Elementary Proof that an Infinite Number of Factorial Primes Exist

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Abstract

This paper presents a complete proof of the Factorial Primes are infinite, even though only 16 of them have been found as of 21 Feb 2017. We use a proof found in Reference 1, that if ***p* > *1*** and ***d* > *0*** are integers, that ***p*** and ***p*+ *d*** are both primes if and only if for integer ***m***:

$$m = (p\text{-}1)!\left(\frac{1}{p} + \frac{(-1)\hat{}d(d!)}{p+d}\right) + \frac{1}{p} + \frac{1}{p+d}$$

We use this proof for $d = n(n!)$ to prove the infinitude of Factorial prime numbers.

The author would like to give many thanks to the authors of 1001 Problems in Classical Number Theory, Jean-Marie De Koninck and Armel Mercier, 2004, Exercise Number 161 (see Reference 1). The proof provided in Exercise 6 is the key to making this paper on the Factorial Prime possible.

Proof

The only factorial that is prime is 2!, so if "factorial primes" are to be worth mentioning, the term must mean something other than a factorial that is prime. In fact, as usually defined, **factorial primes** come in two forms: factorials plus one **(*n*!+1)** and factorials minus one **(*n*!-1)**. It is conjectured that there are infinitely many of each of these.

- *n*!+1 is prime for *n*=1, 2, 3, 11, 27, 37, 41, 73, 77, 116, 154, 320, 340, 399, 427, 872, 1477, 6380, and 26951 (107707 digits).
- *n*!-1 is prime for *n*=3, 4, 6, 7, 12, 14, 30, 32, 33, 38, 94, 166, 324, 379, 469, 546, 974, 1963, 3507, 3610, 6917, 21480, and 34790 (142891 digits).

Both forms have been tested to *n*=37000

For our proof of infinitude we shall assume that the set of factorial primes is a finite set including only the ones known today. Then we will prove there is a factorial prime outside our finite set to prove by contradiction that there are an infinite number of factorial primes.

We will prove the first form and write it in a general equation:

$$p_n = (n!+1), \text{ where } p_n \text{ is prime and } n \text{ is an integer.}$$

We will define a number outside our finite set of factorial primes as follows:

$$p_{n+1} = ((n+1)! +1)$$

Now we need to prove that p_{n+1} is prime to prove there is a factorial prime outside our assumed finite set.

$$p_{n+1} = ((n+1)(n!) +1)$$

$$p_{n+1} = (n)(n!) + n! +1)$$

$$p_{n+1} - (n)(n!) = n! +1 = p_n = \text{prime number}$$

Therefore, p_{n+1} **- (n)*(n*!)** is prime p_n is prime. Now the question we must answer is

$$(p_{n+1} - (n)(n!)) + n(n!) \text{ prime?}$$

$$\text{Since } (p_{n+1} - (n)(n!)) + n(n!) = p_{n+1}$$

Therefore, if we can prove that $(p_{n+1} - (n)(n!)) + n(n!)$ then we will prove that p_{n+1} is prime.

Now we will let $p_{n+1} - (n)(n!) = p$, where p is prime. Also we shall let $n(n!) = d$

We will use the proof, provided in Reference 1, that if $p > 1$ and $d > 0$ are integers, that p and $p + d$, where for our case, $d = n(n!)$, are both primes if and only if for positive integer m:

$$m = (p\text{-}1)!\left(\frac{1}{p} + \frac{(-1)^\wedge d(d!)}{p+d}\right) + \frac{1}{p} + \frac{1}{p+d}$$

Please note that ^ indicates an exponent, $(-1)^\wedge d$ is equivalent to (-1) raised to the d power

For our case p is known to be prime and $d = n(n!)$ for Factorial primes, therefore:

$$m = (p\text{-}1)!\left(\frac{1}{p} + \frac{(-1)^\wedge(n(n!))(n(n!)!))}{p+P2+1}\right) + \frac{1}{p} + \frac{1}{p+n(n!)}$$

Multiplying by p, and since $n(n!)$ is always even since $n!$ is always odd since it is a factorial that is multiplied by 2, then $(-1)^\wedge(n(n!)) = 1$

$$mp = (p)!\left(\frac{1}{p} + \frac{(n(n!))!}{p+n(n!)}\right) + 1 + \frac{p}{p+n(n!)}$$

Multiplying by ***(p + n(n!))***,

$$(p + n(n!))mp = (p + n(n!))(p)!\left(\frac{1}{p} + \frac{(n(n!))!}{p+n(n!)}\right) + p + \; + n(n!) \; + p$$

Reducing again,

$$(p + n(n!))mp = (p)!\left(\frac{(p+n(n!))}{p} + (n(n!))!\right) + 2p + n(n!)$$

Factoring out, **(*p*)!,**

$$(p + n(n!))mp = p(p-1)!\left(\frac{(p+n(n!))}{p} + (n(n!))!\right) + 2p + n(n!)$$

And reducing one final time,

$$(p + n(n!))mp = (p-1)!\left(p + n(n!) \; + p(n(n!))!\right) + 2p + n(n!)$$

We already know **p** is prime, therefore, ***p*** = integer. Since ***p*** is an integer and by definition $n(n!)$ is an integer, the right hand side of the above equation is an integer (likewise the left hand side of the equation must also be an integer). Since the right hand side of the above equation is an integer and ***p*** and $n(n!)$ are integers on the left hand side of the equation, then ***p + n(n!)*** is also an integer. Therefore there are only 4 possibilities (see 1, 2a, 2b, and 2c below) that can hold for ***m*** so the left hand side of the above equation is an integer, they are as follows:

1) ***m*** is an integer, or

2) ***m*** is a rational fraction that is divisible by ***p***. This implies that ***n*** = $\frac{x}{p}$ where, p is prime and ***x*** is an integer. This results in the following three possibilities:

a. Since ***m*** = $\frac{x}{p}$, then **p** = $\frac{x}{m}$, since ***p*** is prime, then ***p*** is only divisible by ***p*** and **1**, therefore, the first possibility is for n to be equal to ***p*** or 1 in this case, which are both integers, thus ***m*** is an integer for this first case.

b. Since ***m*** = $\frac{x}{p}$***,*** and x is an integer, then ***x*** is not evenly divisible by ***p*** unless ***x = p,*** or ***x*** is a multiple of ***p***, where ***x = yp***, for any integer **y**. Therefore ***m*** is an integer for ***x = p*** and ***x = yp***.

c. For all other cases of, integer ***x, m = xp , m*** is not an integer.

To prove there is a Factorial Prime, outside our set of finite Factorial Primes, we only need to prove that there is at least one value of ***m*** that is an integer, outside our finite set. There can be an infinite number of values of ***m*** that are not integers, but that will not negate the existence of one Factorial Prime, outside our finite set of Factorial Primes.

First the only way that n cannot be an integer is if every ***m*** satisfies paragraph 2.c above, namely, ***m*** = $\frac{x}{p}$, where ***x*** is an integer***, x ≠ p, x ≠ yp, m ≠ p, and m ≠ 1*** for any integer ***y***. To prove there exists at least one Factorial Prime outside our finite set, we will assume that no integer ***m*** exists and therefore no Factorial Primes exist outside our finite set. Then we shall prove our assumption to be false.

Proof: Assumption no values of ***m*** are integers, specifically, every value of ***m*** is

$m = \dfrac{x}{p}$,where x is an integer, $x \neq p, x \neq yp, m \neq p,$ and $m \neq 1$, for any integer y. Paragraphs **1, 2.a, and 2.b** prove cases where ***m*** can be an integer, therefore our assumption is false and **there exist values of *m* that are integers.**

Since we have already shown that ***p*** and ***p+ n(n!),*** where ***d = n(n!),*** are both primes if and only if for integer ***m***:

$$m = (p\text{-}1)!\left(\frac{1}{p} + \frac{(-1)\hat{}d(d!)}{p+d}\right) + \frac{1}{p} + \frac{1}{p+d}$$

It suffices to show that there is at least one integer ***m*** to prove there exists a Factorial Prime outside our set of finite set of Factorial Primes.

Since there exists an ***m*** = integer, we have proven that there is at least one ***p*** and $p + n(n!)$ that are both prime. Since ***p*** + ***n(n!)*** is prime and it is also greater than ***p*** then it also is not in the finite set of Factorial primes, therefore, since we have proven that there is at least one ***p*** + ***n(n!)*** that is prime, then we have proven that there is a Factorial prime outside the our assumed finite set of Factorial primes. This is a contradiction from our assumption that the set of Factorial primes is finite, therefore, by contradiction the set of Factorial primes is infinite. Also this same proof can be repeated infinitely for each finite set of Factorial primes, in other words a new Factorial prime can added to each set of finite Factorial primes making the Factorial primes countably infinite. This thoroughly proves that an infinite number of Factorial primes exist, since $p_{n+1} = ((n+1)! + 1)$ is prime. This proof can be repeated for $p_{n+1} = ((n-1)! + 1)$, therefore, this proof proves that both forms of the Factorial Primes exist.

References:

1) 1001 Problems in Classical Number Theory, Jean-Marie De Koninck and Armel Mercier, 2004, Exercise Number 161