Elementary Proof that Hall's Conjecture is False

Stephen Marshall
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Abstract

In mathematics, **Hall's conjecture** is an open question, as of 2015, on the differences cube x^3 that are not equal must lie a substantial distance apart. This question arose from consideration of the Mordell equation in the theory of integer points on elliptic curves. The original version of Hall's conjecture, formulated by Marshall Hall, Jr. in 1970, says that there is a positive constant *C* such that for any integers *x* and *y* for which $y^2 \neq x^3$,

$$|y^2 - x^3| > C\sqrt{|x|}$$

Proof

Rearranging and solving for C is shown below:

$$\left| x^{0.5}\, y^2 - x^{3.5} \right| > C$$

Now we will try an example by letting x = 5 and y = 11, solving for the above equation, $C < 2$

Hall suggested that perhaps *C* could be taken as 1/5, which was consistent with all the data known at that time, and consistent with our example above.

However, the original, *strong*, form of the conjecture with exponent 1/2 has never been disproved, although it is no longer believed to be true. For example, in 1998, Noam Elkies found the following example:

$$\mathbf{447884928428402042307918^2 - 5853886516781223^3 = -1641843,}$$

for which compatibility with Hall's conjecture would require *C* to be less than **.0214 ≈ 1/50**, so roughly 10 times smaller than the original choice of 1/5 that Hall suggested.

The author has a very simple proof to disprove **Hall's conjecture**, specifically that there <u>is</u> not a positive constant *C* such that for any integers *x* and *y* for which $y^2 \neq x^3$,

$$|y^2 - x^3| > C\sqrt{|x|}$$

First, since **Hall's conjecture** must hold for all integers x and *y* for which $y^2 \neq x^3$, therefore, we can set x equal to infinity. We can do this because by definition the smallest ordinal infinity is that of the positive integers, and any set which has the cardinality of the integers is countably infinite. Therefore, we set x = **∞,** then the absolute value of

$$\mathbf{|y^2 - x^3| = \infty}$$

therefore, using Halls Conjecture to calculate the constant C we need to calculate the absolute value of $\mathbf{x^{1/2}}$**,** since x = **∞,** then $\mathbf{x^{1/2} = \infty}$ also. Therefore reducing the equation of **Hall's conjecture** to calculate the value of the constant C, we must divide both sides if the equation by **∞,** to reduce $\mathbf{x^{1/2}}$ from the right side of the Halls equation (see on page 1 above). This results in the following:

$$C < \infty/\infty,$$ which is undefined by definition

Division by infinity is not allowed or defined in mathematics, therefore, there is no positive constant C when x = **∞**. Therefore, there is not a positive constant *C* such that for any integers *x* and *y* for which $y^2 \neq x^3$. Therefore, we have proved that Halls Conjecture is false and that for the infinity number of integers x and y that exists there is not always a Halls constant *C* that exists.