Elementary Proof that an Infinite Number of Pell Primes Exist

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Abstract

This paper presents a complete proof of the Pell Primes are infinite, even though only 16 of them have been found as of 21 Feb 2017. We use a proof found in Reference 1, that if ***p* > *1*** and ***d* > *0*** are integers, that ***p*** and ***p* + *d*** are both primes if and only if for integer ***m***:

$$m = (p-1)!\left(\frac{1}{p} + \frac{(-1)^d (d)!}{p+d}\right) + \frac{1}{p} + \frac{1}{p+d}$$

We use this proof for $d = \frac{\left(1+2^{\frac{1}{2}}\right)^{n+1} - \left(1-2^{\frac{1}{2}}\right)^{n+1}}{2\left(2^{\frac{1}{2}}\right)} - \frac{\left(1+2^{\frac{1}{2}}\right)^{n} - \left(1-2^{\frac{1}{2}}\right)^{n}}{2\left(2^{\frac{1}{2}}\right)}$ to

prove the infinitude of Pell prime numbers. The author would like to give many thanks to the authors of 1001 Problems in Classical Number Theory, Jean-Marie De Koninck and Armel Mercier, 2004, Exercise Number 161 (see Reference 1). The proof provided in Exercise 6 is the key to making this paper on the Pell Prime Conjecture possible.

Introduction

The **Pell numbers** are an infinite sequence of integers, known since ancient times, that comprise the denominators of the closest rational approximations to the square root of 2. This sequence of approximations begins with: $\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}$, so the sequence of Pell numbers begins with 1, 2, 5, 12, and 29.

The Pell numbers are defined by the recurrence relation:

$$P_n = \begin{cases} 0 & \text{if } n = 0; \\ 1 & \text{if } n = 1; \\ 2P_{n-1} + P_{n-2} & \text{otherwise.} \end{cases}$$

In words, the sequence of Pell numbers starts with 0 and 1, and then each Pell number is the sum of twice the previous Pell number and the Pell number before that. The first few terms of the sequence are
0, 1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, 5741, 13860,…

The Pell numbers can also be expressed by the closed form formula

$$\frac{\left(1 + 2^{\frac{1}{2}}\right)^{n} - \left(1 - 2^{\frac{1}{2}}\right)^{n}}{2\left(2^{\frac{1}{2}}\right)}$$

Proof

A **Pell prime** is a Pell number that is prime. The first few Pell primes are 2, 5, 29, 5741,…

The indices of these primes within the sequence of all Pell numbers are 2, 3, 5, 11, 13, 29, 41, 53, 59, 89, 97, 101, 167, 181, 191,… These indices are all themselves prime. As with the Fibonacci numbers, a Pell number P_n can only be prime if n itself is prime, because if d is a divisor of n then P_d is a divisor of P_n.

Our goal is to prove that there are an infinite number of Pell primes. First we shall assume that the set of Pell Primes are finite and then we shall prove that this is false, which will prove that are Pell primes are infinite by contradiction. Even though we have assumed that the set of Pell primes is finite, since there are an infinite number of prime numbers, if we can prove that one Pell prime exists outside our set of finite Pell Primes, then we shall prove the Pell primes are infinite.

Let us define the largest Pell prime in the set of finite Pell primes as having ***n*** values. For the set of finite Pell primes, ***n*** is a positive integer. Therefore, we can define the largest Pell prime as follows:

$$\frac{\left(1 + 2^{\frac{1}{2}}\right)^{n} - \left(1 - 2^{\frac{1}{2}}\right)^{n}}{2\left(2^{\frac{1}{2}}\right)}$$

Where, ***n*** = 1, 2, 3, ………., ***n* – *1, n***

Now we must prove that a Pell prime exists outside our finite set, namely:

$$\frac{\left(1+2^{\frac{1}{2}}\right)^{n+1}-\left(1-2^{\frac{1}{2}}\right)^{n+1}}{2\left(2^{\frac{1}{2}}\right)}$$

must exist. Now we will let:

$$\boldsymbol{d} = \frac{\left(1+2^{\frac{1}{2}}\right)^{n+1}-\left(1-2^{\frac{1}{2}}\right)^{n+1}}{2\left(2^{\frac{1}{2}}\right)} - \frac{\left(1+2^{\frac{1}{2}}\right)^{n}-\left(1-2^{\frac{1}{2}}\right)^{n}}{2\left(2^{\frac{1}{2}}\right)}$$

And we will let a prime ***p***:

$$\mathbf{p} = \frac{\left(1+2^{\frac{1}{2}}\right)^{n}-\left(1-2^{\frac{1}{2}}\right)^{n}}{2\left(2^{\frac{1}{2}}\right)}$$

which we know is prime because it is the last Pell prime.

Therefore a prime ***p + d,*** where,

$$\boldsymbol{d} = \frac{\left(1+2^{\frac{1}{2}}\right)^{n+1}-\left(1-2^{\frac{1}{2}}\right)^{n+1}}{2\left(2^{\frac{1}{2}}\right)} - \frac{\left(1+2^{\frac{1}{2}}\right)^{n}-\left(1-2^{\frac{1}{2}}\right)^{n}}{2\left(2^{\frac{1}{2}}\right)}$$

cannot exist that is outside our finite set of Pell primes otherwise we would have discovered a Pell prime which is outside our assumed set of finite Pell primes. Thus all we need to do is to prove that there exists a Pell prime outside our assumed finite set is to prove that ***p + d*** is prime.

Now we shall proceed to prove at least one ***p + d*** is prime as follows:

We use the proof, provided in Reference 1, that if ***p > 1*** and ***d > 0*** are integers, that ***p*** and ***p+ d*** are both primes if and only if for positive integer ***m***:

$$m = (p-1)!\left(\frac{1}{p} + \frac{(-1)^d\,(d)!}{p+d}\right) + \frac{1}{p} + \frac{1}{p+d}$$

For our case ***p*** is known to be prime and ***d*** **=** $\frac{\left(1+2^{\frac{1}{2}}\right)^{n+1}-\left(1-2^{\frac{1}{2}}\right)^{n+1}}{2\left(2^{\frac{1}{2}}\right)}-\frac{\left(1+2^{\frac{1}{2}}\right)^{n}-\left(1-2^{\frac{1}{2}}\right)^{n}}{2\left(2^{\frac{1}{2}}\right)}$ for Pell primes, where ***n*** is any positive integer, therefore:

$$m = (p-1)!\left(\frac{1}{p}+\frac{(-1)^{\left(\frac{\left(1+2^{\frac{1}{2}}\right)^{n+1}-\left(1-2^{\frac{1}{2}}\right)^{n+1}}{2\left(2^{\frac{1}{2}}\right)}-\frac{\left(1+2^{\frac{1}{2}}\right)^{n}-\left(1-2^{\frac{1}{2}}\right)^{n}}{2\left(2^{\frac{1}{2}}\right)}\right)}\left(\left(\frac{\left(1+2^{\frac{1}{2}}\right)^{n+1}-\left(1-2^{\frac{1}{2}}\right)^{n+1}}{2\left(2^{\frac{1}{2}}\right)}-\frac{\left(1+2^{\frac{1}{2}}\right)^{n}-\left(1-2^{\frac{1}{2}}\right)^{n}}{2\left(2^{\frac{1}{2}}\right)}\right)!\right)}{p+\frac{\left(1+2^{\frac{1}{2}}\right)^{n+1}-\left(1-2^{\frac{1}{2}}\right)^{n+1}}{2\left(2^{\frac{1}{2}}\right)}-\frac{\left(1+2^{\frac{1}{2}}\right)^{n}-\left(1-2^{\frac{1}{2}}\right)^{n}}{2\left(2^{\frac{1}{2}}\right)}}\right)+\frac{1}{p}+\frac{1}{p+\frac{\left(1+2^{\frac{1}{2}}\right)^{n+1}-\left(1-2^{\frac{1}{2}}\right)^{n+1}}{2\left(2^{\frac{1}{2}}\right)}-\frac{\left(1+2^{\frac{1}{2}}\right)^{n}-\left(1-2^{\frac{1}{2}}\right)^{n}}{2\left(2^{\frac{1}{2}}\right)}}$$

Multiplying by ***p***, and since

$$\frac{\left(1+2^{\frac{1}{2}}\right)^{n+1}-\left(1-2^{\frac{1}{2}}\right)^{n+1}}{2\left(2^{\frac{1}{2}}\right)} - \frac{\left(1+2^{\frac{1}{2}}\right)^{n}-\left(1-2^{\frac{1}{2}}\right)^{n}}{2\left(2^{\frac{1}{2}}\right)}$$

can be even or odd, we will prove it as even first, and later prove the odd case later. Demonstrating the even case first:

$$(-1)^{\left(\frac{\left(1+2^{\frac{1}{2}}\right)^{n+1}-\left(1-2^{\frac{1}{2}}\right)^{n+1}}{2\left(2^{\frac{1}{2}}\right)} - \frac{\left(1+2^{\frac{1}{2}}\right)^{n}-\left(1-2^{\frac{1}{2}}\right)^{n}}{2\left(2^{\frac{1}{2}}\right)}\right)} = 1$$

$$mp = (p)!\left(\frac{1}{p} + \frac{\left(\left(\frac{\left(1+2^{\frac{1}{2}}\right)^{n+1}-\left(1-2^{\frac{1}{2}}\right)^{n+1}}{2\left(2^{\frac{1}{2}}\right)} - \frac{\left(1+2^{\frac{1}{2}}\right)^{n}-\left(1-2^{\frac{1}{2}}\right)^{n}}{2\left(2^{\frac{1}{2}}\right)}\right)!\right)}{p+\frac{\left(1+2^{\frac{1}{2}}\right)^{n+1}-\left(1-2^{\frac{1}{2}}\right)^{n+1}}{2\left(2^{\frac{1}{2}}\right)} - \frac{\left(1+2^{\frac{1}{2}}\right)^{n}-\left(1-2^{\frac{1}{2}}\right)^{n}}{2\left(2^{\frac{1}{2}}\right)}}\right) + 1 + \frac{p}{p+\frac{\left(1+2^{\frac{1}{2}}\right)^{n+1}-\left(1-2^{\frac{1}{2}}\right)^{n+1}}{2\left(2^{\frac{1}{2}}\right)} - \frac{\left(1+2^{\frac{1}{2}}\right)^{n}-\left(1-2^{\frac{1}{2}}\right)^{n}}{2\left(2^{\frac{1}{2}}\right)}}$$

Multiplying by:

$$p + \frac{\left(1 + 2^{\frac{1}{2}}\right)^{n+1} - \left(1 - 2^{\frac{1}{2}}\right)^{n+1}}{2\left(2^{\frac{1}{2}}\right)} - \frac{\left(1 + 2^{\frac{1}{2}}\right)^{n} - \left(1 - 2^{\frac{1}{2}}\right)^{n}}{2\left(2^{\frac{1}{2}}\right)}$$

$$(p + \frac{\left(1+ 2^{\frac{1}{2}}\right)^{n+1} - \left(1- 2^{\frac{1}{2}}\right)^{n+1}}{2\left(2^{\frac{1}{2}}\right)} - \frac{\left(1+ 2^{\frac{1}{2}}\right)^{n} - \left(1- 2^{\frac{1}{2}}\right)^{n}}{2\left(2^{\frac{1}{2}}\right)})mp = (p + \frac{\left(1+ 2^{\frac{1}{2}}\right)^{n+1} - \left(1- 2^{\frac{1}{2}}\right)^{n+1}}{2\left(2^{\frac{1}{2}}\right)} - \frac{\left(1+ 2^{\frac{1}{2}}\right)^{n} - \left(1- 2^{\frac{1}{2}}\right)^{n}}{2\left(2^{\frac{1}{2}}\right)})(p)!(\frac{1}{p} + \frac{((p+\frac{\left(1+ 2^{\frac{1}{2}}\right)^{n+1} - \left(1- 2^{\frac{1}{2}}\right)^{n+1}}{2\left(2^{\frac{1}{2}}\right)} - \frac{\left(1+ 2^{\frac{1}{2}}\right)^{n} - \left(1- 2^{\frac{1}{2}}\right)^{n}}{2\left(2^{\frac{1}{2}}\right)})!}{p+(p+\frac{\left(1+ 2^{\frac{1}{2}}\right)^{n+1} - \left(1- 2^{\frac{1}{2}}\right)^{n+1}}{2\left(2^{\frac{1}{2}}\right)} - \frac{\left(1+ 2^{\frac{1}{2}}\right)^{n} - \left(1- 2^{\frac{1}{2}}\right)^{n}}{2\left(2^{\frac{1}{2}}\right)}}) + 2p + (p + \frac{\left(1+ 2^{\frac{1}{2}}\right)^{n+1} - \left(1- 2^{\frac{1}{2}}\right)^{n+1}}{2\left(2^{\frac{1}{2}}\right)} - \frac{\left(1+ 2^{\frac{1}{2}}\right)^{n} - \left(1- 2^{\frac{1}{2}}\right)^{n}}{2\left(2^{\frac{1}{2}}\right)}$$

Reducing again,

$$(p + \frac{\left(1+2^{\frac{1}{2}}\right)^{n+1} - \left(1-2^{\frac{1}{2}}\right)^{n+1}}{2\left(2^{\frac{1}{2}}\right)} - \frac{\left(1+2^{\frac{1}{2}}\right)^{n} - \left(1-2^{\frac{1}{2}}\right)^{n}}{2\left(2^{\frac{1}{2}}\right)})\, mp = (p)!\left(\frac{(p + \frac{\left(1+2^{\frac{1}{2}}\right)^{n+1} - \left(1-2^{\frac{1}{2}}\right)^{n+1}}{2\left(2^{\frac{1}{2}}\right)} - \frac{\left(1+2^{\frac{1}{2}}\right)^{n} - \left(1-2^{\frac{1}{2}}\right)^{n}}{2\left(2^{\frac{1}{2}}\right)})}{p} + \left(\frac{\left(1+2^{\frac{1}{2}}\right)^{n+1} - \left(1-2^{\frac{1}{2}}\right)^{n+1}}{2\left(2^{\frac{1}{2}}\right)} - \frac{\left(1+2^{\frac{1}{2}}\right)^{n} - \left(1-2^{\frac{1}{2}}\right)^{n}}{2\left(2^{\frac{1}{2}}\right)}\right)!\right) + 2p + \frac{\left(1+2^{\frac{1}{2}}\right)^{n+1} - \left(1-2^{\frac{1}{2}}\right)^{n+1}}{2\left(2^{\frac{1}{2}}\right)} - \frac{\left(1+2^{\frac{1}{2}}\right)^{n} - \left(1-2^{\frac{1}{2}}\right)^{n}}{2\left(2^{\frac{1}{2}}\right)}$$

Factoring out, **(*p*)!,**

$$(p + \frac{\left(1+2^{\frac{1}{2}}\right)^{n+1} - \left(1-2^{\frac{1}{2}}\right)^{n+1}}{2\left(2^{\frac{1}{2}}\right)} - \frac{\left(1+2^{\frac{1}{2}}\right)^{n} - \left(1-2^{\frac{1}{2}}\right)^{n}}{2\left(2^{\frac{1}{2}}\right)})\, mp = p(p-1)!\left(\frac{(p + \frac{\left(1+2^{\frac{1}{2}}\right)^{n+1} - \left(1-2^{\frac{1}{2}}\right)^{n+1}}{2\left(2^{\frac{1}{2}}\right)} - \frac{\left(1+2^{\frac{1}{2}}\right)^{n} - \left(1-2^{\frac{1}{2}}\right)^{n}}{2\left(2^{\frac{1}{2}}\right)})}{p} + (\frac{\left(1+2^{\frac{1}{2}}\right)^{n+1} - \left(1-2^{\frac{1}{2}}\right)^{n+1}}{2\left(2^{\frac{1}{2}}\right)} - \frac{\left(1+2^{\frac{1}{2}}\right)^{n} - \left(1-2^{\frac{1}{2}}\right)^{n}}{2\left(2^{\frac{1}{2}}\right)})!\right) + 2p + \frac{\left(1+2^{\frac{1}{2}}\right)^{n+1} - \left(1-2^{\frac{1}{2}}\right)^{n+1}}{2\left(2^{\frac{1}{2}}\right)} - \frac{\left(1+2^{\frac{1}{2}}\right)^{n} - \left(1-2^{\frac{1}{2}}\right)^{n}}{2\left(2^{\frac{1}{2}}\right)}$$

And reducing again,

$$(p + \frac{2^{q(n+1)}+1}{3} - \frac{2^{q(n)}+1}{3})\, mp = (p-1)!(p + \frac{2^{q(n+1)}+1}{3} - \frac{2^{q(n)}+1}{3} + p(\frac{2^{q(n+1)}+1}{3} - \frac{2^{q(n)}+1}{3})!) + 2p + \frac{2^{q(n+1)}+1}{3} - \frac{2^{q(n)}+1}{3}$$

And reducing one final time,

$$(p + \frac{\left(1+2^{\frac{1}{2}}\right)^{n+1} - \left(1-2^{\frac{1}{2}}\right)^{n+1}}{2\left(2^{\frac{1}{2}}\right)} - \frac{\left(1+2^{\frac{1}{2}}\right)^{n} - \left(1-2^{\frac{1}{2}}\right)^{n}}{2\left(2^{\frac{1}{2}}\right)})\, mp = (p-1)!(p + \frac{\left(1+2^{\frac{1}{2}}\right)^{n+1} - \left(1-2^{\frac{1}{2}}\right)^{n+1}}{2\left(2^{\frac{1}{2}}\right)} - \frac{\left(1+2^{\frac{1}{2}}\right)^{n} - \left(1-2^{\frac{1}{2}}\right)^{n}}{2\left(2^{\frac{1}{2}}\right)} +$$

$$p(\frac{\left(1+2^{\frac{1}{2}}\right)^{n+1} - \left(1-2^{\frac{1}{2}}\right)^{n+1}}{2\left(2^{\frac{1}{2}}\right)} - \frac{\left(1+2^{\frac{1}{2}}\right)^{n} - \left(1-2^{\frac{1}{2}}\right)^{n}}{2\left(2^{\frac{1}{2}}\right)})!) + 2p + \frac{\left(1+2^{\frac{1}{2}}\right)^{n+1} - \left(1-2^{\frac{1}{2}}\right)^{n+1}}{2\left(2^{\frac{1}{2}}\right)} - \frac{\left(1+2^{\frac{1}{2}}\right)^{n} - \left(1-2^{\frac{1}{2}}\right)^{n}}{2\left(2^{\frac{1}{2}}\right)}$$

We already know ***p*** is prime, therefore, ***p*** = integer. Since ***p*** is an integer and since $\frac{\left(1+2^{\frac{1}{2}}\right)^{n+1} - \left(1-2^{\frac{1}{2}}\right)^{n+1}}{2\left(2^{\frac{1}{2}}\right)} - \frac{\left(1+2^{\frac{1}{2}}\right)^{n} - \left(1-2^{\frac{1}{2}}\right)^{n}}{2\left(2^{\frac{1}{2}}\right)}$ is an integer because both expressions are forms of Pell numbers and all Pell numbers are integers, therefore, the right hand side of the above equation is an integer (likewise the left hand side of the equation must also be an integer). Since the right hand side of the above equation is an integer and ***p*** and $\frac{\left(1+2^{\frac{1}{2}}\right)^{n+1} - \left(1-2^{\frac{1}{2}}\right)^{n+1}}{2\left(2^{\frac{1}{2}}\right)} - \frac{\left(1+2^{\frac{1}{2}}\right)^{n} - \left(1-2^{\frac{1}{2}}\right)^{n}}{2\left(2^{\frac{1}{2}}\right)}$ are integers on the left hand side of the equation, then the left hand side of the equation is also an integer. Therefore there are only 4 possibilities (see 1, 2a, 2b, and 2c below) that can hold for ***m*** so the left hand side of the above equation is an integer, they are as follows:

1) ***m*** is an integer, or

2) ***m*** is a rational fraction that is divisible by ***p***. This implies that ***n*** **=** $\frac{x}{p}$ where, p is prime and ***x*** is an integer. This results in the following three possibilities:

a. Since ***m*** **=** $\frac{x}{p}$, then **p =** $\frac{x}{m}$, since ***p*** is prime, then ***p*** is only divisible by ***p*** and **1**, therefore, the first possibility is for n to be equal to ***p*** or 1 in this case, which are both integers, thus ***m*** is an integer for this first case.

b. Since ***m*** **=** $\frac{x}{p}$***,*** and x is an integer, then ***x*** is not evenly divisible by ***p*** unless ***x = p,*** or ***x*** is a multiple of ***p***, where ***x = yp***, for any integer **y**. Therefore ***m*** is an integer for ***x = p*** and ***x = yp***.

c. For all other cases of, integer ***x, m = xp , m*** is not an integer.

To prove there is a Pell Prime, outside our set of finite Pell Primes, we only need to prove that there is at least one value of ***m*** that is an integer, outside our finite set. There can be an infinite number of values of ***m*** that are not integers, but that will not negate the existence of one Pell Prime, outside our finite set of Pell Primes.

First the only way that n cannot be an integer is if every ***m*** satisfies paragraph 2.c above, namely, ***m*** **=** $\frac{x}{p}$, where ***x*** is an integer***, x ≠ p, x ≠ yp, m ≠ p, and m ≠ 1*** for any integer ***y***. To prove there exists at least one Pell Prime outside our finite set, we will assume that no integer ***m*** exists and therefore no Pell Primes exist outside our finite set. Then we shall prove our assumption to be false.

Proof: Assumption no values of ***m*** are integers, specifically, every value of ***m*** is

$m = \frac{x}{p}$,where x is an integer, $x \neq p$, $x \neq yp$, $m \neq p$, and $m \neq 1$, for any integer y.

Paragraphs **1, 2.a, and 2.b** prove cases where ***m*** can be an integer, therefore our assumption is false and **there exist values of *m* that are integers.**

Since we have already shown that ***p*** and ***p* +=** $\frac{\left(1+2^{\frac{1}{2}}\right)^{n+1}-\left(1-2^{\frac{1}{2}}\right)^{n+1}}{2\left(2^{\frac{1}{2}}\right)}$ **-** $\frac{\left(1+2^{\frac{1}{2}}\right)^{n}-\left(1-2^{\frac{1}{2}}\right)^{n}}{2\left(2^{\frac{1}{2}}\right)}$, where ***d*** **=** $\frac{\left(1+2^{\frac{1}{2}}\right)^{n+1}-\left(1-2^{\frac{1}{2}}\right)^{n+1}}{2\left(2^{\frac{1}{2}}\right)}$ **-** $\frac{\left(1+2^{\frac{1}{2}}\right)^{n}-\left(1-2^{\frac{1}{2}}\right)^{n}}{2\left(2^{\frac{1}{2}}\right)}$, are both primes if and only if for integer ***m***:

$$m = (p-1)!\left(\frac{1}{p} + \frac{(-1)^{d}\,(d)!}{p+d}\right) + \frac{1}{p} + \frac{1}{p+d}$$

It suffices to show that there is at least one integer n to prove there exists a Pell Prime outside our set of finite set of Pell Primes.

Since there exists an ***m*** = integer, we have proven that there is at least one ***p*** and $P2+1$ that are both prime. Since ***p*** + **=** $\frac{\left(1+2^{\frac{1}{2}}\right)^{n+1}-\left(1-2^{\frac{1}{2}}\right)^{n+1}}{2\left(2^{\frac{1}{2}}\right)}$ **-** $\frac{\left(1+2^{\frac{1}{2}}\right)^{n}-\left(1-2^{\frac{1}{2}}\right)^{n}}{2\left(2^{\frac{1}{2}}\right)}$ is prime and it is also greater than ***p*** then it also is not in the finite set of Pell primes, therefore, since we have proven that there is at least one ***p*** + $\frac{\left(1+2^{\frac{1}{2}}\right)^{n+1}-\left(1-2^{\frac{1}{2}}\right)^{n+1}}{2\left(2^{\frac{1}{2}}\right)}$ **-** $\frac{\left(1+2^{\frac{1}{2}}\right)^{n}-\left(1-2^{\frac{1}{2}}\right)^{n}}{2\left(2^{\frac{1}{2}}\right)}$ that is prime, and since

$$p = \frac{\left(1+2^{\frac{1}{2}}\right)^{n} - \left(1-2^{\frac{1}{2}}\right)^{n}}{2\left(2^{\frac{1}{2}}\right)}, \text{ then:}$$

$$p + d = \frac{\left(1+2^{\frac{1}{2}}\right)^{n} - \left(1-2^{\frac{1}{2}}\right)^{n}}{2\left(2^{\frac{1}{2}}\right)} + \frac{\left(1+2^{\frac{1}{2}}\right)^{n+1} - \left(1-2^{\frac{1}{2}}\right)^{n+1}}{2\left(2^{\frac{1}{2}}\right)} - \frac{\left(1+2^{\frac{1}{2}}\right)^{n} - \left(1-2^{\frac{1}{2}}\right)^{n}}{2\left(2^{\frac{1}{2}}\right)} =$$

$$\frac{\left(1+2^{\frac{1}{2}}\right)^{n+1} - \left(1-2^{\frac{1}{2}}\right)^{n+1}}{2\left(2^{\frac{1}{2}}\right)}$$

And since ***p* + *d* =** prime,

$$\text{then, } \frac{\left(1+2^{\frac{1}{2}}\right)^{n+1} - \left(1-2^{\frac{1}{2}}\right)^{n+1}}{2\left(2^{\frac{1}{2}}\right)} = \text{prime}$$

now we have proven that there is a Pell prime outside our assumed finite set of Pell primes. This is a contradiction from our assumption that the set of Pell primes is finite, therefore, by contradiction the set of Pell primes is infinite. Also this same proof can be repeated infinitely for each finite set of Pell primes, in other words a new Pell prime can added to each set of finite Pell primes making the Pell primes countably infinite. This thoroughly proves that an infinite number of Pell primes exist.

References:

1) 1001 Problems in Classical Number Theory, Jean-Marie De Koninck and Armel Mercier, 2004, Exercise Number 161