Elementary Proof that an Infinite Number of Pierpont Primes Exist

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Abstract

This paper presents a complete proof of the Pierpont Primes are infinite, even though only 16 of them have been found as of 21 Feb 2017. We use a proof found in Reference 1, that if $p > 1$ and $d > 0$ are integers, that p and $p + d$ are both primes if and only if for integer m:

$$m = (p-1)!\left(\frac{1}{p} + \frac{(-1)^{\wedge} d(d!)}{p+d}\right) + \frac{1}{p} + \frac{1}{p+d}$$

We use this proof for $d = 2^{u(n+1)}3^{v(x+1)} - 2^{u(n)}3^{v(x)}$ to prove the infinitude of Pierpont prime numbers. The author would like to give many thanks to the authors of 1001 Problems in Classical Number Theory, Jean-Marie De Koninck and Armel Mercier, 2004, Exercise Number 161 (see Reference 1). The proof provided in Exercise 6 is the key to making this paper on the Pierpont Prime Conjecture possible.

Introduction

James P. Pierpont (June 16, 1866 – December 9, 1938) was a Connecticut-born American mathematician. He did undergraduate studies at Worcester

Polytechnic Institute, initially in mechanical engineering, but turned to mathematics. He prepared his PhD at the University of Vienna under Leopold Gegenbauer and Gustav Ritter von Escherich. The **Pierpont primes** are named after the mathematician James Pierpont, who first studied them.

A **Pierpont prime** is a prime number of the form $\mathbf{2^u3^v + 1}$ for some nonnegative integer's *u* and *v*. That is, they are the prime numbers *p* for which $p - 1$ is 3-smooth. They are named after the mathematician James Pierpont.

The first few Pierpont primes are:
2, 3, 5, 7, 13, 17, 19, 37, 73, 97, 109, 163, 193, 257, 433, 487, 577, 769, 1153, 1297, 1459, 2593, 2917, 3457, 3889, 10369, 12289, 17497, 18433, 39367, 52489, 65537, 139969, 147457, 209953, 331777, 472393, 629857, 746497, 786433, 839809, 995329, ..

As of 2011, the largest known Pierpont prime is $3 \times 2^{7033641} + 1$, whose primality was discovered by Michael Herder in 2011.

Proof

Again, Pierpont primes are of the form $\mathbf{2^u3^v + 1}$**,** first we shall proceed to prove that there are an infinite number of Pierpont primes. First we shall assume that the set of 16 Pierpont Primes are finite and then we shall prove that this is false, which will prove that are Pierpont primes are infinite by contradiction. Even though we have assumed that the set of Pierpont primes is finite since there are an infinite number of prime numbers we can prove that one Pierpont prime exists outside our set of finite Pierpont Primes, then we shall prove the Pierpont primes are infinite.

Let us define the largest Piermont prime in the set of finite Pierpont primes as having ***n*** values for ***u*** and ***x*** values for ***v*** since we cannot assume that there are an equivalent number of values for ***u*** and ***v*** since some values of may be repeated. For the set of finite Piermont primes, ***n*** and ***x*** are positive integers. Therefore, we can define the largest Piermont prime as follows:

$$2^{u(n)}3^{v(x)} + 1$$

Now we must prove that a Piermont prime exists outside our finite set, namely:

$$2^{u(n+1)}3^{v(x+1)} + 1,$$ must exist.

Now we will let:

$$d = (2^{u(n+1)}3^{v(x+1)} + 1) - (2^{u(n)}3^{v(x)} + 1)$$

$$d = 2^{u(n+1)}3^{v(x+1)} - 2^{u(n)}3^{v(x)}$$

And we will let a prime ***p***:

$p = 2^{u(n)}3^{v(m)} + 1$, which we know is prime because it is the last Pierpont prime.

Therefore a prime ***p + d, where*** $d = 2^{u(n+1)}3^{v(x+1)} - 2^{u(n)}3^{v(x)}$ cannot exist that is outside our finite set of Pierpont primes otherwise we would have discovered a Pierpont prime which is outside our assumed set of finite Pierpont primes. Thus all we need to do is to prove that there exists a Pierpont prime outside our assumed finite set is to prove that ***p + d*** is prime.

Now we shall proceed to prove at least one ***p + d*** is prime as follows:

We use the proof, provided in Reference 1, that if $p > 1$ and $d > 0$ are integers, that p and $p + d$, where for our case, $d = 2^{u(n+1)}3^{v(x+1)} - 2^{u(n)}3^{v(x)}$, are both primes if and only if for positive integer m:

$$m = (p\text{-}1)!\left(\frac{1}{p} + \frac{(-1)\hat{}d(d!)}{p+d}\right) + \frac{1}{p} + \frac{1}{p+d}$$

Please note that ^ indicates an exponent, $(-1)\hat{}d$ is equivalent to (-1) raised to the d power

For our case p (also known as $2^{u(n)}3^{v(m)} + 1$, but p will be used for the remainder of this proof) is known to be prime and $d = 2^{u(n+1)}3^{v(m+1)} - 2^{u(n)}3^{v(m)}$ for Pierpont primes, therefore:

$$m = (p\text{-}1)!\left(\frac{1}{p} + \frac{(-1)\hat{}(2^{u(n+1)}3^{v(x+1)} - 2^{u(n)}3^{v(x)})\,(2^{u(n+1)}3^{v(x+1)} - 2^{u(n)}3^{v(x)})!)}{p+(2^{u(n+1)}3^{v(x+1)} - 2^{u(n)}3^{v(x)})}\right) + \frac{1}{p} + \frac{1}{p+(2^{u(n+1)}3^{v(x+1)} - 2^{u(n)}3^{v(x)})}$$

Multiplying by p, and since $2^{u(n+1)}3^{v(x+1)} - 2^{u(n)}3^{v(x)}$ is always even since 2 to any power is even and an even number subtracted from an even number is always even, then $1\hat{}(2^{u(n+1)}3^{v(x+1)} - 2^{u(n)}3^{v(x)}) = 1$

$$mp = (p)!\left(\frac{1}{p} + \frac{(2^{u(n+1)}3^{v(x+1)} - 2^{u(n)}3^{v(x)})!}{p+2^{u(n+1)}3^{v(x+1)} - 2^{u(n)}3^{v(x)}}\right) + 1 + \frac{p}{p+2^{u(n+1)}3^{v(x+1)} - 2^{u(n)}3^{v(x)}}$$

Multiplying by $(p + 2^{u(n+1)}3^{v(x+1)} - 2^{u(n)}3^{v(x)})$,

$$(p + 2^{u(n+1)}3^{v(x+1)} - 2^{u(n)}3^{v(x)})mp = (p + 2^{u(n+1)}3^{v(x+1)} - 2^{u(n)}3^{v(x)})(p)!\left(\frac{1}{p} + \frac{(2^{u(n+1)}3^{v(x+1)} - 2^{u(n)}3^{v(x)})!}{p+2^{u(n+1)}3^{v(x+1)} - 2^{u(n)}3^{v(x)}}\right) + p + 2^{u(n+1)}3^{v(x+1)} - 2^{u(n)}3^{v(x)} + p$$

Reducing again,

$$(p + 2^{u(n+1)}3^{v(x+1)} - 2^{u(n)}3^{v(x)})mp = (p)!\left(\frac{(p+2^{u(n+1)}3^{v(x+1)} - 2^{u(n)}3^{v(x)})}{p} + (2^{u(n+1)}3^{v(x+1)} - 2^{u(n)}3^{v(x)})!\right) + 2p + 2^{u(n+1)}3^{v(x+1)} - 2^{u(n)}3^{v(x)}$$

Factoring out, **(*p*)!,**

$$\left(p + \left(2^{u(n+1)}3^{v(x+1)} - 2^{u(n)}3^{v(x)}\right)\right) mp = p(p-1)!\left(\frac{\frac{\left(p+\left(2^{u(n+1)}3^{v(x+1)} - 2^{u(n)}3^{v(x)}\right)\right)}{p}}{\left(2^{u(n+1)}3^{v(x+1)} - 2^{u(n)}3^{v(x)}\right.} + \left(\left(2^{u(n+1)}3^{v(x+1)} - 2^{u(n)}3^{v(x)}\right)! + 2p + \right.$$

And reducing one final time,

$$\left(p + 2^{u(n+1)}3^{v(x+1)} - 2^{u(n)}3^{v(x)}\right)mp = (p-1)!\left(\frac{p + 2^{u(n+1)}3^{v(x+1)} - 2^{u(n)}3^{v(x)} + p\left(2^{u(n+1)}3^{v(x+1)} - 2^{u(n)}3^{v(x)}\right)!}{2^{u(n+1)}3^{v(x+1)} - 2^{u(n)}3^{v(x)}}\right) + 2p +$$

We already know **p** is prime, therefore, ***p*** = integer. Since ***p*** is an integer and by definition is an integer, the right hand side of the above equation is an integer (likewise the left hand side of the equation must also be an integer). Since the right hand side of the above equation is an integer and ***p*** and $2^{u(n+1)}3^{v(x+1)} - 2^{u(n)}3^{v(x)}$ are integers on the left hand side of the equation, then $p + 2^{u(n+1)}3^{v(x+1)} - 2^{u(n)}3^{v(x)}$ is also an integer. Therefore there are only 4 possibilities (see 1, 2a, 2b, and 2c below) that can hold for ***m*** so the left hand side of the above equation is an integer, they are as follows:

1) ***m*** is an integer, or

2) ***m*** is a rational fraction that is divisible by ***p***. This implies that ***n*** **=** $\frac{x}{p}$ where, p is prime and ***x*** is an integer. This results in the following three possibilities:

a. Since ***m*** **=** $\frac{x}{p}$, then **p =** $\frac{x}{m}$, since ***p*** is prime, then ***p*** is only divisible by ***p*** and **1**, therefore, the first possibility is for n to be equal to ***p*** or 1 in this case, which are both integers, thus ***m*** is an integer for this first case.

b. Since ***m*** **=** $\frac{x}{p}$***,*** and x is an integer, then ***x*** is not evenly divisible by ***p*** unless ***x = p,*** or ***x*** is a multiple of ***p***, where ***x = yp***, for any integer **y**. Therefore ***m*** is an integer for ***x = p*** and ***x = yp***.

c. For all other cases of, integer ***x, m = xp , m*** is not an integer.

To prove there is a Pierpont Prime, outside our set of finite Pierpont Primes, we only need to prove that there is at least one value of ***m*** that is an integer, outside our finite set. There can be an infinite number of values of ***m*** that are not integers, but that will not negate the existence of one Pierpont Prime, outside our finite set of Pierpont Primes.

First the only way that n cannot be an integer is if every ***m*** satisfies paragraph 2.c above, namely, ***m*** **=** $\frac{x}{p}$, where ***x*** is an integer***, x ≠ p, x ≠ yp, m ≠ p, and m ≠ 1*** for any integer ***y***. To prove there exists at least one Pierpont Prime outside our finite set, we will assume that no integer ***m*** exists and therefore no Pierpont Primes exist outside our finite set. Then we shall prove our assumption to be false.

Proof: Assumption no values of ***m*** are integers, specifically, every value of ***m*** is

m = $\frac{x}{p}$,where ***x*** is an integer, ***x ≠ p, x ≠ yp, m ≠ p,*** and ***m ≠ 1***, for any integer ***y***.

Paragraphs **1, 2.a, and 2.b** prove cases where ***m*** can be an integer, therefore our assumption is false and **there exist values of *m* that are integers.**

Since we have already shown that ***p*** and ***p+*** $P2 + 1$**,** where ***d*** **=** $P2 + 1$**,** are both primes if and only if for integer ***m***:

$$m = (p\text{-}1)!\left(\frac{1}{p} + \frac{(-1)^\wedge d(d!)}{p+d}\right) + \frac{1}{p} + \frac{1}{p+d}$$

It suffices to show that there is at least one integer n to prove there exists a Pierpont Prime outside our set of finite set of Pierpont Primes.

Since there exists an ***m*** = integer, we have proven that there is at least one ***p*** and $P2 + 1$ that are both prime. Since ***p*** + $P2 + 1$ is prime and it is also greater than ***p*** then it also is not in the finite set of Pierpont primes, therefore, since we have proven that there is at least one ***p*** + $P2 + 1$ that is prime, then we have proven that there is a Pierpont prime outside the our assumed finite set of Pierpont primes. This is a contradiction from our assumption that the set of Pierpont primes is finite, therefore, by contradiction the set of Pierpont primes is infinite. Also this same proof can be repeated infinitely for each finite set of Pierpont primes, in other words a new Pierpont prime can added to each set of finite Pierpont primes making the Pierpont primes countably infinite. This thoroughly proves that an infinite number of Pierpont primes exist.

References:

1) 1001 Problems in Classical Number Theory, Jean-Marie De Koninck and Armel Mercier, 2004, Exercise Number 161