Fiez identity for interacting four-fermion in four-dimensional space-time

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Abstract

the simple case of Fiez identity for interacting four-fermion in four-dimensional space-time is worked out explicitly.

1 Spinor Algebra

These matrices

$$\epsilon^{AB} \overset{*}{=} \epsilon^{A'B'} \overset{*}{=} \epsilon_{AB} \overset{*}{=} \epsilon_{A'B'} \overset{*}{=} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{1}$$

are using to raising and lowering the spinor indices.

$$\boxed{\begin{array}{c} \xi^A = \epsilon^{AB}\xi_B \ , -\xi_A = \epsilon_{AB}\xi^B \ , \\ \xi^{A'} = \epsilon^{A'B'}\xi_{B'} \ , -\xi_{A'} = \epsilon_{A'B'}\xi^{B'} \ . \end{array}} \tag{2}$$

You must observed that

$$[\epsilon^{AB}][\epsilon_{BC}] = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = -\mathbb{1}_{2\times 2} = [\epsilon^A{}_C] \ . \tag{3}$$

Then it becomes

$$\boxed{\delta^A_B = \epsilon_B{}^A = -\epsilon^A{}_B \ ,} \tag{4}$$

It is worth to notice that

$$M^A N_A = -M_A N^A \quad , M^{A'} N_{A'} = -M_{A'} N^{A'} \ , \tag{5}$$

1.1 Relation to the Metric

$$g_{\mu\nu} = \eta_{IJ} e^I_\mu e^J_\nu \ , \tag{6}$$

$$\eta_{IJ} = diag(-1, 1, 1, 1) \tag{7}$$

The basis one-forms e^a_μ correspond to spinor-valued one-forms

$$e^{AA'}{}_\mu = e^I{}_\mu \sigma_I{}^{AA'} \ , \tag{8}$$

where the soldering , σ, is defi ned to be i times the Infeld-Van de Waerden translation symbol

$$\sigma_0 = \frac{i}{\sqrt{2}} \mathbb{I}_{2\times 2}, \qquad \sigma_i = \frac{i}{\sqrt{2}} \Sigma_i \ ,$$

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where Σ_i is Pauli matrices, $A, B, ... = 0, 1, \quad A', B', ... = 0', 1'$.

$$\Sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \Sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \Sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{9}$$

Numerically, I adopted the notation such that

$$\sigma_I^{AA'} \stackrel{!}{=} \{\sigma_0, \sigma_i\} \equiv \sigma_I , \tag{10}$$

$\stackrel{!}{=}$ means numerically equals. The well known identity is

$$(\sigma_I \bar{\sigma}_J + \sigma_J \bar{\sigma}_I) = \eta_{IJ} \otimes \mathbb{1}_{2\times 2} , \tag{11}$$

where

$$\bar{\sigma}_I \equiv \{\sigma_0, -\sigma_i\} . \tag{12}$$

1.2 soldering (or solder form)

If you want to see the subscripts version of the solder form A.K.A. $\sigma_{I\ AA'}$. The most natural way

$$\begin{aligned} \sigma_{I\ AA'} &:= (-\epsilon_{AB})(-\epsilon_{A'B'})\sigma_I{}^{BB'} , & (13) \\ [\sigma_{I\ AA'}] &= -[\epsilon_{AB}][\sigma_I{}^{BB'}][\epsilon_{B'A'}] , & (14) \\ &\stackrel{*}{=} \bar{\sigma}_I , \quad \text{Just try!} , & (15) \\ &= [\bar{\sigma}_{I\ A'A}] \ ,' \text{ flips positions, needed for matrix mult with } \sigma . & (16) \end{aligned}$$

Now I have

$$\sigma_{I\ AA'} \equiv \bar{\sigma}_{I\ A'A} \tag{17}$$

Now (11) can be written to be

$$\sigma_I{}^{AA'} \sigma_{J\ BA'} + \sigma_J{}^{AA'} \sigma_{I\ BA'} = \eta_{IJ} \otimes \delta^A_B , \tag{18}$$

Consequently,

$$\sigma_{(I}{}^{AA'} \sigma_{J)\ AA'} = \eta_{IJ} \tag{19}$$

Then one can found the reverse of (8) as

$$\boxed{e^I{}_\mu = e^{AA'}_\mu \sigma^I_{AA'}} \tag{20}$$

You will also found that

$$e_{(\mu}{}^{AA'} e_{\nu)\ AA'} = g_{\mu\nu} \tag{21}$$

2 Fierz identity

To perfectly done, we may need to define the bi-spinor

$$a^\mu = \begin{pmatrix} a^A \\ a^{A'} \end{pmatrix} \tag{22}$$

I think everything are almost analogous to the case of two-spinors since now we work on the group of two-spinor⊕two-spinor, for example $SL(2, C) \oplus \overline{SL(2, C)}$. Analogous to ϵ_{AB}, we have $\epsilon_{\mu\nu} = \epsilon_{AB} \oplus \epsilon_{A'B'}$

$$\underbrace{\begin{pmatrix} \epsilon_{AB} & 0 \\ 0 & \epsilon_{A'B'} \end{pmatrix}}_{\epsilon_{\mu\nu}} \underbrace{\begin{pmatrix} a^B \\ a^{B'} \end{pmatrix}}_{a^\nu} = \underbrace{-\begin{pmatrix} a_A \\ a_{A'} \end{pmatrix}}_{-a_\mu} \tag{23}$$

But I will not use these tools properly at this time. I will employ other style(ค่อยปรับแต่งทีหลัง) to obtain the Fierz identity in four dimensional spacetime. The starting point is considering of the quantity

$$MN \equiv \underbrace{\left(\overline{\psi}_1 M\psi_2\right)}_{scalar}\underbrace{\left(\overline{\psi}_3 N\psi_4\right)}_{scalar} = \left(\overline{\psi}_{1\alpha}M^{\alpha}{}_{\beta}\psi_2^{\beta}\right)\left(\overline{\psi}_{3\gamma}N^{\gamma}{}_{\delta}\psi_4^{\delta}\right) , \tag{24}$$

$$= \overline{\psi}_{1\alpha}M^{\alpha}{}_{\beta}\left(\psi_2^{\beta}\otimes\overline{\psi}_{3\gamma}\right)N^{\gamma}{}_{\delta}\psi_4^{\delta} , \tag{25}$$

$$\tag{26}$$

Let us define

$$\psi_2^{\beta}\otimes\overline{\psi}_{3\gamma} =: P^{\beta}{}_{\gamma} , \tag{27}$$

then expand in complete Clifford basis

$$P^{\beta}{}_{\gamma} = C^A\Gamma_A{}^{\beta}{}_{\gamma} . \tag{28}$$

So

$$C^A = P^{\beta}{}_{\gamma}\,\Gamma^{A\gamma}{}_{\beta} . \tag{29}$$

We now have

$$MN = \overline{\psi}_{1\alpha}M^{\alpha}{}_{\beta}\left(C^A\Gamma_I{}^{\beta}{}_{\gamma}\right)N^{\gamma}{}_{\delta}\psi_4^{\delta} , \tag{30}$$

$$= \underbrace{\left(\overline{\psi}_{1\alpha}M^{\alpha}{}_{\beta}\Gamma_I{}^{\beta}{}_{\gamma}N^{\gamma}{}_{\delta}\psi_4^{\delta}\right)}_{scalar}\underbrace{\left(C^A\right)}_{scalar} , \tag{31}$$

$$\left(\overline{\psi}_1 M\psi_2\right)\left(\overline{\psi}_3 N\psi_4\right) = (\overline{\psi}_1 M\Gamma_A N\psi_4)(\psi_2^{\beta}\otimes\overline{\psi}_{3\gamma}\Gamma^{A\gamma}{}_{\beta}) , \tag{32}$$

$$= (\overline{\psi}_1 M\Gamma_I N\psi_4)(\overline{\psi}_{2\,\alpha}\epsilon^{\beta\alpha}\otimes\psi_3^{\delta}\epsilon_{\delta\gamma}\Gamma^{A\gamma}{}_{\beta}) , \tag{33}$$

$$= (\overline{\psi}_1 M\Gamma_I N\psi_4)((\overline{\psi}_{2\,\alpha}\otimes\psi_3^{\delta})\epsilon_{\delta\gamma}\epsilon^{\beta\alpha}\Gamma^{A\gamma}{}_{\beta}) , \tag{34}$$

$$= (\overline{\psi}_1 M\Gamma_I N\psi_4)((\overline{\psi}_{2\,\alpha}\otimes\psi_3^{\delta})\epsilon_{\gamma\delta}\epsilon^{\alpha\beta}\Gamma^{A\gamma}{}_{\beta}) , \tag{35}$$

$$\tag{36}$$

Analogous to (16) we should have

$$\epsilon_{\delta\gamma}\Gamma^{A\gamma}{}_{\beta}\epsilon^{\beta\alpha} = -\Gamma^A{}_{\delta}{}^{\alpha} \tag{37}$$

So

$$\left(\overline{\psi}_1 M\psi_2\right)\left(\overline{\psi}_3 N\psi_4\right) = -(\overline{\psi}_1 M\Gamma_A N\psi_4)(\psi_3^{\delta}\Gamma^A{}_{\delta}{}^{\alpha}\overline{\psi}_{2\,\alpha}) , \tag{38}$$

$$= -(\overline{\psi}_1 M\Gamma_A N\psi_4)(\overline{\psi}_{3\delta}\Gamma^{A\delta}{}_{\alpha}\psi_2{}^{\alpha})(-1)^2 , \tag{39}$$

$$\begin{aligned} = & -\frac{1}{4}(\overline{\psi}_1 MN\psi_4)(\overline{\psi}_{3\,\alpha}\delta^{\alpha}{}_{\delta}\psi_2^{\delta}) \\ & -\frac{1}{4}(\overline{\psi}_1 M\gamma_I N\psi_4)(\overline{\psi}_{3\,\alpha}\gamma^{I\alpha}{}_{\delta}\psi_2^{\delta}) \\ & -\frac{1}{8}(\overline{\psi}_1 M\gamma_{[I}\gamma_{J]}N\psi_4)(\overline{\psi}_{3\,\alpha}(\gamma^{[I}\gamma^{J]})^{\alpha}{}_{\delta}\psi_2^{\delta}) \\ & -\frac{1}{4}(\overline{\psi}_1 M\gamma_5\gamma_I N\psi_4)(\overline{\psi}_{3\,\alpha}(\gamma_5\gamma^I)^{\alpha}{}_{\delta}\psi_2^{\delta}) \\ & -\frac{1}{4}(\overline{\psi}_1 M\gamma_5 N\psi_4)(\overline{\psi}_{3\,\alpha}\gamma_5{}^{\alpha}{}_{\delta}\psi_2^{\delta}) , \end{aligned} \tag{40}$$

$$\begin{aligned} = & -\frac{1}{4}(\overline{\psi}_1 MN\psi_4)(\overline{\psi}_3\psi_2) \\ & -\frac{1}{4}(\overline{\psi}_1 M\gamma_I N\psi_4)(\overline{\psi}_3\gamma^I\psi_2) \\ & -\frac{1}{8}(\overline{\psi}_1 M\gamma_{[I}\gamma_{J]}N\psi_4)(\overline{\psi}_{3\,\alpha}(\gamma^{[I\alpha}{}_{\lambda}\gamma^{J]\lambda}{}_{\delta})\psi_2^{\delta}) \\ & -\frac{1}{4}(\overline{\psi}_1 M\gamma_5\gamma_I N\psi_4)(\overline{\psi}_{3\,\alpha}(\gamma_5{}^{\alpha}{}_{\lambda}\gamma^{I\lambda}{}_{\delta})\psi_2^{\delta}) \end{aligned}$$

$$
\begin{aligned}
& -\frac{1}{4}(\overline{\psi}_1 M\gamma_5 N\psi_4)(\overline{\psi}_3\gamma_5\psi_2)\ , && (41)\\
= \quad & -\frac{1}{4}(\overline{\psi}_1 MN\psi_4)(\overline{\psi}_3\psi_2)\\
& -\frac{1}{4}(\overline{\psi}_1 M\gamma_I N\psi_4)(\overline{\psi}_3\gamma^I\psi_2)\\
& -\frac{1}{8}(\overline{\psi}_1 M\gamma_{[I}\gamma_{J]} N\psi_4)(\overline{\psi}_{3\,\alpha}(-\gamma^{[I\alpha\lambda}\gamma^{J]}{}_{\lambda\delta})\psi_2^\delta)\\
& -\frac{1}{4}(\overline{\psi}_1 M\gamma_5\gamma_I N\psi_4)(\overline{\psi}_{3\,\alpha}(-\gamma_5{}^{\alpha\lambda}\gamma^I{}_{\lambda\delta})\psi_2^\delta)\\
& -\frac{1}{4}(\overline{\psi}_1 M\gamma_5 N\psi_4)(\overline{\psi}_3\gamma_5\psi_2)\ , && (42)\\
& -\frac{1}{4}(\overline{\psi}_1 M\gamma_5 N\psi_4)(\overline{\psi}_3\gamma_5\psi_2)\ , && (43)\\
= \quad & -\frac{1}{4}(\overline{\psi}_1 MN\psi_4)(\overline{\psi}_3\psi_2)\\
& -\frac{1}{4}(\overline{\psi}_1 M\gamma_I N\psi_4)(\overline{\psi}_3\gamma^I\psi_2)\\
& +\frac{1}{8}(\overline{\psi}_1 M\gamma_{[I}\gamma_{J]} N\psi_4)(\overline{\psi}_3(\gamma^{[I}\gamma^{J]})\psi_2)\\
& +\frac{1}{4}(\overline{\psi}_1 M\gamma_5\gamma_I N\psi_4)(\overline{\psi}_3(\gamma_5\gamma^I)\psi_2)\\
& -\frac{1}{4}(\overline{\psi}_1 M\gamma_5 N\psi_4)(\overline{\psi}_3\gamma_5\psi_2)\ , && (44)\\
& && (45)
\end{aligned}
$$

where $\gamma_5 \equiv \gamma^0\gamma^1\gamma^2\gamma^3$, in our original notation it wil replaced by $\star$. So for real representation we have

$$
\psi_2\overline{\psi}_3 = -\frac{1}{4}\overline{\psi}_3\psi_2 - \frac{1}{4}(\overline{\psi}_2\gamma^I\psi_2)\gamma_I + \frac{1}{8}(\overline{\psi}_3\gamma^{[I}\gamma^{J]}\psi_2)\gamma_{[I}\gamma_{J]} + \frac{1}{4}(\overline{\psi}_3\gamma_5\gamma^I\psi_2)\gamma_5\gamma_I - \frac{1}{4}(\overline{\psi}_3\gamma_5\psi_2)\gamma_5\ . \tag{46}
$$

For both real and complex representation we have

$$
\boxed{\psi_2\overline{\psi}_3 = -\frac{1}{4}\overline{\psi}_3\psi_2 - \frac{1}{4}(\overline{\psi}_3\gamma^I\psi_2)\gamma_I + \frac{1}{8}(\overline{\psi}_3\gamma^{[I}\gamma^{J]}\psi_2)\gamma_{[I}\gamma_{J]} + \frac{1}{4}(\overline{\psi}_3\star\gamma^I\psi_2)\star\gamma_I - \frac{1}{4}(\overline{\psi}_3\star\psi_2)\star\ .} \tag{47}
$$

Note that

$$
\overline{\psi}_\beta\otimes\chi^\beta = \psi^\rho\epsilon_{\rho\beta}\otimes\epsilon^{\beta\nu}\overline{\chi}_\nu = -\psi^\rho\epsilon_\rho{}^\nu\otimes\overline{\chi}_\nu = -\psi^\rho\otimes\overline{\chi}_\rho \tag{48}
$$

3 COnclusion

Hope this small paper may helpful, have afun :)