Certain types of graphs in interval-valued intuitionistic fuzzy setting

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Abstract

Interval-valued intuitionistic fuzzy set (IVIFS) as a generalization of intuitionistic fuzzy set (IFS) increase its elasticity drastically. In this paper, some important types of interval-valued intuitionistic fuzzy graphs (IVIFGs) such as regular, irregular, neighbourly irregular, highly irregular and strongly irregular IVIFGs are discussed. The relation among neighbourly irregular, highly irregular and strongly irregular IVIFGs is proved. The notion of interval-valued intuitionistic fuzzy clique (IVIFC) is introduced. A complete characterization of the structure of the IVIFC is presented.

Keywords: Interval-valued intuitionistic fuzzy set; Interval-valued intuitionistic fuzzy graph; Irregular interval-valued intuitionistic fuzzy graph; Interval-valued intuitionistic fuzzy clique.
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1 Introduction

In 1965, Zadeh [26] originally introduced the concept of fuzzy set. Its prominent characteristic is that a membership degree in $[0, 1]$ is assigned to each element in the set. When it is difficult to give the accurate judgments to the things, fuzzy set shows great advantages in expressing uncertain or vague information and depict the indeterminacy of things. It was later understood that a single membership function could not capture

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the ambiguity existing in human mind and the complexity of data. To overcome this shortcoming of the fuzzy set, Atanassov [4] proposed an extension of fuzzy set by introducing non-membership function, and defined IFS. As IFS can describe the uncertainty of an object more reasonably and comprehensively than the FS, lots of research on the IFS have been done, in recent decades. However, in some cases, membership degree or non-membership degree cannot be indicated by using a value, but using an interval. That is why, IFS was extended to the IVIFS by Atanassov and Gargov [5] as a combining concept of IFS and IVFS. It greatly furnishes the additional capability to deal with imprecise information and model non-statistical uncertainty by expressing the variations of membership function and non-membership function and has played a vital role in the vague system and received much attention from researchers. IVIFS has been widely used in many areas, such as decision making [9], pattern recognition [28], medical diagnosis[1], graph theory[12], etc.

It is natural that when there is fuzziness in the description of the items (vertices) or in their relationships (edges) or in both, a fuzzy graph model is designed. Obtaining analogs of several basic graph theoretical concepts, Rosenfeld [22] considered fuzzy relations on fuzzy sets and defined the structure of fuzzy graphs. Applications of fuzzy graphs cover an extensive range such as control theory, information theory, neural networks, expert systems, medical diagnosis, cluster analysis, database theory, decision making and optimization of networks. Nair and Cheng [19] defined the concept of a fuzzy clique in fuzzy graphs. The concept of cycles and cocycles of fuzzy graphs was introduced by Mordeson and Nair [14]. The complement of a fuzzy graph was proposed by Mordeson and Peng [13] and then modified by Sunitha and Vijayakumar [25]. Ghosh et al. [10] introduced fuzzy graph representation of a fuzzy concept lattice. The notion of irregular fuzzy graphs was defined by Gani and Latha [18]. The concept of strongly irregular fuzzy graphs was initiated by Nandhini and Nandhini [20]. Intuitionistic fuzzy graphs were first introduced by Atanassov [6] in 1999 and further discussed in [3] by Akram. The concept of interval-valued fuzzy graphs was initiated by Hongmei and Lianhua in [11]. Product of IVIFGs has been proposed by Mishra and Pal in [16]. The concept of strong IVIFGs was defined by Ismayil and Ali [12]. Rashmanlou and Borzooei [21] introduced the concept of interval-valued intuitionistic (S, T)-fuzzy graphs. Recently, Akram et al. [24, 2] put forward many new concepts, including fuzzy soft graphs and m-polar fuzzy graphs.

This paper is organized as follows: In Section 2, basic concepts related to IVIFSs and IVIFGs are reviewed. In Section 3, we define certain types of IVIFGs like, neighbourly irregular, highly irregular and strongly irregular IVIFGs. In Section 4, we propose the concept of IVIFCs consistent with interval-valued intuitionistic fuzzy cycles in IVIFGs and finally we draw conclusions in section 5.

We have used standard definitions and terminologies, in this paper. For more details and background, the readers are referred to [7, 8, 15, 17].

2 Preliminaries

In the following, some basic concepts are reviewed to facilitate next sections.

A graph is a pair of sets $G = (V, E)$, satisfying $E \subseteq V \times V$. The elements of V are the vertices and the elements of E are the edges of the graph G. A vertex joined by an edge to a vertex x is called a neighbor of x. The (open) neighborhood $\mathcal{N}(x)$ of a vertex x in a graph G is the set of all the neighbors of x, while closed neighborhood $\mathcal{N}[x]$ of x is given by $\mathcal{N}[x] = \mathcal{N}(x) \cup \{x\}$. The degree of a vertex x in G, denoted by $\deg_{\mathcal{G}}(x)$ or $\deg(x)$, is the number of edges incident with x. A graph with no multiple edges and loops is called simple. Throughout this paper we will consider only undirected, simple graphs. A graph G is complete if every two distinct vertices of G are adjacent.

In graph theory, clique is an important concept. A clique in a graph G is a complete subgraph of G. A subgraph H of a graph G is a disjoint union of cliques if $V(H)$ can be partitioned into $H_1, H_2, \ldots, H_k$ such that $xy \in E(H)$ for all $x, y \in V(H)$ if and only if $\{x, y\} \subseteq H_i$, for some $i,\ i = 1, 2, \ldots, k$ [19].

Definition 2.1. [26] A fuzzy subset η of a set V is a function $\eta : V \to [0, 1]$. A fuzzy relation on a set V is a mapping $\mu : V \times V \to [0, 1]$ such that $\mu(x, y) \leq \eta(x) \wedge \eta(y)$ for all $x, y \in V$. A fuzzy relation μ is symmetric if $\mu(x, y) = \mu(y, x)$ for all $x, y \in V$.

Definition 2.2. [4] An IFS X in V is an object of the form

$$X = \{\langle x, \mu_X(x), \nu_X(x)\rangle \mid x \in V\},$$

where the functions $\mu_X : V \to [0, 1]$ and $\nu_X : V \to [0, 1]$ give the degree of membership and the degree of non-membership of the element $x \in V$, respectively, such that $0 \leq \mu_X(x) + \nu_X(x) \leq 1$ for all $x \in V$. The class of all IFSs on V is denoted by $IFS(V)$.

For each IFS X in V, $\pi_X(x) = 1 - \mu_X(x) - \nu_X(x)$ is called a hesitancy degree of x in X. If $\pi_X(x) = 0$ for all $x \in V$, then IFS reduces to Zadeh's fuzzy set.

Definition 2.3. [27] An interval-valued fuzzy set (IVFS) X in V is a mapping

$$M_X : V \to D[0, 1],$$

where $D[0, 1] = \{[a, b] : a \leq b,\ a, b \in [0, 1]\}$. The class of all interval-valued fuzzy sets on V is denoted by $IVFS(V)$.

For any set $X \subseteq [0, 1]$, we introduce the following notation: $X^- = \inf X$ and $X^+ = \sup X$.

Definition 2.4. [5] An IVIFS X in V is an object of the form

$$X = \{\langle x, M_X(x), N_X(x)\rangle \mid x \in V\},$$

where $M_X : V \to D[0, 1]$ and $N_X : V \to D[0, 1]$ such that $M_X^+(x) + N_X^+(x) \leq 1$ for all $x \in V$. The set of all IVIFSs on V is denoted by $IVIFS(V)$.

Definition 2.5. [22] A fuzzy graph $\mathcal{G} = (\eta, \mu)$ is a pair of functions $\eta : V \to [0, 1]$ and $\mu : V \times V \to [0, 1]$ such that $\mu(xy) \leq \eta(x) \wedge \eta(y)$ for all $x, y \in V$. An edge xy of a fuzzy graph is called an effective edge [23] if $\mu(xy) = \eta(x) \wedge \eta(y)$. In a fuzzy graph, the path ρ is a sequence of distinct vertices $x_0, x_1, \ldots, x_n$ such that $\mu(x_{i-1}, x_i) > 0,\ i = 1, 2, \ldots, n$.

Definition 2.6. [3] An intuitionistic fuzzy graph (IFG) of a graph G is defined to be a pair $\mathcal{G} = (X, Y)$, where

(i) the functions $\mu_X : V \to [0, 1]$ and $\nu_X : V \to [0, 1]$ represent the degree of membership and non-membership of the element $x \in V$, respectively, such that $0 \leq \mu_X(x) + \nu_X(x) \leq 1$ for all $x \in V$,

(ii) the functions $\mu_Y : E \subseteq V \times V \to [0, 1]$ and $\nu_Y : E \subseteq V \times V \to [0, 1]$ are defined by

$$\mu_Y(xy) \leq \min(\mu_X(x), \mu_X(y)) \text{ and } \nu_Y(xy) \geq \max(\nu_X(x), \nu_X(y))$$

such that $0 \leq \mu_Y(xy) + \nu_Y(xy) \leq 1$ for all $xy \in E$.

Definition 2.7. [11] An interval-valued fuzzy graph (IVFG) of a graph $G = (V, E)$ is a pair $\mathcal{G} = (X, Y)$, where $X = [\mu_X^-, \mu_X^+]$ is an interval-valued fuzzy set in V and $Y = [\mu_Y^-, \mu_Y^+]$ is an interval-valued fuzzy set in $E \subseteq V \times V$ such that

$$\mu_Y^-(xy) \leq \min(\mu_X^-(x), \mu_X^-(y)) \text{ and } \mu_Y^+(xy) \leq \min(\mu_X^+(x), \mu_X^+(y))$$

for all $xy \in E$.

3 Interval-valued intuitionistic fuzzy graphs (IVIFGs)

Definition 3.1. [16] An IVIFG of a graph G is defined to be a pair $\mathcal{G} = (X, Y)$, where

(i) the functions $M_X : V \to D[0, 1]$ and $N_X : V \to D[0, 1]$ denote the degree of interval-valued membership and interval-valued non-membership of the element $x \in V$, respectively, such that $M_X^+(x) + N_X^+(x) \leq 1$ for all $x \in V$,

(ii) the functions $M_Y : E \subseteq V \times V \to D[0, 1]$ and $N_Y : E \subseteq V \times V \to D[0, 1]$ are defined by

$$\begin{aligned} M_Y^-(xy) &\leq \min(M_X^-(x), M_X^-(y)), M_Y^+(xy) \leq \min(M_X^+(x), M_X^+(y)), \\ N_Y^-(xy) &\geq \max(N_X^-(x), N_X^-(y)) \text{ and } N_Y^+(xy) \geq \max(N_X^+(x), N_X^+(y)) \end{aligned}$$

such that $M_Y^+(xy) + N_Y^+(xy) \leq 1$ for all $xy \in E$.

We call X the interval-valued intuitionistic fuzzy vertex set of $\mathcal{G}$ and Y the interval-valued intuitionistic fuzzy edge set of $\mathcal{G}$.

Example 3.1. Consider a graph $G = (V, E)$, where $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $E = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_2, v_6v_4\}$. Let $\mathcal{G} = (X, Y)$ be an IVIFG of a graph G defined by

$$\begin{aligned} X &= \left\langle \left(\frac{v_1}{0.4}, \frac{v_2}{0.1}, \frac{v_3}{0.2}, \frac{v_4}{0.5}, \frac{v_5}{0.6}, \frac{v_6}{0.2}\right), \left(\frac{v_1}{0.5}, \frac{v_2}{0.6}, \frac{v_3}{0.7}, \frac{v_4}{0.6}, \frac{v_5}{0.8}, \frac{v_6}{0.4}\right),\right. \\ &\quad \left. \left(\frac{v_1}{0.3}, \frac{v_2}{0.2}, \frac{v_3}{0.1}, \frac{v_4}{0.2}, \frac{v_5}{0.1}, \frac{v_6}{0.3}\right), \left(\frac{v_1}{0.6}, \frac{v_2}{0.5}, \frac{v_3}{0.3}, \frac{v_4}{0.4}, \frac{v_5}{0.2}, \frac{v_6}{0.5}\right)\right\rangle, \\ Y &= \left\langle \left(\frac{v_1v_2}{0.1}, \frac{v_2v_3}{0.1}, \frac{v_3v_4}{0.1}, \frac{v_4v_5}{0.4}, \frac{v_5v_2}{0.1}, \frac{v_4v_6}{0.2}\right), \left(\frac{v_1v_2}{0.3}, \frac{v_2v_3}{0.4}, \frac{v_3v_4}{0.5}, \frac{v_4v_5}{0.5}, \frac{v_5v_2}{0.4}, \frac{v_4v_6}{0.3}\right),\right. \\ &\quad \left. \left(\frac{v_1v_2}{0.5}, \frac{v_2v_3}{0.3}, \frac{v_3v_4}{0.3}, \frac{v_4v_5}{0.3}, \frac{v_5v_2}{0.4}, \frac{v_4v_6}{0.5}\right), \left(\frac{v_1v_2}{0.9}, \frac{v_2v_3}{0.7}, \frac{v_3v_4}{0.4}, \frac{v_4v_5}{0.7}, \frac{v_5v_2}{0.6}, \frac{v_4v_6}{0.7}\right)\right\rangle. \end{aligned}$$

The IVIFG is given in Fig. 1. Tabular representation of an IVIFG is given in Table 1.

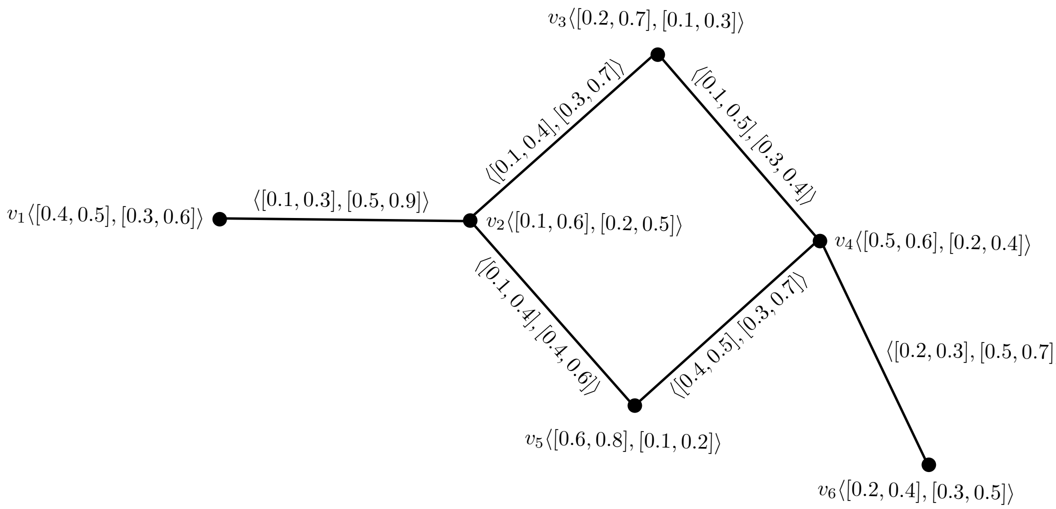

Figure 1: IVIFG.

Table 1: Tabular representation of an IVIFG.

	v_1	v_2	v_3	v_4	v_5	v_6
M_X^-	0.4	0.1	0.2	0.5	0.6	0.2
M_X^+	0.5	0.6	0.7	0.6	0.8	0.4
N_X^-	0.3	0.2	0.1	0.2	0.1	0.3
N_X^+	0.6	0.5	0.3	0.4	0.2	0.5

	v_1v_2	v_2v_3	v_3v_4	v_4v_5	v_5v_2	v_4v_6
M_Y^-	0.1	0.1	0.1	0.4	0.1	0.2
M_Y^+	0.3	0.4	0.5	0.5	0.4	0.3
M_Y^-	0.5	0.3	0.3	0.3	0.4	0.5
M_Y^+	0.9	0.7	0.4	0.7	0.6	0.7

Definition 3.2. The degree of a vertex $x \in V$ in an IVIFG $\mathcal{G}$ is defined as $\deg(x) = \langle [\deg_{M^-}(x), \deg_{M^+}(x)], [\deg_{N^-}(x), \deg_{N^+}(x)] \rangle$, where

$$\begin{aligned} \deg_{M^-}(x) &= \Sigma_{x,y\neq x\in V} M_Y^-(xy), \deg_{M^+}(x) = \Sigma_{x,y\neq x\in V} M_Y^+(xy), \\ \deg_{N^-}(x) &= \Sigma_{x,y\neq x\in V} N_Y^-(xy) \text{ and } \deg_{N^+}(x) = \Sigma_{x,y\neq x\in V} N_Y^+(xy). \end{aligned}$$

For an IVIFG, the degree of a vertex can be generalized in different ways.

Definition 3.3. The sum of the weights of the effective edges incident at a vertex x in an IVIFG is called the effective degree of x. That is, $\mathcal{E}\deg(x) = \langle [\mathcal{E}\deg_{M^-}(x), \mathcal{E}\deg_{M^+}(x)], [\mathcal{E}\deg_{N^-}(x), \mathcal{E}\deg_{N^+}(x)] \rangle$, where

$$\begin{aligned} \mathcal{E}\deg_{M^-}(x) &= \Sigma_{x,y\neq x\in V} M_Y^-(xy), \mathcal{E}\deg_{M^+}(x) = \Sigma_{x,y\neq x\in V} M_Y^+(xy), \\ \mathcal{E}\deg_{N^-}(x) &= \Sigma_{x,y\neq x\in V} N_Y^-(xy) \text{ and } \mathcal{E}\deg_{N^+}(x) = \Sigma_{x,y\neq x\in V} N_Y^+(xy) \end{aligned}$$

for all effective edges $xy \in E$.

Definition 3.4. The neighbourhood degree of a vertex $x \in V$ in an IVIFG $\mathcal{G}$ is defined as $\mathcal{N}\deg(x) = \langle [\mathcal{N}\deg_{M^-}(x), \mathcal{N}\deg_{M^+}(x)], [\mathcal{N}\deg_{N^-}(x), \mathcal{N}\deg_{N^+}(x)] \rangle$, where

$$\begin{aligned} \mathcal{N}\deg_{M^-}(x) &= \Sigma_{y\in\mathcal{N}(x)} M_X^-(y), \mathcal{N}\deg_{M^+}(x) = \Sigma_{y\in\mathcal{N}(x)} M_X^+(y), \\ \mathcal{N}\deg_{N^-}(x) &= \Sigma_{y\in\mathcal{N}(x)} N_X^-(y) \text{ and } \mathcal{N}\deg_{N^+}(x) = \Sigma_{y\in\mathcal{N}(x)} N_X^+(y). \end{aligned}$$

Definition 3.5. The closed neighbourhood degree of a vertex $x \in V$ in an IVIFG $\mathcal{G}$ is defined by $\mathcal{N}\deg[x] = \langle [\mathcal{N}\deg_{M^-}[x], \mathcal{N}\deg_{M^+}[x]], [\mathcal{N}\deg_{N^-}[x], \mathcal{N}\deg_{N^+}[x]] \rangle$, where

$$\begin{aligned} \mathcal{N}\deg_{M^-}[x] &= \mathcal{N}\deg_{M^-}(x) + M_X^-(x), \mathcal{N}\deg_{M^+}[x] = \mathcal{N}\deg_{M^+}(x) + M_X^+(x), \\ \mathcal{N}\deg_{N^-}[x] &= \mathcal{N}\deg_{N^-}(x) + N_X^-(x) \text{ and } \mathcal{N}\deg_{N^+}[x] = \mathcal{N}\deg_{N^+}(x) + N_X^+(x). \end{aligned}$$

Definition 3.6. The vertices of $\mathcal{G}$ which are incident with effective edges are said to be the effective vertices. The sum of the weights of the effective vertices adjacent to a vertex x of an IVIFG $\mathcal{G}$ is called the effective neighbourhood degree of x.

The types of IVIFGs are introduced according to their (open) neighbourhood and closed neighbourhood degree.

Definition 3.7. An IVIFG $\mathcal{G}$ on G, in which each vertex has the same neighbourhood degree is called an interval-valued intuitionistic fuzzy regular graph. If each vertex has degree $\langle [j,k],[s,t] \rangle$, $\mathcal{G}$ is called $\langle [j,k],[s,t] \rangle$-regular.

Example 3.2. Consider a graph $G = (V, E)$, where $V = \{v_1,\ v_2,\ v_3\}$ and $E = \{v_1v_2,\ v_2v_3,\ v_1v_3\}$. Let $\mathcal{G} = (X, Y)$ be an IVIFG of a graph G defined by

$$\begin{aligned} X &= \left\langle \left(\frac{v_1}{0.2}, \frac{v_2}{0.2}, \frac{v_3}{0.2}\right), \left(\frac{v_1}{0.3}, \frac{v_2}{0.3}, \frac{v_3}{0.3}\right), \left(\frac{v_1}{0.4}, \frac{v_2}{0.4}, \frac{v_3}{0.4}\right), \left(\frac{v_1}{0.6}, \frac{v_2}{0.6}, \frac{v_3}{0.6}\right) \right\rangle, \\ Y &= \left\langle \left(\frac{v_1v_2}{0.2}, \frac{v_2v_3}{0.2}, \frac{v_3v_1}{0.1}\right), \left(\frac{v_1v_2}{0.3}, \frac{v_2v_3}{0.3}, \frac{v_3v_1}{0.2}\right), \left(\frac{v_1v_2}{0.5}, \frac{v_2v_3}{0.5}, \frac{v_3v_1}{0.6}\right), \left(\frac{v_1v_2}{0.6}, \frac{v_2v_3}{0.6}, \frac{v_3v_1}{0.7}\right) \right\rangle. \end{aligned}$$

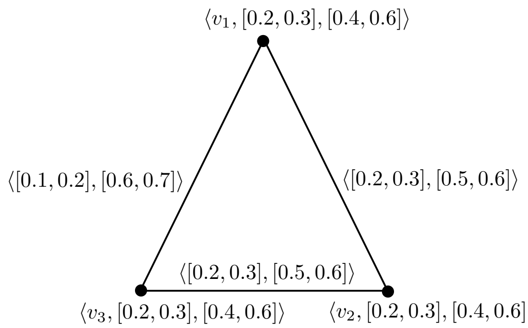

Figure 2: Regular IVIFG.

Here $\deg_{\mathcal{G}}(v_i) = \langle [0.4, 0.6], [0.8, 1.2] \rangle$ for all $i = 1, 2, 3$. Hence $\mathcal{G} = (X, Y)$ is a regular IVIFG. The regular IVIFG is given in Fig. 2. Tabular representation of a regular IVIFG is given in Table 2.

Table 2: Tabular representation of a regular IVIFG.

	v_1	v_2	v_3
M_X^-	0.2	0.2	0.2
M_X^+	0.3	0.3	0.3
N_X^-	0.4	0.4	0.4
N_X^+	0.6	0.6	0.6

	v_1v_2	v_2v_3	v_1v_3
M_Y^-	0.2	0.2	0.1
M_Y^+	0.3	0.3	0.2
N_Y^-	0.5	0.5	0.6
N_Y^+	0.6	0.6	0.7

Definition 3.8. An IVIFG $\mathcal{G}$ is said to be irregular, if there is a vertex which is adjacent to vertices with distinct neighbourhood degrees. That is, $\deg_{\mathcal{G}}(l) \neq \langle [j, k], [s, t] \rangle$ for all $l \in V$.

Example 3.3. Consider a graph $G = (V, E)$, where $V = \{v_1, v_2, v_3\}$ and $E = \{v_1v_3, v_2v_3\}$. Let $\mathcal{G} = (X, Y)$ be an IVIFG of a graph G defined by

$$\begin{aligned} X &= \left\langle \left(\frac{v_1}{0.2}, \frac{v_2}{0.2}, \frac{v_3}{0.4} \right), \left(\frac{v_1}{0.3}, \frac{v_2}{0.3}, \frac{v_3}{0.5} \right), \left(\frac{v_1}{0.5}, \frac{v_2}{0.5}, \frac{v_3}{0.2} \right), \left(\frac{v_1}{0.6}, \frac{v_2}{0.6}, \frac{v_3}{0.4} \right) \right\rangle, \\ Y &= \left\langle \left(\frac{v_1v_3}{0.1}, \frac{v_2v_3}{0.1} \right), \left(\frac{v_1v_3}{0.3}, \frac{v_2v_3}{0.2} \right), \left(\frac{v_1v_3}{0.5}, \frac{v_2v_3}{0.6} \right), \left(\frac{v_1v_3}{0.6}, \frac{v_2v_3}{0.7} \right) \right\rangle. \end{aligned}$$

Clearly, $\deg_{\mathcal{G}}(v_1) = \deg_{\mathcal{G}}(v_2) = \langle [0.4, 0.5], [0.2, 0.4] \rangle$ and $\deg_{\mathcal{G}}(v_3) = \langle [0.4, 0.6], [1.0, 1.2] \rangle$. Hence $\mathcal{G} = (X, Y)$ is an irregular IVIFG. The irregular IVIFG is given in Fig. 3. Tabular representation of an irregular IVIFG is given in Table 3.

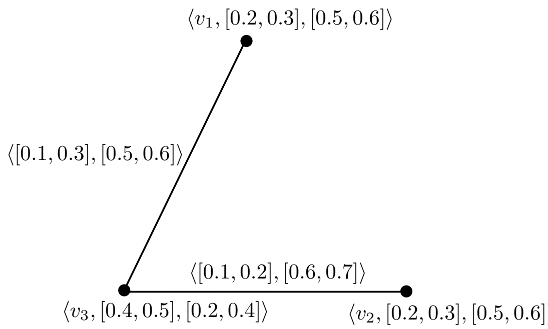

Figure 3: Irregular IVIFG.

Table 3: Tabular representation of an irregular IVIFG.

	v_1	v_2	v_3
M_X^-	0.2	0.2	0.4
M_X^+	0.3	0.3	0.5
N_X^-	0.5	0.5	0.2
N_X^+	0.6	0.6	0.4

	v_1v_3	v_2v_3
M_Y^-	0.1	0.1
M_Y^+	0.3	0.2
N_Y^-	0.5	0.6
N_Y^+	0.6	0.7

Definition 3.9. Let $\mathcal{G}$ be a connected IVIFG on G. $\mathcal{G}$ is called neighbourly irregular, if no two adjacent vertices of $\mathcal{G}$ have same neighbourhood degree. That is, $\deg_{\mathcal{G}}(l) \neq deg_{\mathcal{G}}(m)$ for all $lm \in E$.

Example 3.4. Consider a graph $G = (V, E)$, where $V = \{v_1, v_2, v_3, v_4\}$ and $E = \{v_1v_2,\ v_1v_3,\ v_2v_4,\ v_3v_4\}$. Let $\mathcal{G} = (X, Y)$ be an IVIFG of a graph G, given in Fig. 4, defined by

$$\begin{aligned}
X &= \left\langle \left(\frac{v_1}{0.2}, \frac{v_2}{0.1}, \frac{v_3}{0.4}, \frac{v_4}{0.3}\right), \left(\frac{v_1}{0.4}, \frac{v_2}{0.5}, \frac{v_3}{0.5}, \frac{v_4}{0.5}\right), \left(\frac{v_1}{0.3}, \frac{v_2}{0.4}, \frac{v_3}{0.2}, \frac{v_4}{0.1}\right), \left(\frac{v_1}{0.5}, \frac{v_2}{0.5}, \frac{v_3}{0.3}, \frac{v_4}{0.3}\right) \right\rangle, \\
Y &= \left\langle \left(\frac{v_1v_2}{0.1}, \frac{v_1v_3}{0.1}, \frac{v_2v_4}{0.1}, \frac{v_3v_4}{0.2}\right), \left(\frac{v_1v_2}{0.2}, \frac{v_1v_3}{0.3}, \frac{v_2v_4}{0.4}, \frac{v_3v_4}{0.4}\right), \right. \\
&\quad \left. \left(\frac{v_1v_2}{0.5}, \frac{v_1v_3}{0.4}, \frac{v_2v_4}{0.4}, \frac{v_3v_4}{0.3}\right), \left(\frac{v_1v_2}{0.6}, \frac{v_1v_3}{0.6}, \frac{v_2v_4}{0.5}, \frac{v_3v_4}{0.5}\right) \right\rangle,
\end{aligned}$$

where $\deg_{\mathcal{G}}(v_1) = \deg_{\mathcal{G}}(v_4) = \langle [0.5, 1.0], [0.6, 0.8] \rangle$ and $\deg_{\mathcal{G}}(v_2) = \deg_{\mathcal{G}}(v_3) = \langle [0.5, 0.9], [0.4, 0.8] \rangle$. Hence $\mathcal{G} = (X, Y)$ is a neighbourly irregular IVIFG.

Definition 3.10. A connected IVIFG $\mathcal{G}$ is said to be a highly irregular if every vertex of $\mathcal{G}$ is adjacent to vertices with distinct neighbourhood degrees. That is, $l, m \in \mathcal{N}(x)$, $l \neq m \implies \deg_{\mathcal{G}}(l) \neq \deg_{\mathcal{G}}(m)$ for all $x \in V$.

Example 3.5. Consider a graph $G = (V, E)$, where $V = \{v_1, v_2, v_3, v_4, v_5\}$ and $E = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_1, v_2v_4\}$. Let $\mathcal{G} = (X, Y)$ be an IVIFG of a graph G, as shown in

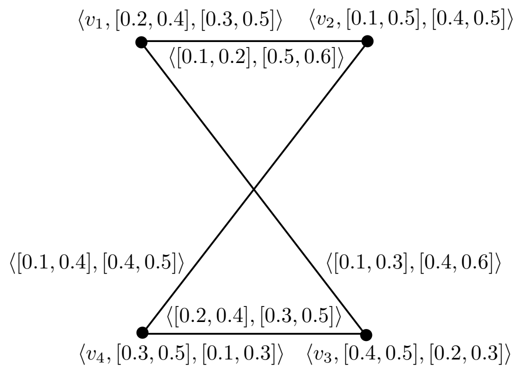

Figure 4: Neighbourly irregular IVIFG.

Fig. 5 defined by

$$\begin{aligned} X &= \Big\langle \Big(\frac{v_1}{0.2}, \frac{v_2}{0.1}, \frac{v_3}{0.2}, \frac{v_4}{0.3}, \frac{v_5}{0.2}\Big), \Big(\frac{v_1}{0.5}, \frac{v_2}{0.2}, \frac{v_3}{0.4}, \frac{v_4}{0.4}, \frac{v_5}{0.3}\Big), \\ &\quad \Big(\frac{v_1}{0.3}, \frac{v_2}{0.2}, \frac{v_3}{0.3}, \frac{v_4}{0.3}, \frac{v_5}{0.1}\Big), \Big(\frac{v_1}{0.4}, \frac{v_2}{0.3}, \frac{v_3}{0.4}, \frac{v_4}{0.6}, \frac{v_5}{0.3}\Big)\Big\rangle, \\ Y &= \Big\langle \Big(\frac{v_1v_2}{0.1}, \frac{v_2v_3}{0.1}, \frac{v_3v_4}{0.1}, \frac{v_4v_5}{0.2}, \frac{v_5v_1}{0.1}, \frac{v_2v_4}{0.1}\Big), \Big(\frac{v_1v_2}{0.2}, \frac{v_2v_3}{0.2}, \frac{v_3v_4}{0.4}, \frac{v_4v_5}{0.3}, \frac{v_5v_1}{0.3}, \frac{v_2v_4}{0.2}\Big), \\ &\quad \Big(\frac{v_1v_2}{0.4}, \frac{v_2v_3}{0.4}, \frac{v_3v_4}{0.5}, \frac{v_4v_5}{0.5}, \frac{v_5v_1}{0.5}, \frac{v_2v_4}{0.4}\Big), \Big(\frac{v_1v_2}{0.6}, \frac{v_2v_3}{0.5}, \frac{v_3v_4}{0.6}, \frac{v_4v_5}{0.6}, \frac{v_5v_1}{0.7}, \frac{v_2v_4}{0.7}\Big)\Big\rangle, \end{aligned}$$

where $\deg_{\mathcal{G}}(v_1) = \langle[0.3, 0.5], [0.3, 0.6]\rangle$, $\deg_{\mathcal{G}}(v_2) = \langle[0.7, 1.3], [0.9, 1.4]\rangle$, $\deg_{\mathcal{G}}(v_3) = \langle[0.4, 0.6]$ $, [0.5, 0.9]\rangle$ and $\deg_{\mathcal{G}}(v_4) = \deg_{\mathcal{G}}(v_5) = \langle[0.5, 0.9], [0.6, 1.0]\rangle$. Therefore, $\mathcal{G} = (X, Y)$ is a highly irregular IVIFG.

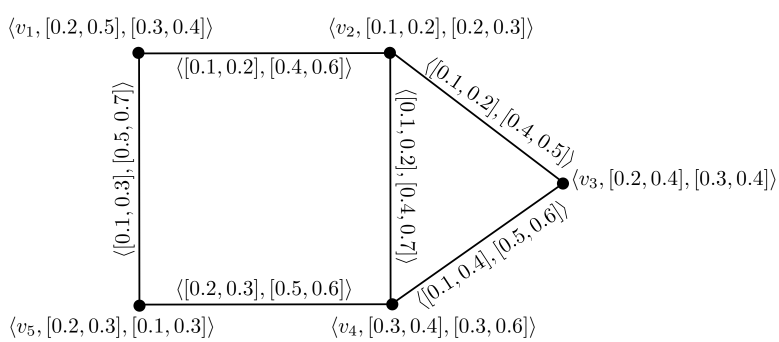

Figure 5: Highly irregular IVIFG

Remark 3.1. A neighbourly irregular IVIFG may not be a highly irregular IVIFG.

Remark 3.2. A highly irregular IVIFG may not be a neighbourly irregular IVIFG.

Definition 3.11. A connected IVIFG $\mathcal{G}$ on G is called strongly irregular if every pair of vertices in $\mathcal{G}$ have distinct neighborhood degrees. That is, $\deg_{\mathcal{G}}(l) \neq \deg_{\mathcal{G}}(m)$ for all $l, m \in V$.

Example 3.6. Consider a graph $G = (V, E)$, where $V = \{v_1, v_2, v_3\}$ and $E = \{v_1v_2, v_2v_3, v_1v_3\}$. Let $\mathcal{G} = (X, Y)$ be an IVIFG of a graph G defined by

$$\begin{aligned} X &= \left\langle \left(\frac{v_1}{0.2}, \frac{v_2}{0.5}, \frac{v_3}{0.6}\right), \left(\frac{v_1}{0.7}, \frac{v_2}{0.6}, \frac{v_3}{0.8}\right), \left(\frac{v_1}{0.1}, \frac{v_2}{0.2}, \frac{v_3}{0.1}\right), \left(\frac{v_1}{0.3}, \frac{v_2}{0.4}, \frac{v_3}{0.2}\right) \right\rangle, \\ Y &= \left\langle \left(\frac{v_1v_2}{0.1}, \frac{v_2v_3}{0.2}, \frac{v_3v_1}{0.1}\right), \left(\frac{v_1v_2}{0.3}, \frac{v_2v_3}{0.4}, \frac{v_3v_1}{0.2}\right), \left(\frac{v_1v_2}{0.5}, \frac{v_2v_3}{0.3}, \frac{v_3v_1}{0.6}\right), \left(\frac{v_1v_2}{0.6}, \frac{v_2v_3}{0.5}, \frac{v_3v_1}{0.7}\right) \right\rangle, \end{aligned}$$

where $\deg_{\mathcal{G}}(v_1) = \langle [1.1, 1.4], [0.3, 0.6] \rangle$, $\deg_{\mathcal{G}}(v_2) = \langle [0.8, 1.5], [0.2, 0.5] \rangle$ and $\deg_{\mathcal{G}}(v_3) = \langle [0.7, 1.3], [0.3, 0.7] \rangle$. Therefore $\mathcal{G} = (X, Y)$ is a strongly regular IVIFG. The strongly regular IVIFG is shown in Fig. 6.

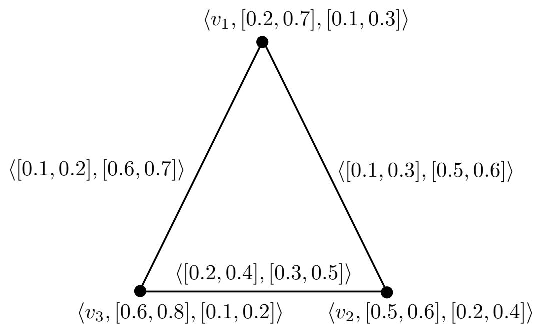

Figure 6: Strongly irregular IVIFG.

Theorem 3.1. *Every strongly irregular IVIFG is both neighbourly irregular IVIFG and highly irregular IVIFG.*

Proof. Suppose that $\mathcal{G}$ is a strongly irregular IVIFG. That is, degrees of every pair of vertices in $\mathcal{G}$ are distinct. Then every two adjacent vertices of $\mathcal{G}$ have distinct degrees and every vertex of $\mathcal{G}$ is adjacent to vertices with distinct degrees . Hence $\mathcal{G}$ is neighbourly irregular IVIFG and highly irregular IVIFG. $\square$

The converse of above statement does not hold. That is, a highly irregular IVIFG and neighbourly irregular IVIFG may not be a strongly irregular IVIFG. The following example illustrate the assertion above.

Example 3.7. Consider a graph $G = (V, E)$, where $V = \{v_1, v_2, v_3, v_4\}$ and $E = \{v_1v_2,\ v_1v_4,\ v_3v_4\}$. Let $\mathcal{G} = (X, Y)$ be an IVIFG of a graph G, as shown in Fig. 7, defined by

$$\begin{aligned} X &= \left\langle \left(\frac{v_1}{0.5}, \frac{v_2}{0.3}, \frac{v_3}{0.1}, \frac{v_4}{0.5}\right), \left(\frac{v_1}{0.7}, \frac{v_2}{0.5}, \frac{v_3}{0.3}, \frac{v_4}{0.7}\right), \left(\frac{v_1}{0.1}, \frac{v_2}{0.2}, \frac{v_3}{0.3}, \frac{v_4}{0.1}\right), \left(\frac{v_1}{0.2}, \frac{v_2}{0.4}, \frac{v_3}{0.5}, \frac{v_4}{0.2}\right) \right\rangle, \\ Y &= \left\langle \left(\frac{v_1v_2}{0.1}, \frac{v_1v_4}{0.2}, \frac{v_3v_4}{0.1}\right), \left(\frac{v_1v_2}{0.3}, \frac{v_1v_4}{0.4}, \frac{v_3v_4}{0.2}\right), \left(\frac{v_1v_2}{0.2}, \frac{v_1v_4}{0.3}, \frac{v_3v_4}{0.4}\right), \left(\frac{v_1v_2}{0.5}, \frac{v_1v_4}{0.4}, \frac{v_3v_4}{0.7}\right) \right\rangle. \end{aligned}$$

Clearly $\mathcal{G}$ is neighbourly irregular IVIFG and highly irregular IVIFG, but not strongly irregular IVIFG, as $\deg_{\mathcal{G}}(v_2) = \deg_{\mathcal{G}}(v_3)$.

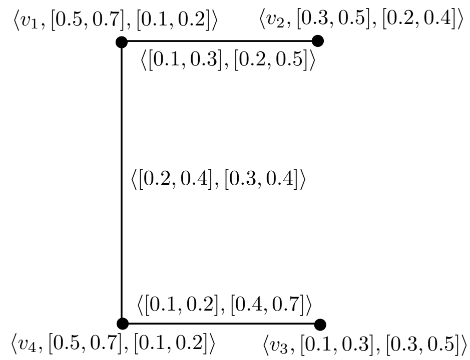

Figure 7: Neighbourly irregular and highly irregular IVIFG

Definition 3.12. An IVIFG $\mathcal{G} = (X, Y)$ is said to be complete if

$$\begin{aligned} M_Y^-(xy) &= \min(M_X^-(x), M_X^-(y)), M_Y^+(xy) = \min(M_X^+(x), M_X^+(y)), \\ N_Y^-(xy) &= \max(N_X^-(x), N_X^-(y)), N_Y^+(xy) = \max(N_X^+(x), N_X^+(y)) \end{aligned}$$

such that $0 < M_Y^+(xy) + N_Y^+(xy) \leq 1$ for all $x, y \in V$.

Proposition 3.1. *A complete IVIFG may not be a strongly irregular IVIFG.*

Definition 3.13. An IVIFG $\mathcal{G}$ is said to be totally irregular, if there is a vertex which is adjacent to vertices with distinct closed neighbourhood degrees. That is, $\deg_{\mathcal{G}}[l] \neq \langle [j, k], [s, t] \rangle$ for all $l \in V$.

An IVIFG in Fig. 3. is totally irregular as, $\deg_{\mathcal{G}}[v_1] = \deg_{\mathcal{G}}[v_2] = \langle [0.6, 0.8], [0.7, 1.0] \rangle$ and $\deg_{\mathcal{G}}[v_3] = \langle [0.8, 1.1], [1.2, 1.6] \rangle$.

Definition 3.14. A connected IVIFG $\mathcal{G}$ is said to be a neighbourly totally irregular, if no two adjacent vertices of $\mathcal{G}$ have same closed neighbourhood degree. That is, $\deg_{\mathcal{G}}[l] \neq \deg_{\mathcal{G}}[m]$ for all $lm \in E$.

Definition 3.15. A connected IVIFG $\mathcal{G}$ is said to be a highly totally irregular if every vertex of $\mathcal{G}$ is adjacent to vertices with distinct closed neighbourhood degrees. That is, $l, m \in \mathcal{N}(x)$, $l \neq m \implies \deg_{\mathcal{G}}[l] \neq \deg_{\mathcal{G}}[m]$ for all $x \in V$.

Definition 3.16. A connected IVIFG $\mathcal{G}$ on G is called strongly totally irregular if every pair of vertices in $\mathcal{G}$ have distinct closed neighborhood degrees.

Example 3.8. Consider a graph $G = (V, E)$, where $V = \{v_1, v_2, v_3\}$ and $E = \{v_1v_3, v_2v_3\}$. Let $\mathcal{G} = (X, Y)$ be an IVIFG of a graph G defined by

$$\begin{aligned} X &= \left\langle \left(\frac{v_1}{0.1}, \frac{v_2}{0.2}, \frac{v_3}{0.4} \right), \left(\frac{v_1}{0.2}, \frac{v_2}{0.5}, \frac{v_3}{0.5} \right), \left(\frac{v_1}{0.3}, \frac{v_2}{0.1}, \frac{v_3}{0.2} \right), \left(\frac{v_1}{0.6}, \frac{v_2}{0.4}, \frac{v_3}{0.4} \right) \right\rangle, \\ Y &= \left\langle \left(\frac{v_1v_3}{0.1}, \frac{v_2v_3}{0.2} \right), \left(\frac{v_1v_3}{0.2}, \frac{v_2v_3}{0.4} \right), \left(\frac{v_1v_3}{0.5}, \frac{v_2v_3}{0.3} \right), \left(\frac{v_1v_3}{0.6}, \frac{v_2v_3}{0.5} \right) \right\rangle, \end{aligned}$$

where $\deg_{\mathcal{G}}(v_1) = \langle [0.5, 0.7], [0.5, 1.0] \rangle$, $\deg_{\mathcal{G}}(v_2) = \langle [0.6, 1.0], [0.3, 0.8] \rangle$ and $\deg_{\mathcal{G}}(v_3) = \langle [0.7, 1.2], [0.6, 1.4] \rangle$. Therefore $\mathcal{G} = (X, Y)$ is a strongly totally irregular IVIFG, as given in Fig. 8.

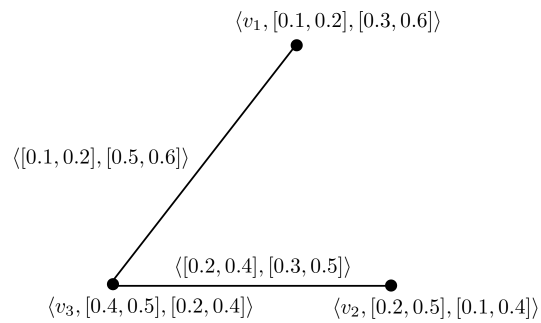

Figure 8: Strongly totally irregular IVIFG.

Remark 3.3. A neighbourly irregular IVIFG need not be a neighbourly totally irregular IVIFG.

Remark 3.4. A neighbourly totally irregular IVIFG need not be a neighbourly irregular IVIFG.

Theorem 3.2. *Let $\mathcal{G} = (X, Y)$ be an IVIFG. If $\mathcal{G}$ is neighbourly irregular and $\langle [M_X^-, M_X^+], [N_X^-, N_X^+] \rangle$ is a constant function. Then $\mathcal{G}$ is a neighbourly totally irregular IVIFG.*

Proof. Suppose that $\mathcal{G}$ is a neighbourly irregular IVIFG. That is, no two adjacent vertices of $\mathcal{G}$ have same neighbourhood degree. Let x and y be the adjacent vertices of $\mathcal{G}$ with distinct neighborhood degrees $\langle [j_1, k_1], [s_1, t_1] \rangle$ and $\langle [j_2, k_2], [s_2, t_2] \rangle$, respectively. Also take $\langle [M_X^-(x_i), M_X^+(x_i)], [N_X^-(x_i), N_X^+(x_i)] \rangle = \langle [c_1, c_2], [c_3, c_4] \rangle$ for all $x_i \in V$, where $c_1, c_2, c_3, c_4 \in [0, 1]$ are constants. Therefore,

$$\begin{aligned}
\deg[x] &= \langle [\deg_{M^-}[x], \deg_{M^+}[x]], [\deg_{N^-}[x], \deg_{N^+}[x]] \rangle \\
&= \langle [\deg_{M^-}(x) + M_X^-(x), \deg_{M^+}(x) + M_X^+(x)], \\
&\quad [\deg_{N^-}(x) + N_X^-(x), \deg_{N^+}(x) + N_X^+(x)] \rangle \\
&= \langle [j_1 + c_1, k_1 + c_2], [s_1 + c_3, t_1 + c_4] \rangle,
\end{aligned}$$

$$\begin{aligned}
\deg[y] &= \langle [\deg_{M^-}[y], \deg_{M^+}[y]], [\deg_{N^-}[y], \deg_{N^+}[y]] \rangle \\
&= \langle [\deg_{M^-}(y) + M_X^-(y), \deg_{M^+}(y) + M_X^+(y)], \\
&\quad [\deg_{N^-}(y) + N_X^-(y), \deg_{N^+}(y) + N_X^+(y)] \rangle \\
&= \langle [j_2 + c_1, k_2 + c_2], [s_2 + c_3, t_2 + c_4] \rangle.
\end{aligned}$$

To prove that closed neighborhood degrees of every two adjacent vertices are distinct. Assume that, $\deg[x] = \deg[y]$.

$\langle [j_1+c_1, k_1+c_2], [s_1+c_3, t_1+c_4] \rangle = \langle [j_2+c_1, k_2+c_2], [s_2+c_3, t_2+c_4] \rangle$
$\Rightarrow \langle [j_1, k_1], [s_1, t_1] \rangle = \langle [j_2, k_2], [s_2, t_2] \rangle$,
a contradiction. Therefore, no two adjacent vertices of $\mathcal{G}$ have same closed neighbourhood degree. Hence $\mathcal{G}$ is a neighbourly totally irregular IVIFG. $\square$

Theorem 3.3. *Let $\mathcal{G} = (X, Y)$ be an IVIFG. If $\mathcal{G}$ is a neighbourly totally irregular and $\langle [M_X^-, M_X^+], [N_X^-, N_X^+] \rangle$ is a constant function, then $\mathcal{G}$ is a neighbourly irregular IVIFG.*

Proof. Suppose that $\mathcal{G}$ is a neighbourly totally irregular IVIFG. That is, no two adjacent vertices of $\mathcal{G}$ have same closed neighbourhood degrees. Let x and y be the adjacent vertices of $\mathcal{G}$ with distinct closed neighborhood degrees $\langle [j_1+c_1, k_1+c_2], [s_1+c_3, t_1+c_4] \rangle$ and $\langle [j_2+c_1, k_2+c_2], [s_2+c_3, t_2+c_4] \rangle$, respectively. Also take $\langle [M_X^-(x_i), M_X^+(x_i)], [N_X^-(x_i), N_X^+(x_i)] \rangle = \langle [c_1, c_2], [c_3, c_4] \rangle$ for all $x_i \in V$, where $c_1, c_2, c_3, c_4 \in [0, 1]$ are constants. We show that no two adjacent vertices of $\mathcal{G}$ have same neighbourhood degrees.
As $\deg[x] \neq \deg[y]$
$\Rightarrow \langle [j_1+c_1, k_1+c_2], [s_1+c_3, t_1+c_4] \rangle \neq \langle [j_2+c_1, k_2+c_2], [s_2+c_3, t_2+c_4] \rangle$
$\Rightarrow \langle [j_1, k_1], [s_1, t_1] \rangle \neq \langle [j_2, k_2], [s_2, t_2] \rangle$.
Therefore, no two adjacent vertices of $\mathcal{G}$ have same neighbourhood degrees. Hence $\mathcal{G}$ is a neighbourly irregular IVIFG. $\square$

Proposition 3.2. *If an IVIFG $\mathcal{G}$ is both neighbourly irregular and neighbourly totally irregular, then $\langle [M_X^-, M_X^+], [N_X^-, N_X^+] \rangle$ may not be a constant function.*

Proposition 3.3. *The interval-valued intuitionistic fuzzy subgraph $H = (X', Y')$ of a neighbourly (totally) irregular IVIFG $\mathcal{G} = (X, Y)$ may not be neighbourly (totally) irregular.*

4 Interval-valued intuitionistic fuzzy cliques

In this section, we propose the notion of IVIFC consistent with interval-valued intuitionistic fuzzy cycles in IVIFGs and present a complete characterization of the structure of the IVIFC. To do this, we firstly introduce the concept of interval-valued intuitionistic fuzzy cycles.

Definition 4.1. Let $\mathcal{G} = (X, Y)$ be an IVIFG. Then

1. $\mathcal{G}$ is a cycle if and only if $G = (V, E)$ is a cycle.

2. $\mathcal{G}$ is called an interval-valued intuitionistic fuzzy cycle if and only if G is a cycle and there does not exist unique edge lm of G such that

$$\begin{aligned} M_Y^-(lm) &= \min\{M_Y^-(xy) \mid xy \in E\}, M_Y^+(lm) = \min\{M_Y^+(xy) \mid xy \in E\}, \\ N_Y^-(lm) &= \max\{N_Y^-(xy) \mid xy \in E\}, N_Y^+(lm) = \max\{N_Y^+(xy) \mid xy \in E\}. \end{aligned}$$

Definition 4.2. Let $\mathcal{G} = (X, Y)$ be an IVIFG of a graph $G = (V, E)$ and $H = (X', Y')$ be a subgraph induced by $S \subseteq V$. Then H is a clique if $H^* = (S, T)$ is a clique and H is an IVIFC if H is a clique and every cycle in H is an interval-valued intuitionistic fuzzy cycle.

Example 4.1. Consider an IVIFG $\mathcal{G}$ as shown in Fig. 2. Take $S = V$, then H is the same as $\mathcal{G}$. Routine computations show that H is a cycle but not an interval-valued intuitionistic fuzzy cycle. Hence H is a clique but not an IVIFC.

Example 4.2. Consider a graph $G = (V, E)$, where $V = \{v_1, v_2, v_3, v_4\}$ and $E = \{v_1v_2, v_2v_3, v_3v_4, v_4v_1, v_1v_3, v_2v_4\}$. Let $\mathcal{G} = (X, Y)$ be an IVIFG of a graph G with $M_X^-(v) = 0.4, M_X^+(v) = 0.6, N_X^-(v) = 0.1$ and $N_X^+(v) = 0.2$ for all $v \in V$ and set of interval-valued intuitionistic fuzzy relations

$$
\begin{aligned}
Y \;=\; \Big\langle & \Big(\frac{v_1v_2}{0.1}, \frac{v_2v_3}{0.4}, \frac{v_3v_4}{0.3}, \frac{v_4v_1}{0.1}, \frac{v_1v_3}{0.1}, \frac{v_2v_4}{0.3}\Big), \Big(\frac{v_1v_2}{0.2}, \frac{v_2v_3}{0.6}, \frac{v_3v_4}{0.5}, \frac{v_4v_1}{0.2}, \frac{v_1v_3}{0.2}, \frac{v_2v_4}{0.5}\Big), \\
& \Big(\frac{v_1v_2}{0.4}, \frac{v_2v_3}{0.1}, \frac{v_3v_4}{0.2}, \frac{v_4v_1}{0.4}, \frac{v_1v_3}{0.4}, \frac{v_2v_4}{0.2}\Big), \Big(\frac{v_1v_2}{0.6}, \frac{v_2v_3}{0.3}, \frac{v_3v_4}{0.4}, \frac{v_4v_1}{0.6}, \frac{v_1v_3}{0.6}, \frac{v_2v_4}{0.4}\Big)\Big\rangle.
\end{aligned}
$$

Take $S = V$, then H is the same as $\mathcal{G}$. Routine computations show that every cycle in H is an interval-valued intuitionistic fuzzy cycle. Hence H is a clique and is also an IVIFC. The IVIFC is given in Fig. 9. Tabular representation of an IVIFC is given in Table 4.

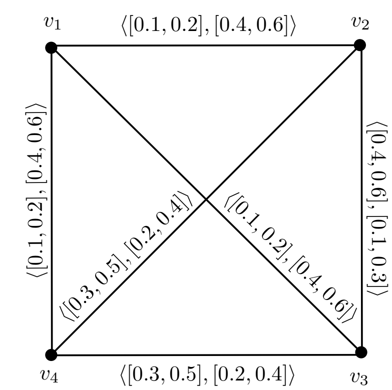

Figure 9: IVIFC

Table 4: Tabular representation of an IVIFC.

	v_1v_2	v_2v_3	v_3v_4	v_4v_1	v_1v_3	v_2v_4
M_Y^-	0.1	0.4	0.3	0.1	0.1	0.3
M_Y^+	0.2	0.6	0.5	0.2	0.2	0.5
M_Y^-	0.4	0.1	0.2	0.4	0.4	0.2
M_Y^+	0.6	0.3	0.4	0.6	0.6	0.4

Theorem 4.1. *Let $\mathcal{G} = (X, Y)$ be an IVIFG of a graph $G = (V, E)$ and $H = (X', Y')$ be a subgraph induced by $S \subseteq V$. Then H is an IVIFC if and only if every cycle of length 3 in H is an interval-valued intuitionistic fuzzy cycle.*

Proof. Suppose that H is an IVIFC. Then by above definition every cycle in H is an interval-valued intuitionistic fuzzy cycle and so every cycle of length 3 in H is also an interval-valued intuitionistic fuzzy cycle.

Conversely, assume that every cycle of length 3 is an interval-valued intuitionistic fuzzy cycle. To prove that H is an IVIFC, we have to show that every cycle in H of length $n \geq 3$ is an interval-valued intuitionistic fuzzy cycle. The proof is by induction on the length of interval-valued intuitionistic fuzzy cycles in H. By assumption, every cycle of length 3 is an interval-valued intuitionistic fuzzy cycle. Induction hypothesis is that every cycle of length n is an interval-valued intuitionistic fuzzy cycle. Let $v_0, v_1, ..., v_n, v_{n+1}$ be any cycle C_{n+1} of length $n+1$ in H. Since H is a clique, H contains a cycle C_n of length n i.e $v_0, v_1, ..., v_n$ and is an interval-valued intuitionistic fuzzy cycle in H. Therefore $\exists$ at least two edges, say e_1 and e_2 in an interval-valued intuitionistic fuzzy cycle C_n such that

$$\begin{aligned}
M_Y^-(e_1) &= M_Y^-(e_2) = \min\{M_Y^-(e) \mid e \text{ is an edge in } C_n\}, \\
M_Y^+(e_1) &= M_Y^+(e_2) = \min\{M_Y^+(e) \mid e \text{ is an edge in } C_n\}, \\
N_Y^-(e_1) &= N_Y^-(e_2) = \max\{N_Y^-(e) \mid e \text{ is an edge in } C_n\}, \\
N_Y^+(e_1) &= N_Y^+(e_2) = \max\{N_Y^+(e) \mid e \text{ is an edge in } C_n\}.
\end{aligned}$$

Also v_0, v_n, v_{n+1} is an interval-valued intuitionistic fuzzy cycle and hence $\exists$ at least two edges, say e_3 and e_4 in an interval-valued intuitionistic fuzzy cycle v_0, v_n, v_{n+1} such that

$$\begin{aligned}
M_Y^-(e_3) &= M_Y^-(e_4) = \min\{M_Y^-(e) \mid e \text{ is an edge in } v_0, v_n, v_{n+1}\}, \\
M_Y^+(e_3) &= M_Y^+(e_4) = \min\{M_Y^+(e) \mid e \text{ is an edge in } v_0, v_n, v_{n+1}\}, \\
N_Y^-(e_3) &= N_Y^-(e_4) = \max\{N_Y^-(e) \mid e \text{ is an edge in } v_0, v_n, v_{n+1}\}, \\
N_Y^+(e_3) &= N_Y^+(e_4) = \max\{N_Y^+(e) \mid e \text{ is an edge in } v_0, v_n, v_{n+1}\}.
\end{aligned}$$

Then two cases arise, firstly, if one of the edges e_1 or e_2 is the same as one of the edges e_3 or e_4. In this case, take $e_1 = e_3$. Then e_2 and e_4 are the edges in C_{n+1} such that

$$\begin{aligned}
M_Y^-(e_2) &= M_Y^-(e_4) = \min\{M_Y^-(e) \mid e \text{ is an edge in } C_{n+1}\}, \\
M_Y^+(e_2) &= M_Y^+(e_4) = \min\{M_Y^+(e) \mid e \text{ is an edge in } C_{n+1}\}, \\
N_Y^-(e_2) &= N_Y^-(e_4) = \max\{N_Y^-(e) \mid e \text{ is an edge in } C_{n+1}\}, \\
N_Y^+(e_2) &= N_Y^+(e_4) = \max\{N_Y^+(e) \mid e \text{ is an edge in } C_{n+1}\}
\end{aligned}$$

as required.
Secondly, all four edges e_1, e_2, e_3, e_4 are edges in C_{n+1} and either

$$\begin{aligned}
M_Y^-(e_1) &= M_Y^-(e_2) = \min\{M_Y^-(e) \mid e \text{ is an edge in } C_{n+1}\}, \\
M_Y^+(e_1) &= M_Y^+(e_2) = \min\{M_Y^+(e) \mid e \text{ is an edge in } C_{n+1}\}, \\
N_Y^-(e_1) &= N_Y^-(e_2) = \max\{N_Y^-(e) \mid e \text{ is an edge in } C_{n+1}\}, \\
N_Y^+(e_1) &= N_Y^+(e_2) = \max\{N_Y^+(e) \mid e \text{ is an edge in } C_{n+1}\}
\end{aligned}$$

or

$$
\begin{aligned}
M_Y^-(e_3) &= M_Y^-(e_4) = \min\{M_Y^-(e) \mid e \text{ is an edge in } C_{n+1}\}, \\
M_Y^+(e_3) &= M_Y^+(e_4) = \min\{M_Y^+(e) \mid e \text{ is an edge in } C_{n+1}\}, \\
N_Y^-(e_3) &= N_Y^-(e_4) = \max\{N_Y^-(e) \mid e \text{ is an edge in } C_{n+1}\}, \\
N_Y^+(e_3) &= N_Y^+(e_4) = \max\{N_Y^+(e) \mid e \text{ is an edge in } C_{n+1}\}.
\end{aligned}
$$

Hence in both cases, H is an IVIFC. □

Lemma 4.1. *Let $\mathcal{G} = (X, Y)$ be an IVIFG of a graph $G = (V, E)$ and $H = (X', Y')$ be a subgraph induced by $S \subseteq V$. Then every cycle of length 3 in H is an interval-valued intuitionistic fuzzy cycle if and only if for any three vertices u, v, w in H such that the edges $uv, vw \in E(H_t)$ implies $uw \in E(H_t)$ for all $t \in [0, 1]$.*

Lemma 4.2. *Let $\mathcal{G} = (X, Y)$ be an IVIFG of a graph $G = (V, E)$ and $H = (X', Y')$ be a subgraph induced by $S \subseteq V$. Then H_t is a disjoint union of cliques if and only if for any three vertices u, v, w in H such that the edges $uv, vw \in E(H_t)$ implies $uw \in E(H_t)$ for all $t \in [0, 1]$.*

As a consequence of Lemmas 4.1 and 4.2, we obtain

Theorem 4.2. *Let $\mathcal{G} = (X, Y)$ be an IVIFG of a graph $G = (V, E)$ and $H = (X', Y')$ be a subgraph induced by $S \subseteq V$. Then H is an IVIFC if and only if every cut set of H is a disjoint union of cliques.*

5 Conclusion

IVIFG is an extended structure of a fuzzy graph which gives more precision, flexibility, and compatibility to the system when compared with the classical, fuzzy and intuitionistic fuzzy models. In this paper, we have mainly provided specific types of IVIFGs. Firstly, regular, irregular, neighbourly irregular, highly irregular and strongly irregular IVIFGs have been introduced and some of their properties have been investigated. Then, the concept of IVIFC consistent with interval-valued intuitionistic fuzzy cycles in IVIFGs is proposed and a complete characterization of the structure of the IVIFC is presented.

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