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Mathematical model of a plasma crystal

Abstract

A mathematical model of the plasma crystal built using Maxwell's equations is given.

Content

1. Problem statement

Dusty plasma (see the [1]) is a set of charged particles. These "particles can arrange in space in a certain way and form the so-called plasma crystal" [2]. The mechanism of formation, behavior and form of such crystals is difficult to predict. Observation of these processes and forms under low gravity conditions sets at the gaze – see illustration (Fig 1.) of the experiments in space in the [3].

Therefore, they were simulated on computer in 2007. The results surprised even greater, which was reflected in the name of a corresponding article [4]: "From plasma crystals and helical structures towards inorganic living matter". The [5] gives a summary and discussion of the simulation results.

I like such comparisons too. But, nevertheless, it should be noted that the method used by the authors of the molecular dynamics simulations does not fully take into account all the features of the dusty plasma. To describe the motion of the particles this method uses classical mechanics and considers only electrostatic forces between the charged particles. In fact, the charged particles motion causes occurrence of charge currents – electrical currents and electromagnetic fields as a consequence. They should be considered during simulation.

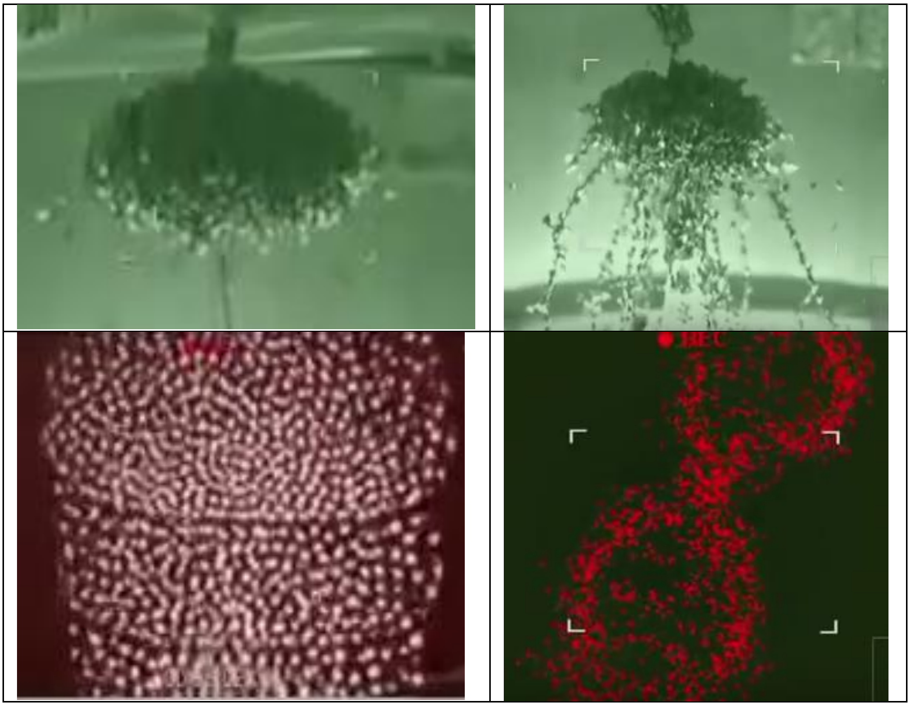


Fig. 1.

In absence of gravity the plasma particles are not affected by gravitational forces. If we exclude radiation energy, then it can be said that the dusty plasma is electric charges, electric currents and electromagnetic fields. Moreover, at its formation (filling a vessel with a set of charged particles) the plasma receives some energy. This energy may be only electromagnetic and kinetic energy of the particles, since there is no mechanical interaction between the particles: they are charged with like charges. Thus, the dusty plasma should meet the following conditions:

- to meet the Maxwell's equations,
- to maintain the total energy as a sum of electromagnetic and kinetic energy of the particles,
- to become stable in terms of the particles structure and motion in some time; it follows, for example, from the said experiments in space – see fig. 1.

The charged particles obviously push off from each other by Coulomb forces. However, the experiments show that these forces do not act on the periphery of a particles cloud. Consequently, they are compensated by other forces. It will be shown below that these forces are Lorentz forces arising during charged particles motion (although it seems strange at first sight that these forces direct into the cloud, opposing the Coulomb forces). The particles cannot be fixed, since then the Coulomb forces will prevail. But then these forces will move the particles, which causes the Lorentz forces, etc.

In the mathematical model shown below we will not take into account the Coulomb forces, believing that their role is only to ensure that the particles are isolated from each other (just as these forces are not considered in electrical engineering problems).

Thus, we will consider the dusty plasma as an area with flowing electrical currents and analyze it using the Maxwell's equations. Since the particles are in vacuum and are always isolated from each other, there is no ohmic resistance and no electrical voltage proportional to the current – it should not be taken into account in the Maxwell's equations. In addition, in the first stage, we will assume that the currents change slowly – they are constant currents. Considering these remarks, the Maxwell's equations are as follows:

$$\mathrm{rot}(H) - J = 0, \tag{1}$$

$$\mathrm{div}(J) = 0, \tag{2}$$

$$\mathrm{div}(H) = 0, \tag{3}$$

where the $J,\ H$ is the current and magnetic intensity, respectively. In addition, we need to add to these equations an equation uniting the plasma energy W with the $J,\ H$:

$$W = f(J,\ H). \tag{4}$$

In this equation, the energy W is known since the plasma receives this energy at its formation.

In scalar form, the system of equations (1-4) is a system of 6 equations with 6 unknowns and should have only one solution. However, there is no regular algorithm for solving such a system. Therefore, below we propose another approach:

1. Search for analytical solutions of underdetermined system of equations (1-3) with this plasma cloud form. There can be multiple solutions.
2. Calculation of energy W using the (4). If the solution of the system (1-4) is the only one then this solves the system (1-4) with the data of the W and cloud form.

2. System of equations

In the cylindrical coordinates $r,\ \varphi,\ z$, as is well-known [6], the divergence and curl of the vector H are as follows:

$$\operatorname{div}(H)=\left(\frac{H_r}{r}+\frac{\partial H_r}{\partial r}+\frac{1}{r}\cdot\frac{\partial H_\varphi}{\partial \varphi}+\frac{\partial H_z}{\partial z}\right), \tag{a}$$

$$\operatorname{rot}_r(H)=\left(\frac{1}{r}\cdot\frac{\partial H_z}{\partial \varphi}-\frac{\partial H_\varphi}{\partial z}\right), \tag{b}$$

$$\operatorname{rot}_\varphi(H)=\left(\frac{\partial H_r}{\partial z}-\frac{\partial H_z}{\partial r}\right), \tag{c}$$

$$\operatorname{rot}_z(H)=\left(\frac{H_\varphi}{r}+\frac{\partial H_\varphi}{\partial r}-\frac{1}{r}\cdot\frac{\partial H_r}{\partial \varphi}\right). \tag{d}$$

Considering the equations (a-d) we rewrite the equations (1.1-1.3) as follows:

$$\frac{H_r}{r}+\frac{\partial H_r}{\partial r}+\frac{1}{r}\cdot\frac{\partial H_\varphi}{\partial \varphi}+\frac{\partial H_z}{\partial z}=0, \tag{1}$$

$$\frac{1}{r}\cdot\frac{\partial H_z}{\partial \varphi}-\frac{\partial H_\varphi}{\partial z}=J_r, \tag{2}$$

$$\frac{\partial H_r}{\partial z}-\frac{\partial H_z}{\partial r}=J_\varphi, \tag{3}$$

$$\frac{H_\varphi}{r}+\frac{\partial H_\varphi}{\partial r}-\frac{1}{r}\cdot\frac{\partial H_r}{\partial \varphi}=J_z, \tag{4}$$

$$\frac{J_r}{r}+\frac{\partial J_r}{\partial r}+\frac{1}{r}\cdot\frac{\partial J_\varphi}{\partial \varphi}+\frac{\partial J_z}{\partial z}=0 \tag{5}$$

The system of 5 equations (1-5) with respect to the 6 unknowns $\left(H_r,\ H_\varphi,\ H_z,\ J_r,\ J_\varphi,\ J_z\right)$ is overdetermined and may have multiple solutions. It is shown below that such solutions exist and for different cases some of possible solutions can be identified.

We will first look for a solution for this system of equations (1-5) as functions separable relative to the coordinates. These functions are as follows:

$$H_r.=h_r(r)\cdot\cos(\chi z), \tag{6}$$

$$H_\varphi.=h_\varphi(r)\cdot\sin(\chi z), \tag{7}$$

$$H_z.=h_z(r)\cdot\sin(\chi z), \tag{8}$$

$$J_{r}. = j_r(r)\cdot\cos(\chi z),\tag{9}$$
$$J_{\varphi}. = j_\varphi(r)\cdot\sin(\chi z),\tag{10}$$
$$J_z = j_z(r)\cdot\sin(\chi z),\tag{11}$$
where the χ is a constant, while the $h_r(r),\ h_\varphi(r),\ h_z(r),\ j_r(r),\ j_\varphi(r),\ j_z(r)$ are the functions of the coordinate r; derivatives of these functions will be denoted by strokes.

By putting the (6-11) into the (1-5) we get:

$$\frac{h_r}{r}+h_r'+\chi h_z=0,\tag{12}$$
$$-\chi h_\varphi = j_r,\tag{13}$$
$$-\chi h_r - h_z' = j_\varphi\tag{14}$$
$$\frac{h_\varphi}{r}+h_\varphi' = j_z,\tag{15}$$
$$\frac{j_r}{r}+j_r'+\chi j_z=0.\tag{16}$$

Let's put the (13) and (15) into the (16). Then we get:

$$\frac{-\chi h_\varphi}{r}-\chi h_\varphi'+\chi\left(\frac{h_\varphi}{r}+h_\varphi'\right)=0.\tag{17}$$

The expression (17) is an identity 0=0. Therefore, the (16) follows from the (13, 15) and can be excluded from the system of equations (12-16). The rest of the equations can be rewritten as:

$$h_z=-\frac{1}{\chi}\left(\frac{h_r}{r}+h_r'\right),\tag{18}$$
$$j_z=\frac{h_\varphi}{r}+h_\varphi',\tag{19}$$
$$j_r=-\chi h_\varphi,\tag{20}$$
$$j_\varphi=-\chi h_r-h_z'\tag{21}$$

3. The first mathematical model

In this system of 4 differential equations (18-21) with 6 unknown functions we can define two functions arbitrarily. For further study we define the following two functions:

$$h_\varphi = q\cdot r\cdot\sin(\pi\cdot r/\chi),\tag{22}$$
$$h_r = h\cdot r\cdot\sin(\pi\cdot r/\chi),\tag{23}$$

where the $q,\ h$ are some constants. Then using the (18-23) we find:

$$h_z = -\frac{h}{\chi}\left(2\sin(\pi\cdot r/\chi) + \frac{\pi\cdot r}{\chi}\cos(\pi\cdot r/\chi)\right), \tag{24}$$

$$j_z = q\left(2\sin(\pi\cdot r/\chi) + \frac{\pi\cdot r}{\chi}\cdot\cos(\pi\cdot r/\chi)\right), \tag{25}$$

$$j_r = -\chi\cdot q\cdot r\cdot\sin(\pi\cdot r/\chi) \tag{26}$$

$$j_\varphi = h\cdot\left(\frac{\pi^2}{\chi R^2} - \chi\right)\cdot r\cdot\sin(\pi\cdot r/\chi) + \frac{h}{\chi}\left(2 - \frac{\pi}{\chi}\right)\cdot\cos(\pi\cdot r/\chi). \tag{27}$$

Thus, the functions $j_r(r)$, $j_\varphi(r)$, $j_z(r)$, $h_r(r)$, $h_\varphi(r)$, $h_z(r)$ can be defined using the (26, 27, 25, 23, 22, 24), respectively.

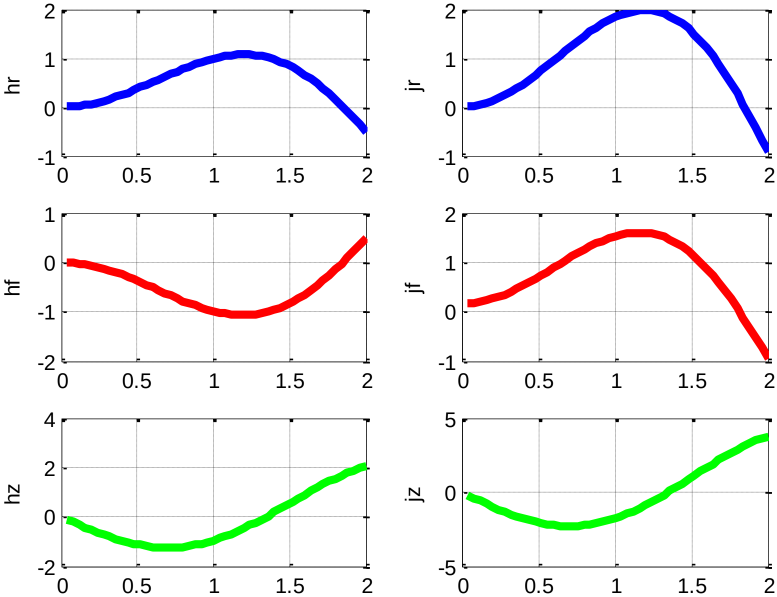


FIG. 2 (figPlazma.m)

Example 1.

Fig. 2 shows function graphs $j_r(r)$, $j_\varphi(r)$, $j_z(r)$, $h_r(r)$, $h_\varphi(r)$, $h_z(r)$. These functions can be calculated with data $\chi = 2$, $h = 1$, $q = -1$. The first column shows the functions $h_r(r)$, $h_\varphi(r)$, $h_z(r)$, the second column shows the functions $j_r(r)$, $j_\varphi(r)$, $j_z(r)$.

It is important to note that there is a point in the function graph $j_r(r)$, $j_\varphi(r)$ where $j_r(r) = 0$ and $j_\varphi(r) = 0$. Physically, this means that there are radial currents $J_r(r)$ in the area $r < \chi$ directed from the center (with $\chi q < 0$). There are no currents $J_r(r)$, $J_\varphi(r)$, in the point $r = \chi$. Therefore, the value $R = \chi$ is the radius of a crystal. The specks of dust outside this radius experience radial currents $J_r(r)$ directed towards the center. This creates a stable boundary of the crystal.

The built model describes a cylindrical crystal of infinite length, which, of course, is inconsistent with reality. Let's now consider a more complex model.

4. The second mathematical model

The root of the equation $j_r(r) = 0$ determines the value $R = \chi$ of the cylindrical crystal radius. Let's now change the value χ. If the value χ is dependent on the z, then the radius R will depend on the z. But this very dependence is observed in the experiments – see, for example, the first fragment in Fig. 1.

With this in mind, let's consider the mathematical model which differs from the above used by the fact that the function $\chi(z)$ is used instead of the constant χ. Let's rewrite the (6-11) with this in mind:

$$H_r. = h_r(r) \cdot \cos(\chi(z)), \tag{28}$$

$$H_\varphi. = h_\varphi(r) \cdot \sin(\chi(z)), \tag{29}$$

$$H_z. = h_z(r) \cdot \sin(\chi(z)), \tag{30}$$

$$J_r. = j_r(r) \cdot \cos(\chi(z)), \tag{31}$$

$$J_\varphi. = j_\varphi(r) \cdot \sin(\chi(z)), \tag{32}$$

$$J_z = j_z(r) \cdot \sin(\chi(z)). \tag{33}$$

The system of equations (1-6) differs from the system (2.9-2.14) only by the fact that instead of the constant χ we use the derivative $\chi'(z)$ along the z of the function $\chi(z)$. Consequently, the solution of the system (28-33) will be different from that of the previous system only by using the derivative $\chi'(z)$ in instead of the constant χ. Thus, the solution in this case will be as follows:

$$j_r = -\chi'(z) \cdot q \cdot r \cdot \sin(\pi \cdot r / \chi'(z)), \tag{34}$$

$$j_\varphi = \begin{pmatrix} h \cdot \left(\dfrac{\pi^2}{\chi'(z) R^2} - \chi'(z) \right) \cdot r \cdot \sin(\pi \cdot r / \chi'(z)) + \\ + \dfrac{h}{\chi'(z)} \left(2 - \dfrac{\pi}{\chi'(z)} \right) \cdot \cos(\pi \cdot r / \chi'(z)) \end{pmatrix}, \tag{35}$$

$$j_z = q\left(2\sin(\pi \cdot r/\chi'(z)) + \frac{\pi \cdot r}{R} \cdot \cos(\pi \cdot r/\chi'(z))\right), \qquad (36)$$

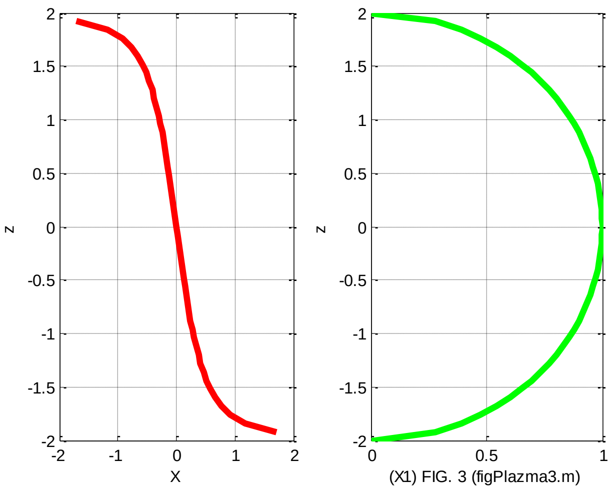
$$h_r = h \cdot r \cdot \sin(\pi \cdot r/\chi'(z)), \qquad (37)$$

$$h_\varphi = q \cdot r \cdot \sin(\pi \cdot r/\chi'(z)), \qquad (38)$$

$$h_z = -\frac{h}{\chi'(z)}\left(2\sin(\pi \cdot r/\chi'(z)) + \frac{\pi \cdot r}{R}\cos(\pi \cdot r/\chi'(z))\right). \qquad (39)$$

The said functions will depend on the $\chi'(z)$. With the $\chi(z) = \eta z$ the equations (34-39) are transformed into the equations (22-27).

For example, Fig. 3 shows the functions $\chi(z)$ and $\chi'(z)$ where the $\chi'(z)$ is an equation of ellipse.



(X1) FIG. 3 (figPlazma3.m)

We can suggest that the current of the specks of dust is such that their average speed does not depend on the current direction. In particular, the path covered by a speck of dust per a unit of time in a circumferential direction and the path covered by it in a vertical direction are equal with a fixed radius. Consequently, in this case with a fixed radius we may assume that

$$\Delta\varphi \equiv \Delta z. \qquad (40)$$

The dust trajectory in the above considered system is described by the following formulas

$$co = \cos(\chi(z)), \qquad (41)$$

$$si = \sin(\chi(z)). \qquad (42)$$

Thus, there is a point trajectory described by the formulas (40-42) in such system on the rotation figure with a radius of $r = \chi'(z)$. This trajectory is a helix. All the tensions and densities of currents do not depend on the φ in this trajectory.

Based on this assumption, we can construct a movement trajectory for specks of dust in accordance with the functions (1-3). Fig. 4 shows the two helices described by the current functions $j_r(r)$ and $j_z(r)$: with $r_1 = \chi'(z)$ with $r_2 = 0.5\chi'(z)$, where the $\chi'(z)$ is defined in Fig. 3.

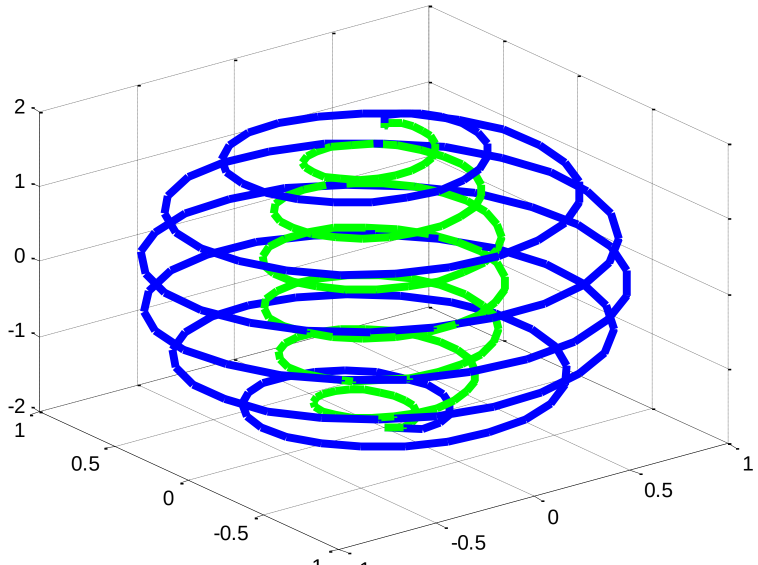


Fig 4 (Plazma4.m)

5. The plasma crystal energy

Under certain magnetic strengths and current densities we can find the plasma crystal energy. The magnetic field energy density

$$W_H = \frac{\mu}{2}\left(H_r^2 + H_\varphi^2 + H_z^2\right). \tag{43}$$

The specks of dust kinetic energy density W_J can be found in the assumption that all the specks of dust have equal mass m. Then

$$W_J = \frac{1}{m}\left(J_\varphi^2 + J_\varphi^2 + J_\varphi^2\right). \tag{44}$$

To determine the full crystal energy we need to integrate the (43, 44) by the volume of the crystal, which form is defined. Thus, with a defined form of the crystal and assumed mathematical model we can find all the characteristics of the crystal.

References

1. V.Ye. Fortov, A.G. Khrapak, S.A. Khrapak, V.I. Molotkov, O.F. Petrov./ Dusty plasma, UFN, May 2004, in Russian, http://ufn.ru/ru/articles/2004/5/b/
2. Dusty plasma, https://en.wikipedia.org/wiki/Dusty_plasma
3. Experiments with plasma in space, in Rusian, https://www.youtube.com/watch?v=SI406HKLYkM
4. V.N. Tsytovich, G.E. Morfill, V.E. Fortov, N.G. Gusein-Zade, B.A. Klumov and S.V. Vladimirov. From plasma crystals and helical structures towards inorganic living matter. New Journal of Physics, Volume 9, August 2007, http://iopscience.iop.org/article/10.1088/1367-2630/9/8/263/meta
5. Dusty plasma hints at a molecule of life, in Russian, http://www.membrana.ru/particle/693.
6. Andre Angot. Mathematics for Electrical and Radio Engineers, ed. “Nauka”, Moscow, 1964, p. 772 (in Russian).