Was the Vector Field in Weyl's 1918 Theory Unnecessary?

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Introduction

In 1918, several years after Einstein's announcement of the general theory of relativity, the great German mathematical physicist Hermann Weyl proposed a generalization of Riemannian geometry that appeared to show that electromagnetism—like gravitation—is a purely geometrical construct. Weyl accomplished this by eliminating the constraint of invariant vector length under parallel transport in Riemannian geometry, a restriction that he believed was unnecessary. By sidestepping this constraint, Weyl was able to derive all of Maxwell's equations from a formalism that was initially hailed as the first workable unification of gravity and electromagnetism, the only forces of Nature known at the time.

Weyl's effort led directly to the notion of *gauge invariance*, which today is a cornerstone of modern quantum theory. Although Weyl's original 1918 theory was found to be unphysical (by Einstein, no less), it also introduced the notion of *conformal invariance* in general relativity, an idea that may ultimately lead to breakthroughs in the understanding of dark matter and dark energy.

In this paper we show that the so-called Weyl vector field ϕ_μ, which Weyl used to fashion a conformally invariant connection $\Gamma^\lambda_{\mu\nu}$ for his action Lagrangian, is unnecessary. By assuming only that the Ricci scalar R be a non-zero universal constant, we demonstrate that the Weyl Lagrangian $\sqrt{-g}\,R^2$ in ordinary Riemannian space is conformally invariant. As a consequence, the invariance of vector length under parallel transport is preserved, as is *metricity* (the vanishing of the covariant derivative of the metric tensor $g_{\mu\nu}$ and its variants), and thus Einstein's primary objection to Weyl's theory is removed.

Notation

Following Adler et al., we will denote covariant differentiation with a double subscripted bar and ordinary partial differentiations with a single subscripted bar, as in the usual identity

$$\xi_{\mu||\nu} = \xi_{\mu|\nu} - \xi_\lambda \begin{Bmatrix} \lambda \\ \mu\nu \end{Bmatrix}$$

where the quantity in braces is the Christoffel connection

$$\begin{Bmatrix} \lambda \\ \mu\nu \end{Bmatrix} = \frac{1}{2}\, g^{\lambda\beta}\left(g_{\beta\nu|\mu} + g_{\mu\beta|\nu} - g_{\mu\nu|\beta}\right)$$

1. Conformal Variations

We define a conformal (or scale) variation by way of the local transformations

$$\bar{g}_{\mu\nu} = e^{\pi} g_{\mu\nu}, \quad \bar{g}^{\mu\nu} = e^{-\pi} g^{\mu\nu}$$

where $\pi(x)$ is an arbitrary, smooth scalar field. For simplicity we shall consider only infinitesimal variations $\pi \to \varepsilon\pi$, where $\varepsilon \ll 1$. We then have

$$\bar{g}_{\mu\nu} = (1+\varepsilon\pi)\, g_{\mu\nu} \quad \text{or} \quad \delta g_{\mu\nu} = \varepsilon\pi g_{\mu\nu} \tag{1.1}$$

where $\delta g_{\mu\nu} = \bar{g}_{\mu\nu} - g_{\mu\nu}$. Similarly,

$$\delta g^{\mu\nu} = -\varepsilon\pi g^{\mu\nu} \tag{1.2}$$

We will also have need of the variation of the metric determinant $\sqrt{-g}$, which is given by

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\, g_{\mu\nu}\,\delta g^{\mu\nu} \tag{1.3}$$

where $\delta g^{\mu\nu}$ here is *any* arbitrary variation of the metric tensor. For a conformal variation in four dimensions, this becomes

$$\delta\sqrt{-g} = 2\varepsilon\pi\sqrt{-g} \tag{1.4}$$

2. Review of Weyl's Theory

Weyl's 1918 theory has an extensive literature, and it will be assumed that the reader is already familiar with it. The basic idea is the replacement of the Riemannian connection

$$\Gamma^{\lambda}_{\mu\nu} = \left\{ {\lambda \atop \mu\nu} \right\} = \frac{1}{2}\, g^{\lambda\beta}\left(g_{\beta\nu|\mu} + g_{\mu\beta|\nu} - g_{\mu\nu|\beta}\right)$$

with the non-Riemannian form

$$\Gamma^{\lambda}_{\mu\nu} = \left\{ {\lambda \atop \mu\nu} \right\} - \delta^{\lambda}_{\mu}\phi_{\nu} - \delta^{\lambda}_{\nu}\phi_{\mu} + g_{\mu\nu}g^{\lambda\beta}\phi_{\beta} \tag{2.1}$$

where ϕ_{μ} is a vector field whose conformal variation is

$$\delta\phi_{\mu} = \frac{1}{2}\varepsilon\pi_{|\mu}$$

(It was the resemblance of this variation with the *gauge transformation* property of the electromagnetic four-potential that led Weyl to believe his theory might be the path to the unification of gravitation and electromagnetism.) It is easily shown that the Weyl connection is conformally invariant ($\delta\Gamma^{\lambda}_{\mu\nu} = 0$) as are the Riemann-Christoffel curvature tensor $R^{\lambda}_{\mu\nu\alpha}$ and the Ricci tensor $R_{\mu\nu}$. However, the Ricci scalar $R = g^{\mu\nu}R_{\mu\nu}$ is not invariant, and so Weyl was forced to assume the form

$$S = \int \sqrt{-g}\, R^2\, d^4x \tag{2.2}$$

for his action. By comparison, the Einstein-Hilbert action for free space,

$$S = \int \sqrt{-g}\, R\, d^4x$$

is not invariant for either the Christoffel or Weyl connection.

One complication of Weyl's theory (and the one that motivated Einstein's objection to it) is the fact that the covariant derivative of the metric tensor does not vanish:

$$g_{\mu\nu||\lambda} = 2g_{\mu\nu}\,\phi_{\lambda} \tag{2.3}$$

This identity presents a problem. The magnitude or length L of an arbitrary vector ξ^{μ} is given by

$$L^2 = g_{\mu\nu}\,\xi^{\mu}\xi^{\nu}$$

Under parallel transport, it can be shown that the length changes according to

$$2LdL = g_{\mu\nu||\lambda}\,\xi^{\mu}\xi^{\nu}dx^{\lambda}$$

which goes over to

$$\frac{dL}{L} = \phi_{\lambda}\, dx^{\lambda}$$

where we've used Weyl's identity for $g_{\mu\nu||\lambda}$. Thus, for $\phi_\lambda \neq 0$ the length of a vector can never be truly invariant. But there are certain vectors whose magnitudes can never change, such as the unit vector dx^λ/ds (and any vector proportional to this, like the four-momentum). This was the gist of Einstein's objection, who argued that certain vectors can be treated as clocks marking the histories of atoms, whose spectral lines never change with time. Since atomic spectra are always the same, Einstein's argument effectively killed off Weyl's theory.

3. Equations of Motion of the Weyl Action in Riemannian Space

Let us now assume Weyl's action

$$S = \int \sqrt{-g}\, R^2\, d^4x$$

but with $\phi_\mu = 0$, so that we're working in ordinary Riemannian space. By taking an arbitrary variation of the action with respect to $g^{\mu\nu}$, we can derive the equations of motion associated with this action, which will incidentally provide an identity we can use to show that the action is also conformally invariant.

We therefore have

$$\delta S = \int \left(R^2\, \delta\sqrt{-g} + 2\sqrt{-g}\, R\, \delta R\right) d^4x$$

where $\delta R = R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}$. We can now use Palatini's identity

$$\delta R_{\mu\nu} = \left(\delta \begin{Bmatrix} \alpha \\ \mu\alpha \end{Bmatrix}\right)_{||\nu} - \left(\delta \begin{Bmatrix} \alpha \\ \mu\nu \end{Bmatrix}\right)_{||\alpha} \tag{3.1}$$

(which is applicable for any variation of a symmetric connection) to greatly simplify the calculation. In addition, the variation is further simplified by assuming a locally inertial frame, so that quantities like $g^{\alpha\beta}\delta g_{\mu\nu|\lambda}$ are non-zero while quantities like $g^{\alpha\beta}_{\ \ |\lambda}\delta g_{\mu\nu}$ can be ignored. Integration by parts will also be needed to isolate the $\delta g^{\mu\nu}$ terms. After some simple algebra, the calculation leads to

$$\delta S = 2\int \sqrt{-g}\left(R\,(R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R) + R_{|\mu||\nu} - g_{\mu\nu}\, g^{\alpha\beta}\, R_{|\alpha||\beta}\right)\delta g^{\mu\nu}\, d^4x \tag{3.2}$$

Setting the integrand to zero and dropping the $\sqrt{-g}\,\delta g^{\mu\nu}$ term then gives us the equations of motion

$$R\left(R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R\right) + R_{|\mu||\nu} - g_{\mu\nu}\, g^{\alpha\beta}\, R_{|\alpha||\beta} = 0 \tag{3.3}$$

Contracting this expression with $g^{\mu\nu}$ gives the trace, which leaves the condition

$$g^{\mu\nu}R_{|\mu||\nu} \equiv \frac{1}{\sqrt{-g}}\left(\sqrt{-g}\, g^{\mu\nu}R_{|\mu}\right)_{|\nu} = 0 \tag{3.4}$$

so that (3.3) reduces to the fourth-order expression

$$R\left(R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R\right) + R_{|\mu||\nu} = 0 \tag{3.5}$$

The trivial solution is just $R = 0$, but if R is a non-zero constant we can simply write

$$R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R = 0 \tag{3.6}$$

Like the Einstein equations of motion, this is of second order in the metric tensor and its derivatives. The general solution of (3.6) is easily obtained. For the Schwarzschild-like line element

$$ds^2 = e^\nu c^2 dt^2 - e^\lambda dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta\, d\varphi^2$$

we have

$$e^{\nu} = 1 - \frac{2GM}{c^2 r} + \kappa r^2, \quad e^{\lambda} = e^{-\nu} \tag{3.7}$$

where κ is a constant. In addition, solving for $R = g^{\mu\nu} R_{\mu\nu}$ gives $R = -12\kappa$, consistent with the requirement that R be a non-zero constant. It is also interesting to note that even if we had assumed the Ricci scalar to be some function $R(r)$ and tried to solve (3.5) by brute force, we would have gotten the same result.

4. Conformal Invariance of the Weyl Action in Riemannian Space

We note in (3.2) that the conformal variation $\delta g^{\mu\nu} = -\varepsilon\pi g^{\mu\nu}$ trivially provides the same result as above. However, we can instead conformally vary the Weyl action directly using the identities

$$\delta\sqrt{-g} = 2\varepsilon\pi\sqrt{-g}, \quad \delta \begin{Bmatrix} \alpha \\ \mu\nu \end{Bmatrix} = \frac{1}{2}\varepsilon\,\delta^{\alpha}_{\mu}\,\pi_{|\nu} + \frac{1}{2}\varepsilon\,\delta^{\alpha}_{\nu}\,\pi_{|\mu} - \frac{1}{2}\varepsilon g_{\mu\nu} g^{\alpha\beta}\pi_{|\beta}, \quad \delta \begin{Bmatrix} \alpha \\ \alpha\nu \end{Bmatrix} = 2\varepsilon\pi_{|\nu} \tag{4.1}$$

Using the Palatini identity as before along with a few integrations by parts gives

$$\delta I = -6\varepsilon \int \sqrt{-g}\, g^{\mu\nu} R_{|\mu}\,\pi_{|\nu}\, d^4x$$

and a final integration by parts over the parameter π gives us

$$\delta I = 6\varepsilon \int \left(\sqrt{-g}\, g^{\mu\nu} R_{|\mu}\right)_{|\nu} \pi\, d^4x \tag{4.2}$$

The integrand vanishes by virtue of (3.4), indicating that the Weyl action in Riemannian space is indeed conformally invariant.

5. Inclusion of the Electromagnetic Stress-Energy Tensor

The Einstein equations in the presence of matter and energy are

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{8\pi G}{c^4} T_{\mu\nu} \tag{5.1}$$

where $T_{\mu\nu}$ is the energy-momentum tensor. For the electromagnetic field, we have the (traceless) tensor

$$T_{\mu\nu} = F_{\alpha\mu} F^{\alpha}_{\ \nu} - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}$$

where $F_{\mu\nu}$ is the antisymmetric electromagnetic tensor. It is tempting to see if we can find a solution to the set of associated Weyl equations

$$R\left(R_{\mu\nu} - \frac{1}{4} g_{\mu\nu} R\right) = -\frac{8\pi G}{c^4}\left(F_{\alpha\mu} F^{\alpha}_{\ \nu} - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}\right) \tag{5.2}$$

It is remarkable that this can indeed be solved using the assumption that $e^{\lambda} = e^{-\nu}$ and $R =$ constant as before. Given a radial electric charge Q at the origin, it can be shown that (see Adler et al.)

$$T_{\mu\nu} = -\frac{GQ^2}{4\pi c^4 r^4} \begin{bmatrix} e^{\nu} & 0 & 0 & 0 \\ 0 & -e^{\lambda} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{bmatrix}$$

We now assume a solution of the form

$$e^{\nu} = 1 - \frac{2GM}{r} + \kappa r^2 + \beta r^n$$

where β and n are constants to be determined. For brevity, we will compute only the expression

$$R\left(R_{22}-\frac{1}{4}g_{22}R\right)=-\frac{8\pi G}{c^4}T_{22}$$

(where $g_{22}=-r^2$), although the solution we obtain can be shown to be consistent for the other equations of motion. Using $\nu=-\lambda$ we have

$$R_{22}=re^{\nu}\nu'+e^{\nu}-1$$

where $\nu'=d\nu/dr$. The problem is then to solve

$$re^{\nu}\nu'+e^{\nu}-1+\frac{1}{4}r^2R=-\frac{GQ^2}{4\pi c^4r^2R}$$

It is not difficult to show that

$$n=-2,\quad \beta=-\frac{GQ^2}{2\pi c^4 R}$$

so that

$$e^{\nu}=1-\frac{2GM}{c^2r}+\kappa r^2-\frac{GQ^2}{2\pi c^4 r^2R} \tag{5.3}$$

Remarkably, $R=-12\kappa$ as before. With the exception of the r^2 term, this is nearly identical to the solution of the conventional Einstein equations (5.1) for a radial electric field.

6. Comments

The alert reader will note that the r^2 term in (3.7) represents an effective acceleration in the Schwarzschild solution, in which the classical gravitational potential $\varphi(r)$ changes according to

$$\varphi=-\frac{GM}{r}\to-\frac{GM}{r}-\frac{c^2r^2R}{24}$$

Even in the absence of the central mass M, a test particle would experience an acceleration equal to

$$a=\frac{c^2rR}{12},$$

an effect that may help to explain dark energy and the Pioneer spacecraft anomaly.

The reduced Weyl equations of motion for an electromagnetic field

$$RR_{\mu\nu}-\frac{1}{4}g_{\mu\nu}R^2=-\frac{8\pi G}{c^4}\left(F_{\alpha\mu}F^{\alpha}_{\ \nu}-\frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right)$$

have a particularly pleasing look to them. Both sides are quadratic, structurally similar and traceless, characteristics that appear to show a deep correspondence between gravitation and electromagnetism.

It is also interesting to note that our simplified Weyl formalism preserves an intriguing remnant of Weyl's original theory. In that theory, Weyl showed that the electromagnetic source vector density $\sqrt{-g}\,\rho^{\nu}$ appears as

$$\sqrt{-g}\,\rho^{\nu}=\sqrt{-g}\,g^{\mu\nu}\left(R_{|\mu}+2R\phi_{\mu}\right)$$

whose divergence vanishes due to conservation of charge. Setting $\phi_{\mu}=0$, the conservation condition is then

$$\left(\sqrt{-g}\,\rho^{\nu}\right)_{|\nu}=\left(\sqrt{-g}\,g^{\mu\nu}R_{|\mu}\right)_{|\nu}=0$$

which is in agreement with (3.4).

Lastly, we note that the solutions of the revised Weyl equations of motion are remarkably similar to those of the standard Einstein equations. Indeed, if we take $\kappa=0$ in (3.7), all the predictions of Einstein's gravity theory—including the deflection of light, the precession of the orbit of Mercury and the gravitational redshift effect—are reproduced by (3.6). More importantly, we see that the formalism is, like Weyl dreamed, fully derivable from a conformally invariant action principle.

Reference

R. Adler, M. Bazin and M. Schiffer, *Introduction to general relativity*, 2nd ed., McGraw-Hill, 1975. An older and dated text, but it details all the $R_{\mu\nu}$ connection terms needed to follow this paper. Included are detailed analyses of the Schwarzschild and Kerr black holes. A comprehensive account of Weyl's 1918 theory is also provided, along with an analysis of the field surrounding a charged mass point.