Ramanujan's Issues N°1: the radical $r=\frac{\sqrt[4]{5}-1}{\sqrt[4]{5}+1}$

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Abstract

In this note we presents some formulas related with the radical:

$$r=\frac{\sqrt[4]{5}-1}{\sqrt[4]{5}+1}$$

1. Introducción

En esta nota mostramos algunas fórmulas relacionadas con el radical de Ramanujan:

$$r=\sqrt[4]{\frac{3-2\sqrt[4]{5}}{3+2\sqrt[4]{5}}}=\frac{\sqrt[4]{5}-1}{\sqrt[4]{5}+1}$$

2. Ecuación polinomial asociada

El radical r satisface la ecuación:

$$r^4-6r^3+6r^2-6r+1=0$$

3. Soluciones de la ecuación: $p(z)=z^4-6z^3+6z^2-6z+1=0$

Las soluciones de la ecuación $p(z)=0$ son:

$$z_1=\frac{3+\sqrt{5}-\sqrt{10+6\sqrt{5}}}{2}=r$$

$$z_2=\frac{3+\sqrt{5}+\sqrt{10+6\sqrt{5}}}{2}=\frac{1}{r}$$

$$z_3=\frac{3-\sqrt{5}+i\sqrt{6\sqrt{5}-10}}{2}$$

$$z_4=\frac{3-\sqrt{5}-i\sqrt{6\sqrt{5}-10}}{2}=\frac{1}{z_3}$$

4. Factorización del polinomio $p(z)$

$$p(z) = z^4 - 6z^3 + 6z^2 - 6z + 1 = \left(z^2 - \left(3+\sqrt{5}\right)z + 1\right)\left(z^2 - \left(3-\sqrt{5}\right)z + 1\right)$$

5. Fórmulas para r

$$r = \frac{3 - \sqrt[4]{5} - \sqrt[4]{125} + \sqrt{5}}{2}$$

$$r = \frac{2}{3 + \sqrt[4]{5} + \sqrt[4]{125} + \sqrt{5}}$$

$$r = \left(\sqrt[4]{5} - 1\right)^2 \left(\frac{\varphi}{2}\right) = \frac{1}{\left(\sqrt[4]{5} + 1\right)^2}\left(\frac{2}{\varphi}\right) \ , \varphi = \frac{1+\sqrt{5}}{2}$$

$$r = \frac{3 - g + g^2 - g^3}{2} = \frac{2}{3 + g + g^2 + g^3} \ , g = \sqrt[4]{5}$$

$$r = \sqrt{3 - 2\sqrt[4]{5}} \ \sqrt[4]{9 + 4\sqrt{5}}$$

$$r = \sqrt[4]{161 - 108\sqrt[4]{5} + 72\sqrt{5} - 48\sqrt[4]{125}}$$

6. Sucesión asociada a r

Sea u_n , $n \in \mathbb{N}_0 = \{0,1,2,3, \dots\}$,la sucesión definida por:

$$u_{n+4} = 6u_{n+3} - 6u_{n+2} + 6u_{n+1} - u_n$$

donde

$$u_0 = 1, u_1 = 6, u_2 = 30, u_3 = 150$$

Se tiene:

$$\lim_{n \to \infty} \frac{u_n}{u_{n+1}} = r$$

7. La sucesión $v_n = \alpha_n + \beta_n\sqrt{5}$

Sea la sucesión v_n , definida por:

$$v_{n+2} = \left(3+\sqrt{5}\right)v_{n+1} - v_n \ , v_0 = 1 \ , v_1 = 3 + \sqrt{5}$$

Se tiene:

$$\lim_{n\to\infty} \frac{v_n}{v_{n+1}} = r$$

Poniendo $v_n = \alpha_n + \beta_n\sqrt{5}$, con $\alpha_n, \beta_n \in \mathbb{N}_0$ se tiene:

$$\alpha_{n+2} = 3\alpha_{n+1} + 5\beta_{n+1} - \alpha_n$$

$$\beta_{n+2} = \alpha_{n+1} + 3\beta_{n+1} - \beta_n$$

donde

$$\alpha_0 = 1, \beta_0 = 0, \alpha_1 = 3, \beta_1 = 1$$

Una fórmula general para la sucesión v_n es:

$$v_n = \left(\frac{1}{2} - \frac{\sqrt[4]{5}}{4} - \frac{\sqrt[4]{125}}{20}\right) r^n + \left(\frac{1}{2} + \frac{\sqrt[4]{5}}{4} + \frac{\sqrt[4]{125}}{20}\right) r^{-n} \ , n \in \mathbb{N}_0$$

8. La sucesión $t_n = \delta_n + \gamma_n\varphi$

Sea t_n la sucesión definida por:

$$t_{n+2} = (2 + 2\varphi)t_{n+1} - t_n \ , t_0 = 1, t_1 = 2 + 2\varphi \ , \varphi = \frac{1 + \sqrt{5}}{2}$$

Se tiene:

$$\lim_{n\to\infty} \frac{t_n}{t_{n+1}} = r$$

Poniendo $t_n = \delta_n + \gamma_n\varphi$, con $\delta_n, \gamma_n \in \mathbb{N}_0$ se tiene:

$$\delta_{n+2} = 2\delta_{n+1} + 2\gamma_{n+1} - \delta_n$$

$$\gamma_{n+2} = 2\delta_{n+1} + 4\gamma_{n+1} - \gamma_n$$

$$\delta_0 = 1, \gamma_0 = 0, \delta_1 = 2, \gamma_1 = 2$$

9. Los números r^n y r^{-n}

Para $n \in \mathbb{N}_0$ se tiene:

$$r^n = a_n + b_n r + c_n r^2 + d_n r^3$$

$$r^{-n} = a_n + b_n r^{-1} + c_n r^{-2} + d_n r^{-3}$$

donde

$$a_{n+1} = -d_n$$

$$b_{n+1} = a_n + 6d_n$$
$$c_{n+1} = b_n - 6d_n$$
$$d_{n+1} = c_n + 6d_n$$
$$a_0 = 1, b_0 = c_0 = d_0 = 0$$

10. Los números r^n y r^{-n} en función de $g = \sqrt[4]{5}$

Para $n \in \mathbb{N}_0$ se tiene:

$$r^n = \frac{1}{2^n}(a_n - b_n g + c_n g^2 - d_n g^3)$$
$$r^{-n} = \frac{1}{2^n}(a_n + b_n g + c_n g^2 + d_n g^3)$$

donde

$$a_{n+1} = 3a_n + 5b_n + 5c_n + 5d_n$$
$$b_{n+1} = a_n + 3b_n + 5c_n + 5d_n$$
$$c_{n+1} = a_n + b_n + 3c_n + 5d_n$$
$$d_{n+1} = a_n + b_n + c_n + 3d_n$$
$$a_0 = 1, b_0 = c_0 = d_0 = 0$$

11. Identidades con el número r

$$\frac{1-r}{1+r} = \frac{1}{\sqrt[4]{5}}$$
$$5(1-r)^4 = (1+r)^4$$
$$\frac{1-r}{1+r} = \frac{9 - 13r + 7r^2 - r^3}{10}$$
$$\frac{1+r}{1-r} = \frac{3 - r + 5r^2 - r^3}{2}$$
$$\left(\frac{1-r^4}{1+r^4}\right)^4 = 5\left(\frac{2}{3}\right)^4$$
$$r^2 - (2+2\varphi)r + 1 = 0 \ , \varphi = \frac{1+\sqrt{5}}{2}$$
$$r + r^{-1} = 3 + \sqrt{5}$$

$$\frac{1}{r} = 6 - 6r + 6r^2 - r^3$$

$$r = 6 - \frac{6}{r} + \frac{6}{r^2} - \frac{1}{r^3}$$

$$\frac{1-2r}{1+2r} = \frac{69 - 148r + 104r^2 - 16r^3}{101}$$

$$\frac{1-3r}{1+3r} = \frac{77 - 219r + 171r^2 - 27r^3}{158}$$

$$\frac{r}{1-r} = \frac{\sqrt[4]{5} - 1}{2}$$

$$\frac{1}{1-r} = \frac{\sqrt[4]{5} + 1}{2}$$

12. Fracciones continuas para r

Para $\varphi = \frac{1+\sqrt{5}}{2}$, $g = \sqrt[4]{5}$, $g + g^3 = \sqrt[4]{5} + \sqrt[4]{125}$, se tiene:

$$r = \cfrac{1}{3 + \sqrt{5} - \cfrac{1}{3 + \sqrt{5} - \cfrac{1}{3 + \sqrt{5} - \cdots}}} = \cfrac{1}{2\varphi^2 - \cfrac{1}{2\varphi^2 - \cfrac{1}{2\varphi^2 - \cdots}}}$$

$$r = \cfrac{1}{g + g^3 + \cfrac{1}{g + g^3 + \cfrac{1}{g + g^3 + \cdots}}} = \cfrac{1}{2g\varphi + \cfrac{1}{2g\varphi + \cfrac{1}{2g\varphi + \cdots}}}$$

13. Series para r

$$r = \frac{5}{24} + \frac{1}{24}\sum_{n=1}^{\infty} \frac{(-1/4)_n}{n!}\left(27\left(\frac{161}{3^8}\right)^n - 40\left(-\frac{1}{80}\right)^n - 18\left(\frac{1}{81}\right)^n\right)$$

$$r = \sum_{n=1}^{\infty} \frac{c_n}{\left(3+\sqrt{5}\right)^{2n-1}} = \sum_{n=1}^{\infty} c_n\, 2^{-2n+1}\varphi^{-4n+2}$$

donde

$$c_1 = 1 \ , c_{n+2} = \frac{2}{n+2} \sum_{k=0}^{n} (k+1)\, c_{k+1} c_{n-k+1}$$

$$c_n = \{1,1,2,5,14,42,132,429,1430,4862, \dots\}$$

14. Aproximaciones racionales

Sean $n, p_n, q_n \in \mathbb{N}$, tales que $\lim_{n\to\infty} \frac{p_n}{q_n} = \sqrt[4]{5}$, entonces se tiene:

$$r = \lim_{n\to\infty} \frac{p_n - q_n}{p_n + q_n}$$

$$r = \lim_{n\to\infty} \frac{3q_n^3 - p_n q_n^2 + p_n^2 q_n - p_n^3}{2q_n^3}$$

15. Una relación con los números de Fibonacci

Sea F_n la sucesión de Fibonacci:

$$F_{n+2} = F_{n+1} + F_n \ , F_0 = F_1 = 1$$

$$F_n = \{1,1,2,3,5,8,13,21, \dots\}$$

Sea w_n la sucesión definida por:

$$w_{n+2} = 2\frac{F_{n+2}}{F_n} w_{n+1} - w_n \ , w_0 = 1 \, , w_1 = 4$$

Se tiene:

$$\lim_{n\to\infty} \frac{w_n}{w_{n+1}} = r$$

16. Algunas sucesiones convergentes a r

$$z_{n+1} = \frac{1 + z_n^2}{3 + \sqrt{5}} \ , z_1 = 0 \ \Rightarrow \ z_n \to r$$

$$z_{n+1} = \frac{1 + 6z_n^2 - 6z_n^3 + z_n^4}{6} \ , z_1 = 0 \ \Rightarrow \ z_n \to r$$

$$z_{n+1} = \frac{1}{6 - 6z_n + 6z_n^2 - z_n^3} \ , z_1 = 0 \ \Rightarrow \ z_n \to r$$

$$z_{n+1} = \frac{1 + 6z_n^2 + z_n^4}{6 + 6z_n^2} \quad , z_1 = 0 \;\Rightarrow\; z_n \to r$$

$$z_{n+1} = \frac{1 + z_n^4}{6 - 6z_n + 6z_n^2} \quad , z_1 = 0 \;\Rightarrow\; z_n \to r$$

$$z_{n+1} = \frac{1 - 2z_n + 6z_n^2 - 6z_n^3 + z_n^4}{4} \quad , z_1 = 0 \;\Rightarrow\; z_n \to r$$

$$z_{n+1} = \frac{1 - z_n + 6z_n^2 - 6z_n^3 + z_n^4}{5} \quad , z_1 = 0 \;\Rightarrow\; z_n \to r$$

$$z_{n+1} = \frac{1 - z_n + 6z_n^2 - z_n^3 + z_n^4}{5 + 5z_n^2} \quad , z_1 = 0 \;\Rightarrow\; z_n \to r$$

$$z_{n+1} = \frac{1 - 6z_n^2 + 12z_n^3 - 3z_n^4}{6 - 12z_n + 18z_n^2 - 4z_n^3} \quad , z_1 = 0 \;\Rightarrow\; z_n \to r$$

$$z_{n+1} = \frac{1 - \left(\sqrt{5} - 2\right)z_n + z_n^2}{5} \quad , z_1 = 0 \;\Rightarrow\; z_n \to r$$

17. Integrales para r

$$r = 3 + \sqrt{5} - \frac{2}{\pi}\int_0^{2\pi} \frac{3 + \sqrt{5} - \left(6 + 3\sqrt{5}\right)e^{xi}}{4 - \left(6 + 2\sqrt{5}\right)e^{xi} + e^{2xi}}\,dx$$

$$r = 6 - \frac{4}{\pi}\int_0^{2\pi} \frac{12 - 35e^{xi} + 9e^{2xi} - 6e^{3xi}}{16 - 48e^{xi} + 24e^{2xi} - 12e^{3xi} + e^{4xi}}\,dx$$

18. Fórmulas para pi

$$\pi = 4\tan^{-1}(r) + 4\tan^{-1}\left(\frac{1}{\sqrt[4]{5}}\right)$$

$$\pi = 4\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} r^{2n+1} + \frac{4}{\sqrt[4]{5}}\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\left(\frac{1}{\sqrt{5}}\right)^n$$

$$\pi = 4\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\left(\sqrt[4]{5} - 1\right)^{4n+2}\left(\frac{\varphi}{2}\right)^{2n+1} + \frac{4}{\sqrt[4]{5}}\sum_{n=0}^{\infty} \frac{5^{-n}}{4n+1} - \frac{4}{\sqrt[4]{125}}\sum_{n=0}^{\infty} \frac{5^{-n}}{4n+3}$$

$$\pi = 2\sin^{-1}\left(\frac{3 - \sqrt{5}}{2}\right) + 2\sin^{-1}\left(\frac{\sqrt{6\sqrt{5} - 10}}{2}\right)$$

$$\pi = 4\sin^{-1}\left(\frac{3-\sqrt{5}}{2}\right) + 4\tan^{-1}\left(\frac{\left(\sqrt{10}-\sqrt{3\sqrt{5}-5}\right)^2\left(5+\sqrt{5}\right)}{60}\right)$$

$$\pi = 4\tan^{-1}(r) + 4\sin^{-1}\left(\frac{1}{\sqrt{1+\sqrt{5}}}\right)$$

19. Identidades

$$\sqrt[4]{\frac{3+2\sqrt[4]{5}}{3-2\sqrt[4]{5}}} + \sqrt[4]{\frac{3-2\sqrt[4]{5}}{3+2\sqrt[4]{5}}} = 3+\sqrt{5}$$

$$\sqrt[4]{\frac{3+2\sqrt[4]{5}}{3-2\sqrt[4]{5}}} - \sqrt[4]{\frac{3-2\sqrt[4]{5}}{3+2\sqrt[4]{5}}} = \sqrt[4]{5}+\sqrt[4]{125}$$

$$\sqrt{\frac{\sqrt[4]{5}-1}{\sqrt[4]{5}+1}} = \left(\sqrt[4]{5}-1\right)\sqrt{\frac{\varphi}{2}} = \frac{1}{\sqrt[4]{5}+1}\sqrt{\frac{2}{\varphi}} \quad , \qquad \varphi = \frac{1+\sqrt{5}}{2}$$

$$6\sqrt[4]{\frac{3-2\sqrt[4]{5}}{3+2\sqrt[4]{5}}} - \sqrt{\frac{3-2\sqrt[4]{5}}{3+2\sqrt[4]{5}}} - \sqrt{\frac{3+2\sqrt[4]{5}}{3-2\sqrt[4]{5}}} + 6\sqrt[4]{\frac{3+2\sqrt[4]{5}}{3-2\sqrt[4]{5}}} = 6$$

$$6\left(\frac{\sqrt[4]{5}-1}{\sqrt[4]{5}+1}\right) - \left(\frac{\sqrt[4]{5}-1}{\sqrt[4]{5}+1}\right)^2 - \left(\frac{\sqrt[4]{5}+1}{\sqrt[4]{5}-1}\right)^2 + 6\left(\frac{\sqrt[4]{5}+1}{\sqrt[4]{5}-1}\right) = 6$$

20. Algunos polinomios

$$z = \sqrt[4]{5}-1 \;\Rightarrow\; z^4+4z^3+6z^2+4z-4=0$$

$$z = \sqrt[4]{5}+1 \;\Rightarrow\; z^4-4z^3+6z^2-4z-4=0$$

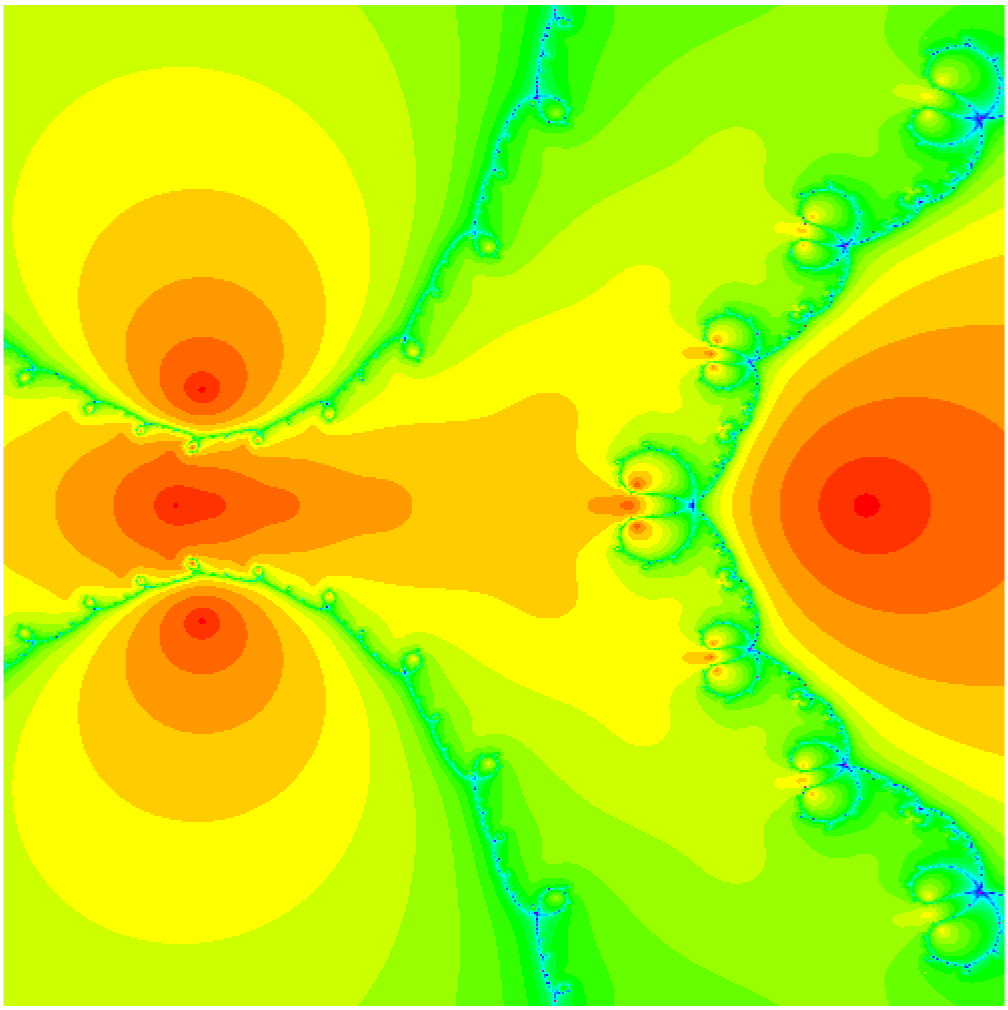
$$z = 3-2\sqrt[4]{5} \;\Rightarrow\; z^4-12z^3+54z^2-108z+1=0$$

$$z = 3+2\sqrt[4]{5} \;\Rightarrow\; z^4-12z^3+54z^2-108z+1=0$$

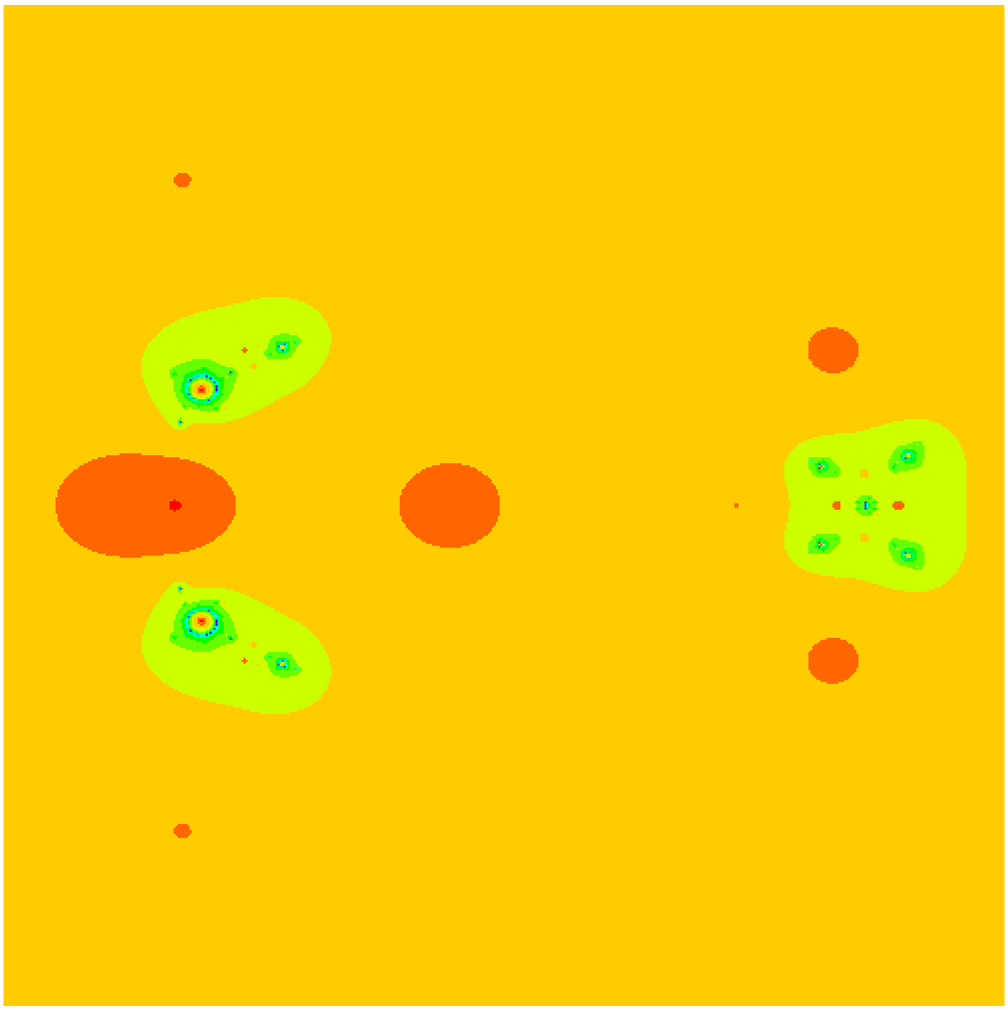
$$z = \frac{3-2\sqrt[4]{5}}{3+2\sqrt[4]{5}} \;\Rightarrow\; z^4-644z^3+6z^2-644z+1=0$$

21. Fractales relacionados

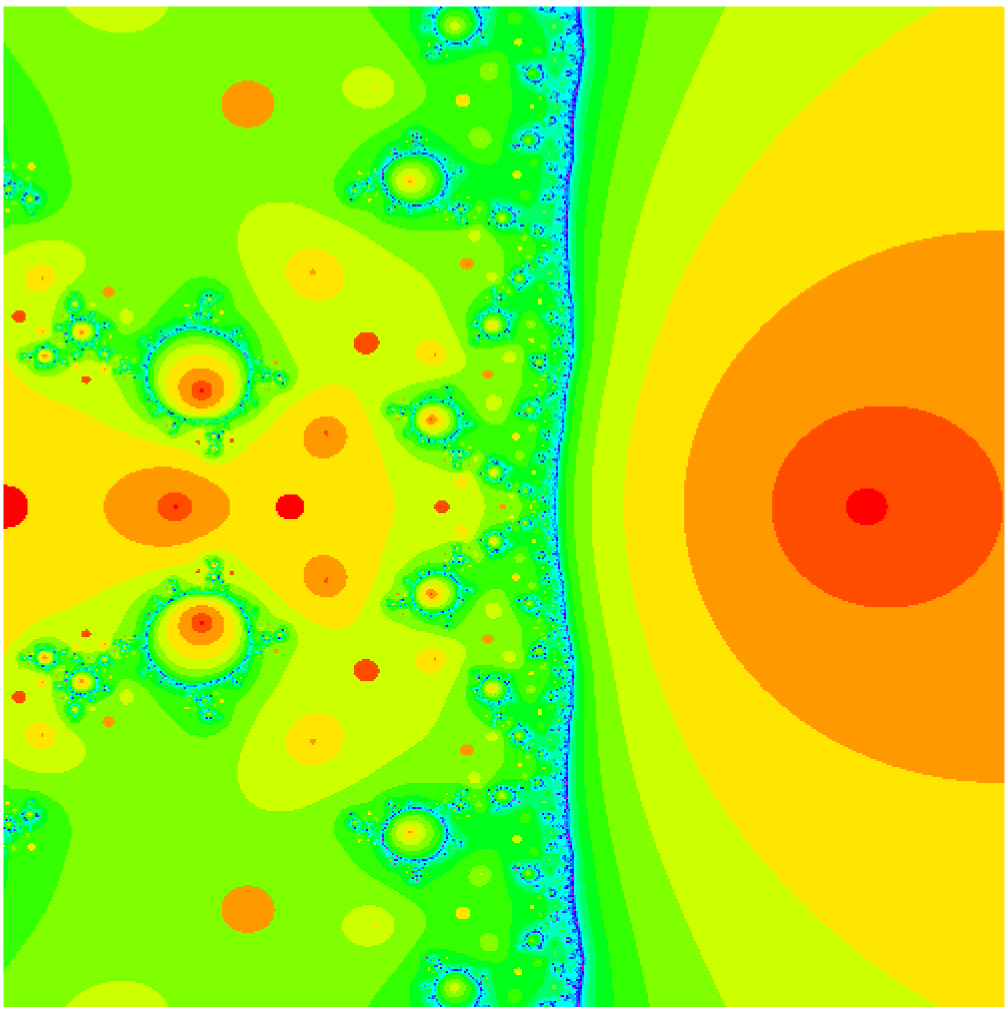
$$f(z) = z - \frac{1 + 6z^2 - 6z^3 + z^4}{6}$$



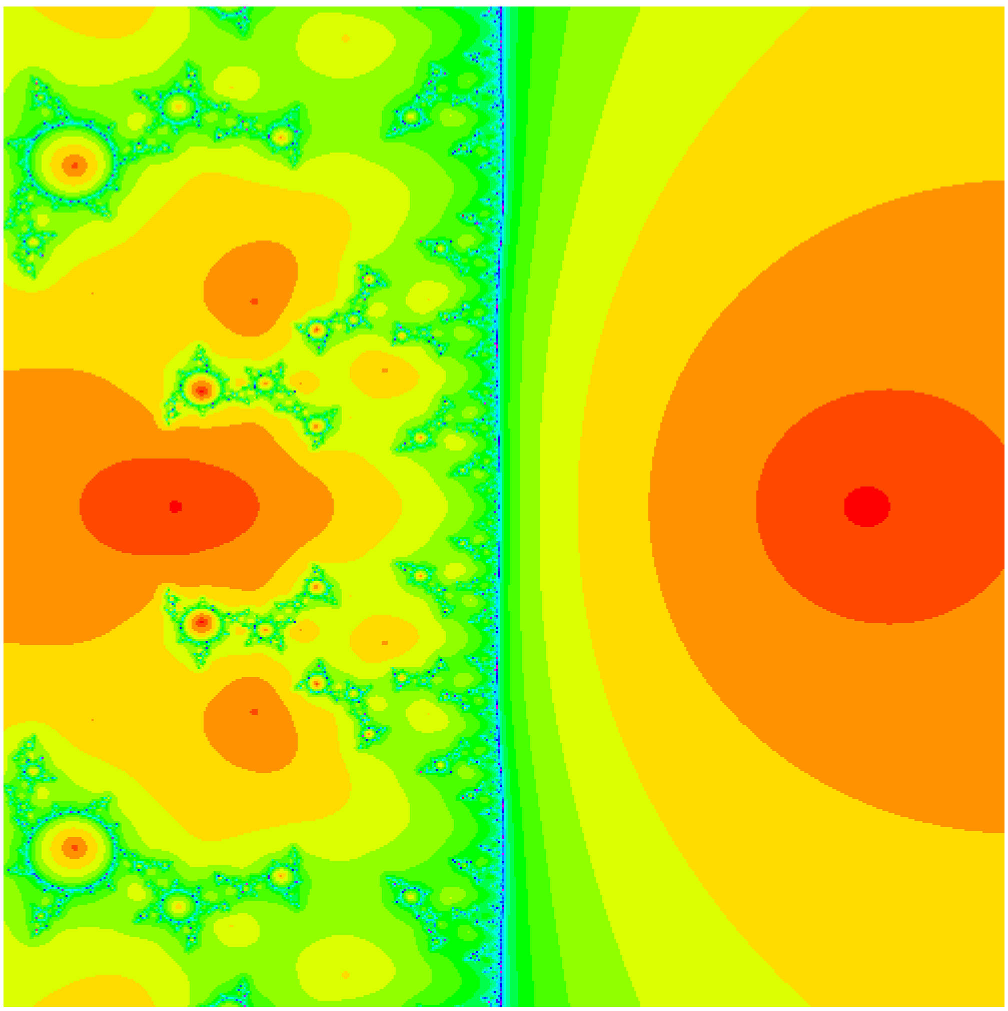
$$f(z) = z - \frac{1}{6 - 6z + 6z^2 - z^3}$$



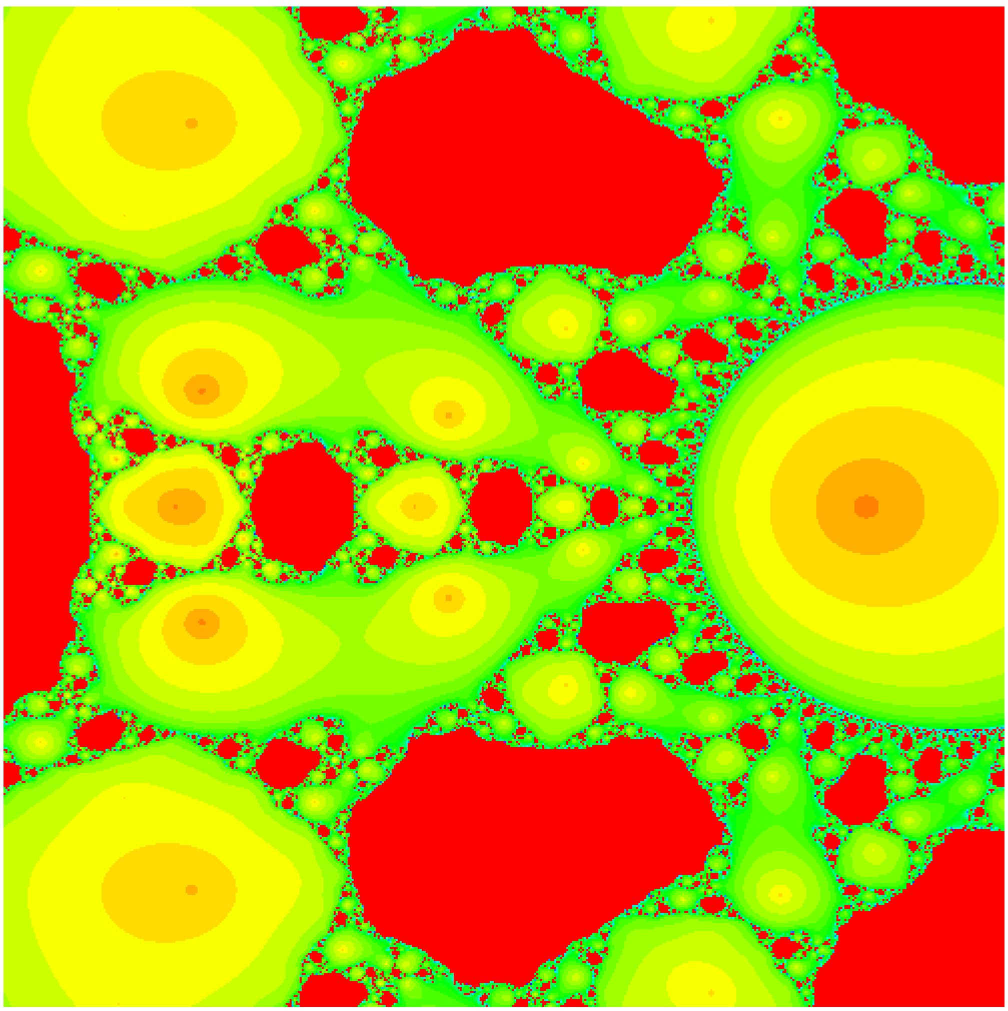
$$f(z) = z - \frac{1 + 6z^2 + z^4}{6 + 6z^2}$$



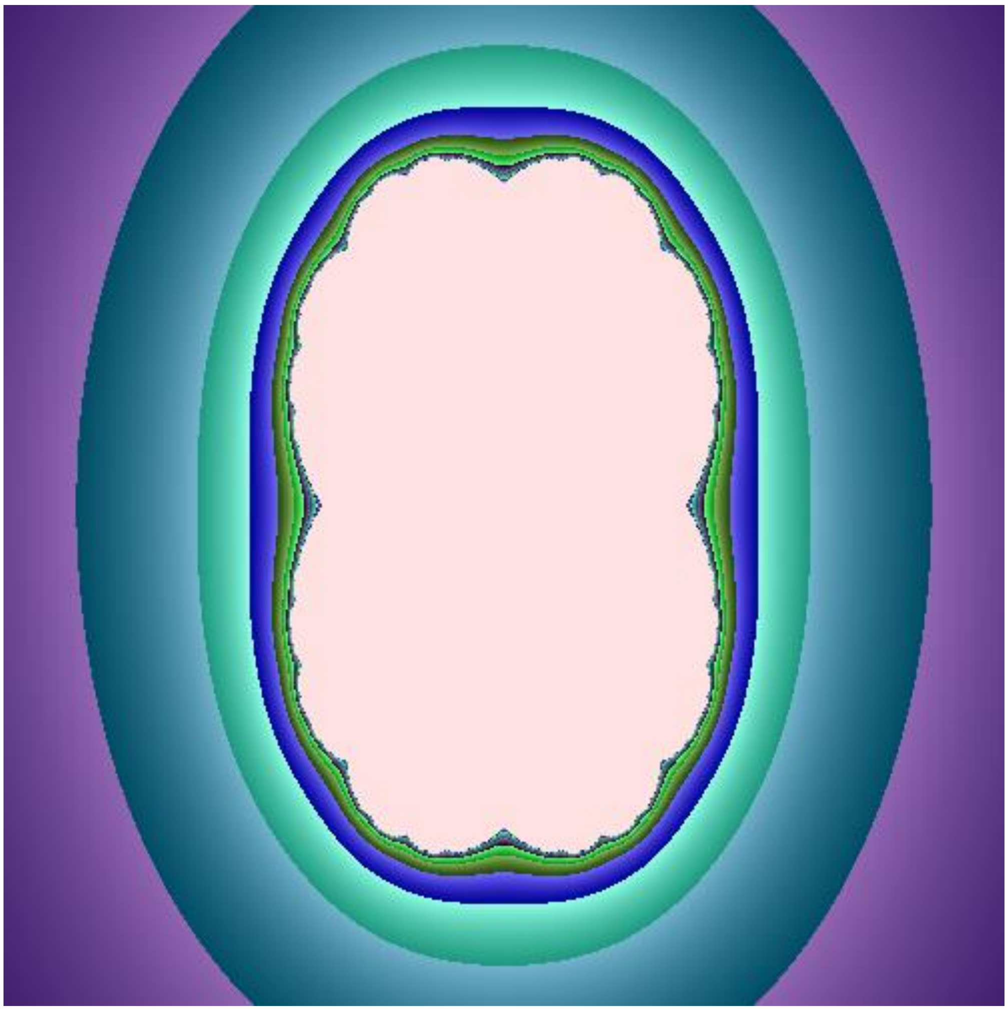
$$f(z) = z - \frac{1 + z^4}{6 - 6z + 6z^2}$$



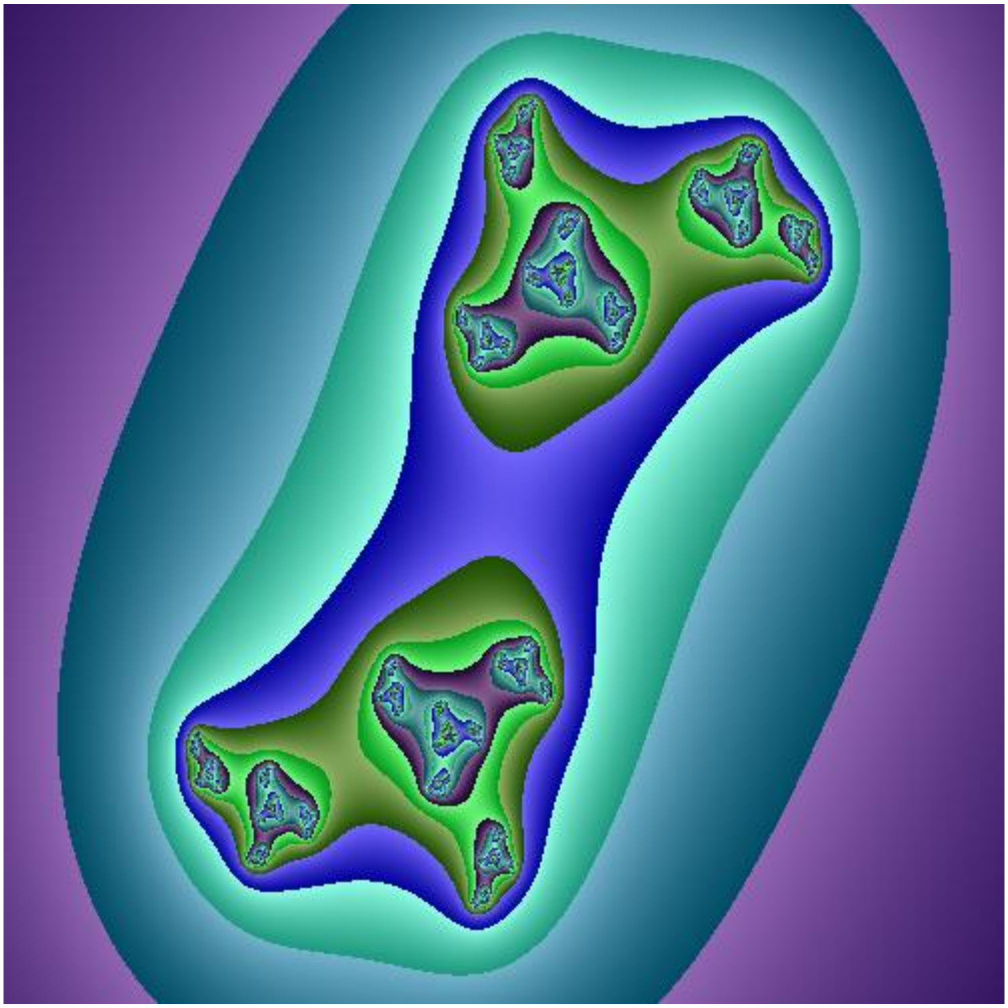
$$f(z) = z - \frac{1 - 6z^2 + 12z^3 - 3z^4}{6 - 12z + 18z^2 - 4z^3}$$



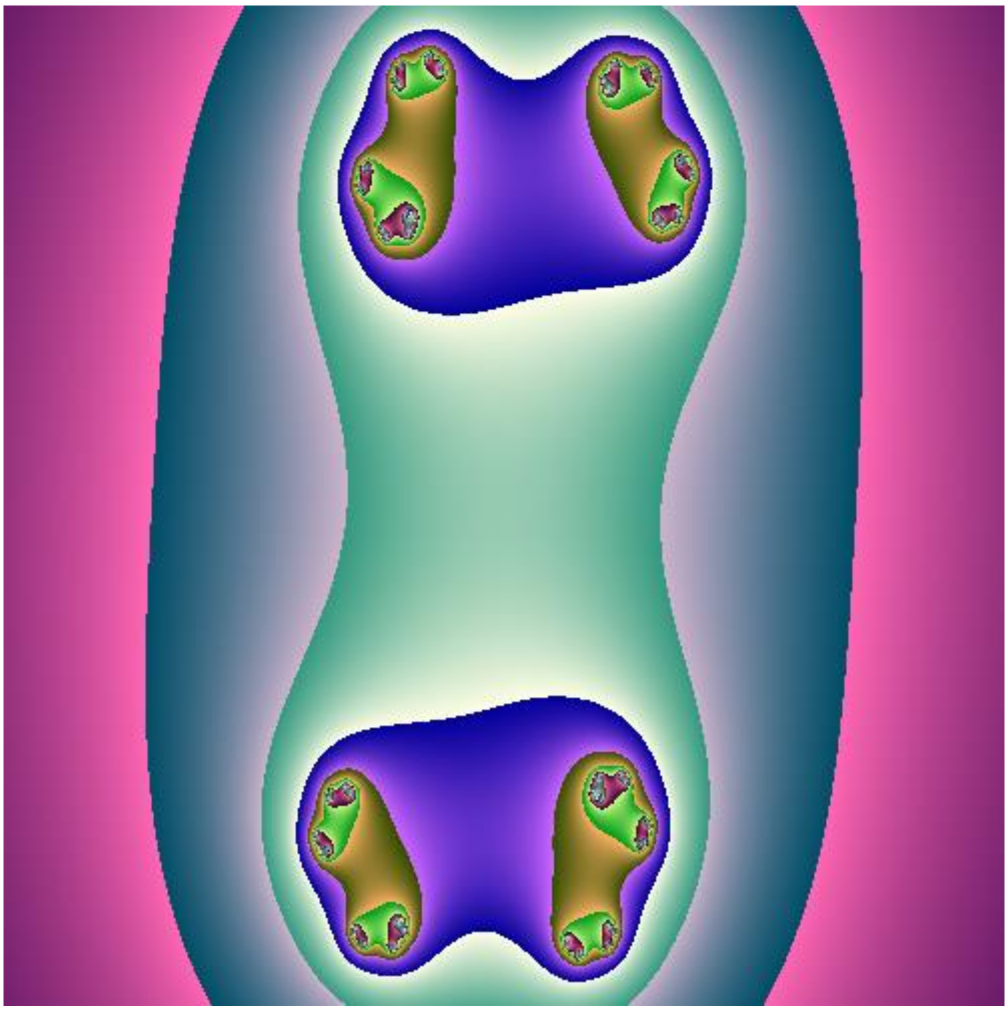
Julia set for $z = \frac{\sqrt[4]{5}-1}{\sqrt[4]{5}+1}$



Julia set for $z = \frac{\sqrt[4]{5}-1}{\sqrt[4]{5}+1} + i$



Julia set for $z = 1 + i\ \frac{\sqrt[4]{5}-1}{\sqrt[4]{5}+1}$



Referencias

1. B.C. Berndt, H.H. Chan and L.-C. Zhang, Radicals and units in Ramanujan's work, Acta Arith. 87 (1998), 145.
2. G.B. Campbell, and A. Zujev, Variations on Ramanujan's nested radicals, arxiv:1511.06865v1 [math.NT] 21 Nov 2015.
3. S. Ramanujan, Collected Papers, Chelsea, New York, 1962.