Elementary Proof Grimm's Conjecture

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Abstract

In mathematics, and in particular number theory, Grimm's Conjecture (named after Karl Albert Grimm) states that to each element of a set of consecutive composite numbers one can assign a distinct prime that divides it. It was first published in *American Mathematical Monthly*, 76(1969) 1126-1128.

The Formal statement defining Grimm's Conjecture, still unproved, is as follows:

Suppose $n + 1, n + 2, \ldots, n + k$ are all composite numbers, then there are k distinct primes p_i such that p_i divides $n + i$ for $1 \leq i \leq k$.

A weaker, though still unproven, version of this conjecture goes: If there is no prime in the interval $[n+1, n+k]$, then $\prod_{x \leq k}(n+x)$ has at least k distinct prime divisors.

Proof

In number theory, the prime factors of a positive integer are the prime numbers that divide that integer exactly (see reference 1). The prime factorization of a positive integer is a list of the integer's prime factors, together with their multiplicities; the process of determining these factors is called integer factorization. The fundamental theorem of arithmetic says that every positive integer has a single unique prime factorization (see reference 2).

To shorten prime factorizations, factors are often expressed in powers (multiplicities). For example,

$$\mathbf{360 = 2 \times 2 \times 2 \times 3 \times 3 \times 5 = 2^3 \times 3^2 \times 5}$$

in which the factors 2, 3 and 5 have multiplicities of 3, 2 and 1, respectively.
For a prime factor p of n, the multiplicity of p is the largest exponent a for which p^a divides n exactly.

A strait forward proof of Grimm's Conjecture is provided by the author, however, this proof is totally dependent on the fundamental theorem of arithmetic. Since by this theorem every positive integer has a single unique prime factorization, then by deduction, every positive composite integer has a single unique prime factorization. Therefore, every $n + 1, n + 2, \ldots, n + k$ are all composite numbers, therefore each has a single unique prime factorization. Furthermore, since every consecutive set of composite numbers in the form $n + 1, n + 2, \ldots, n + k$ has a unique prime number that divides it since they have a single unique prime factorization, according to the fundamental theorem of arithmetic. As a result, the Grimm's Conjecture is proven to be true since a unique prime divisor makes the following true for all k, where, $1 \leq i \leq k$, therefore, if:

$n + 1, n + 2, \ldots, n + k$ are all composite numbers, then there are k distinct primes p_i such that p_i divides $n + i$ for $1 \leq i \leq k$ is true for all n and k. This conclusively proves that Grimm's Conjecture is true.

References:

1. Jensen, Gary R. (2004). Arithmetic for Teachers: With Applications and Topics from Geometry. American Mathematical Society.

2. Riesel, Hans (1994), Prime numbers and computer methods for factorization, Basel, Switzerland: Birkhäuser, ISBN 978-0-8176-3743-9