .

We have now to develop the argument giving the title of the present paper .

Previously we have given different indications on the fact that using the Clifford algebra we were reasoning about our mind entities, It is now missing the final evidence that we are considering an algebra relating our consciousness. To reach this objective we need to give proof that it is possible an algebraic representation of the two consciousness peculiarities that are :

(-) It is an entity that has self-awareness and this is to say that it has in its inner the image of itself. In most cases we speak of self-image to represent such peculiar feature.

(-) The other marking property is that it has awareness of an external space -time located abstract entity.

In brief consciousness is an abstract entity that at the same entity has self-awareness (self image) of itself and self image of the outside

We will develop such algebraic features here now.

Until here we considered only basic Clifford abstract elements at their elementary order. Let us expand our formulation introducing the Clifford algebra at any order n.

First consider Clifford $A(Si)$ algebra at order n=4 (for details see our previous papers and references therein). One has

$E_{0i} = I^1 \otimes e_i$; $\qquad E_{i0} = e_i \otimes I^2$ (2.72a)

The notation $\otimes$ denotes direct product of matrices, and I^i is the *i*th 2x2 unit matrix. Thus, in the case of *n= 4* we have two distinct sets of Clifford basic unities, E_{0i} and E_{i0}, with

$$E_{0i}^2 = 1 \ ; \qquad E_{i0}^2 = 1, \qquad i = 1, 2, 3; \qquad (2.72b)$$

$E_{0i} E_{0j} = i\, E_{0k}$; $\qquad E_{i0} E_{j0} = i\, E_{k0}$, $j = 1, 2, 3$; $\qquad i \neq j$

and

$E_{i0}\, E_{0j} = E_{0j}\, E_{i0}$ (2.73)

with (i, j, k) cyclic permutation of $(1, 2, 3)$.

Let us examine now the following result

$(I^1 \otimes e_i)\,(e_j \otimes I^2) = E_{0i}\, E_{j0} = E_{ji}$ (2.74)

It is obtained according to our basic rule on cyclic permutation required for Clifford basic unities. We have that $E_{0i} E_{j0} = E_{ji}$ with $i = 1, 2, 3$ and $j = 1, 2, 3$, with $E^2_{ji} = 1$, $E_{ij} E_{km} \neq E_{km} E_{ij}$, and $E_{ij} E_{km} = E_{pq}$ where p results from the cyclic permutation (i, k, p) of $(1, 2, 3)$ and q results from the cyclic permutation (j, m, q) of $(1, 2, 3)$.

In the case $n = 4$ we have two distinct basic set of unities E_{0i}, E_{i0} and, in addition, basic sets of unities (E_{ij}, E_{ip}, E_{0m}) with (j, p, m) basic permutation of $(1, 2, 3)$.

This is the Clifford algebra A at order n=4.

We may now give the explicit expressions of E_{0i}, E_{i0}, and E_{ij} at the order n=4.

The starting elements are:

$$e_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\; e_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\; e_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We consider that these are the starting elements to interact with the outside entity to have awareness . The outside entity is represented by

$$e_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

At the order n= 4 we have

$$E_{01} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix};\; E_{02} = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix};\; E_{03} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad (2.76)$$

Comparing the starting representation at order n=2 with the new abstract entities given in (2.76) we see that this new abstract entity maintains in its inner structure the inner image of itself. This three entities maintain the first required structure, the proper inner self- image but with reference to the outside self - image . This is the first requirement we posed for an algebraic representation of consciousness.

$$E_{10}=\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix};\ E_{20}=\begin{pmatrix}0&0&-i&0\\0&0&0&-i\\i&0&0&0\\0&i&0&0\end{pmatrix};E_{30}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}; \qquad (2.77)$$

In this second set of abstract elements we have instead the self image of the outside but with reference to the inner self - image of itself .

In the mind of the subject consequently coexist the three self image of him (her)-self with reference to the three outside entities (awareness of itself with reference to the outside) (2.76) and the three abstract elements given in (2.77) represent that in the mind of the subject coexist the three self image of the outside (awareness of the outside) as seen in the inner awareness of this subject. Both the (2.76 and the (2.77) coexist in the mind inner structure of the subject. This result totally agrees with the algebraic elaboration realized by Lefebvre who studied in detail the algebra of conscience [35].

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