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Elliptical Fly-By and Expected Gyro-gravitational Orbit Accelerations

described by using
the Maxwell Analogy for gravitation.

T. De Mees - thierrydm @ pandora.be

Abstract

Following to the two former papers *"The Gyro-Gravitational Spin Vector Torque Dynamics of Main Belt Asteroids in relationship with their Tilt and their Orbital Inclination"* and *"Cyclic Tilt Spin Vector Variations of Main Belt Asteroids due to the Solar Gyro-Gravitation"*, wherein we theoretically studied the tilt motions and variations of spinning asteroids, we continue the analysis with the study of the orbit anomalies of satellites. The equations for the fly-by of satellites near the Earth, or near planets in general are deduced.

Keywords: Fly-by – satellite – planet – gravitation – gyrotation – prograde – retrograde – orbit.
Method: Analytical.

1. Basic equations of the former papers.

In the former paper *"The Gyro-Gravitational Spin Vector Torque Dynamics of Main Belt Asteroids in relationship with their Tilt and their Orbital Inclination"*, a physical deduction is found for the motion and the variations of the tilt of asteroids. This deduction is based upon the Maxwell Analogy for Gravitation.
As explained, the gravito-magnetic field of the Earth indeed influences the path of satellites because of their velocity, by the following equation, which is the analogue Lorentz force for gravity:

$$\boldsymbol{F} \Leftarrow m\,(\,\boldsymbol{g} + \boldsymbol{v} \times \boldsymbol{\Omega}\,) \qquad (1.1)$$

Herein $\boldsymbol{g}$ is the gravity field vector of the Earth, $\boldsymbol{\Omega}$ its gravito-magnetic field vector (also called *gyrotation*), and m and $\boldsymbol{v}$ the mass and the velocity vector of the satellite. As explained the gravito-magnetic field vector is found out of the Earth's data (see eq.(3.8.a) in that paper and eq.(1.2) below).
The equations are totally valid for a spinning Earth that is surrounded by orbiting satellites. The Earth's angular velocity is $\boldsymbol{\omega}$, its moment of inertia is I.

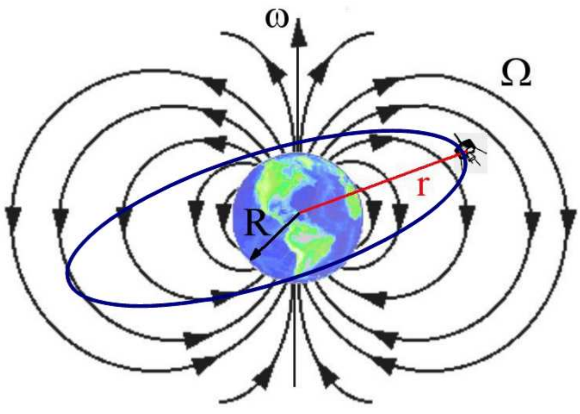


Fig. 1.1 : A spinning sphere with radius R and rotation velocity $\boldsymbol{\omega}$ is generating a rotary gravitation field (or "gyrotation" field) $\boldsymbol{\Omega}$ at a distance $\boldsymbol{r}$ from the sphere's centre.

$$\vec{\Omega} = \frac{G\,I}{2\,r^3 c^2}\left(\vec{\omega} - \frac{3\,\vec{r}\,(\vec{\omega}\cdot\vec{r}\,)}{r^2}\right) \qquad (1.2.a)$$

wherein for a sphere : $$I = \frac{2}{5} m\,R^2 \qquad (1.2.b)$$

The value of the gyrotation can be found at each place in the universe, and is decreasing with the third power of the distance r . The factor $\boldsymbol{\omega} \bullet \boldsymbol{r}$ represents the scalar vector-product, and this value is zero at the equatorial level.

If we want to understand the accelerations of the satellites due to the second field, gyrotation, we need to know the vector product $\vec{v} \times \vec{\Omega}$ in the vector equation (1.1) with the help of the vector equation (1.2). Therefore, we need some definitions of orbit angles, see fig.1.2.

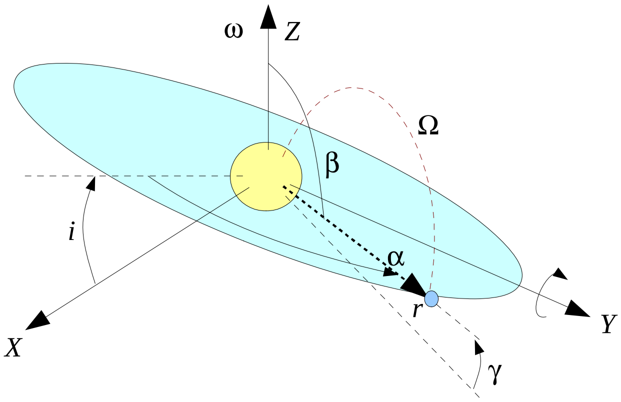


Fig. 1.2 : Definition of the angles α and i . The orbital plane is defined by the orbital inclination i in relation to the X axis. The location of the asteroid inside the orbit is defined by the angle α . The equipotential line of the gyrotation $\boldsymbol{\Omega}$ through the asteroid has been shown as well.

In order to find the vector product $v \times \Omega$, we need to know the angle β in terms of the inclination i and the position angle α, since the scalar vector-product of (1.2 a) is defined by $\omega\, r \cos\beta$.

Therefore we notice that (see fig.1.2): $$r \sin\gamma = r_z = r \cos\alpha \sin i \qquad (1.3.a)$$

And since $\sin\gamma = \cos\beta$, we get : $$\cos\beta = \cos\alpha \sin i \qquad (1.3.b)\ (1.3.c)$$

Hence, $$\left(\Omega_x, \Omega_y, \Omega_z\right) = \frac{G\, m R^2}{5 r^3 c^2}\left[\left(0,0,\omega\right) - \frac{3}{r^2}\left(r_x, r_y, r_z\right)\left(\omega r \cos\alpha \sin i\right)\right] \qquad (1.3.d)$$

wherein $$\left(r_x, r_y, r_z\right) = r\left(\cos\alpha \cos i, \sin\alpha, \cos\alpha \sin i\right) \qquad (1.3.e)$$

The equations (1.3) constitute the detailed vector formula of the equation (1.2). Remark that $\omega = \omega_{\text{earth}}$.

2. Accelerations due to the Earth’s or planet’s spin.

In this paper, we will make abstraction of the satellite’s elliptic exact orbit shape, but the reader can implement that by defining an angle α_0 that defines the location of the orbit’s pericenter. Then, by applying the angle α_0 in the equation (1.3.e), the correct variability of the radius can be expressed. By using the classical velocity equations for elliptical orbits, defined by the angles α_0 and α, the reader can find any primary velocity of the orbit.
The analytical equations below are valid for $\alpha_0 = 0$. This means that the orbit’s pericenter coincides with the position of $\alpha = 0$. They allow us to get graphical representations of the satellite accelerations due to the Earth’s gyrotation field.

Rotation of coordinate system to the elliptical plane.

With (1.1), we find the accelerations $\vec{v} \times \vec{\Omega}$ due to gyrotation.

In order to see more easily what really happens with a satellite, let us make a transform in the plane of the satellite’s orbit. More precisely a rotation of the system over the orbit inclination i. The coordinate system *X’ Y’ Z’* is given by a clockwise rotation over the angle i :

$$\left(X', Y', Z'\right) = \left(X \cos i + Z \sin i, Y, -X \sin i + Z \cos i\right) \qquad (2.1)$$

By doing this, we have put the satellite orbit in the *X’ Y’* plane, and we can easily find the corresponding gyrotation

$$\left(\Omega_x', \Omega_y', \Omega_z'\right) = \left(\Omega_x \cos i + \Omega_z \sin i, \Omega_y, -\Omega_x \sin i + \Omega_z \cos i\right) \qquad (2.2)$$

Equation (2.2) is written in full in Appendix A.

Below, we will define the equations that cover elliptical orbits and then we find the gyrotational accelerations, which are explicitly written down in the Appendix B.

3. Elliptical equations.

In order to adapt the equations for an elliptic path, we apply the following Keplerian equations:

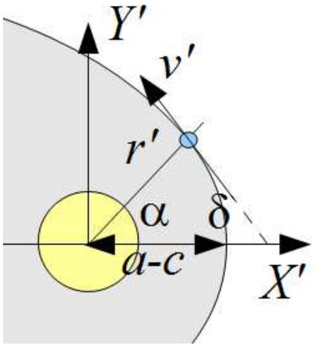


Fig. 3.1 : Definition of the angle α for elliptic orbits.

$$r' = \frac{a\left(1-\varepsilon^2\right)}{1+\varepsilon\cos\alpha} \qquad \text{and} \qquad v' = \sqrt{GM\left(\frac{2}{r'}-\frac{1}{a}\right)} \qquad (3.1)\ (3.2)$$

wherein a is the ellipse's major radius and ε is the eccentricity given by

$$\varepsilon = \sqrt{1-\left(b/a\right)^2} = c/a\,. \qquad (3.3)$$

Herein, b is the ellipse's minor radius, c the coordinate of the focus (the planet) if the center is taken in the middle of the ellipse, and a-c the shortest distance between the ellipse and the planet's center.

Remark that we have defined the angle α as the angle between the major axis and the satellite's position.

Furthermore, the satellite's position can be written as (fig.3.1):

$$\vec{r}\,' = (r_x', r_y', r_z') = (r'\cos\alpha, r'\sin\alpha, 0) \qquad (3.4)$$

If we want to find the coordinates of the orbit's velocity vector in the coordinate system (X',Y',Z'), we need the slope of the tangent, which is given by the angle δ (see fig.3.1). Therefore we take the basic equation of the ellipse whereof the center of the coordinate system coincides with the planet: $\frac{\left(x'-c\right)^2}{a^2}+\frac{y'^2}{b^2}=1$. (3.5)

By differentiating this equation, we come to : $\frac{\left(x'-c\right)}{a^2}+\frac{y'}{b^2}\frac{\mathrm{d}\,y}{\mathrm{d}\,x}=0$, or with (3.1) and (3.3) this gives:

$$\frac{\mathrm{d}\,y}{\mathrm{d}\,x} = \tan\delta = -\frac{b^2\left(x'-c\right)}{a^2 y'} = -\frac{b^2\left(r'\cos\alpha - c\right)}{a^2 r'\sin\alpha} = -\frac{b^2\left(\cos\alpha - a\varepsilon/r'\right)}{a^2\sin\alpha} = -\frac{1-\varepsilon-\varepsilon^2\left(1+\cos\alpha\right)}{\sin\alpha} \qquad (3.6)$$

From (3.6) follows the following initial orbit velocities:

$$v_x' = v'\cos\delta = \frac{v'\sin\alpha}{\sqrt{\sin^2\alpha + \left(1-\varepsilon-\varepsilon^2\left(1+\cos\alpha\right)\right)^2}}$$

$$v_y' = v'\sin\delta = -\frac{v'\left(1-\varepsilon-\varepsilon^2\left(1+\cos\alpha\right)\right)}{\sqrt{\sin^2\alpha + \left(1-\varepsilon-\varepsilon^2\left(1+\cos\alpha\right)\right)^2}}$$

$$v_z' = 0$$

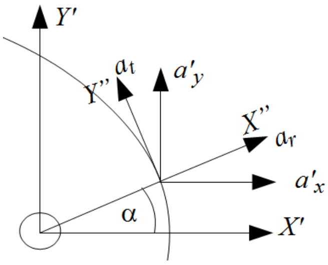
(3.7.a) (3.7.b) (3.7.c)

4. Further equations.

The satellite's gyrotational accelerations $\vec{v}\times\vec{\Omega}$ in the (X',Y',Z') system due to the Earth's rotation are then given by:

$$\left(a_x', a_y', a_z'\right) = \left(v_y'\Omega_z' - v_z'\Omega_y', v_z'\Omega_x' - v_x'\Omega_z', v_x'\Omega_y' - v_y'\Omega_x'\right) = \left(v_y'\Omega_z', -v_x'\Omega_z', v_x'\Omega_y' - v_y'\Omega_x'\right) \quad 4.1)$$

Option 1 : Rotation of coordinate system to polar coordinates versus the planet.



What interests us are the values of the tangential and the radial accelerations versus the planet, and finally the accelerations that are perpendicular to the orbital plane. To see these accelerations, let us make a transform in the plane of the satellite's orbit. More precisely a rotation of the system over the angle α. The coordinate system *X" Y" Z"* is given by a counter-clockwise rotation over the angle α :

Fig. 3.2. rotation of the coordinate system.

$$\left(X'',Y'',Z''\right)=\left(X'\cos\alpha+Y'\sin\alpha,Y'\cos\alpha-X'\sin\alpha,Z'\right) \tag{4.2}$$

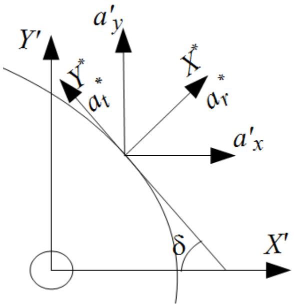
Or, for the accelerations:

$$\left(a_x'',a_y'',a_z''\right)=\left(a_x'\cos\alpha+a_y'\sin\alpha\,,\,a_y'\cos\alpha-a_x'\sin\alpha\,,\,a_z'\right)$$

Wherein we find the radial and the tangential accelerations (see fig.3.2):

$$\left(a_x'',\,a_y'',\,a_z''\right)=\left(a_r\,,\,a_t\,,\,a_z'\right) \tag{4.3}$$

Option 2 : Rotation of coordinate system to polar coordinates versus the orbital path.



Another interest thing are the values of the tangential and the radial accelerations to the orbital path, and finally the accelerations that are perpendicular to the orbital plane. To see these accelerations, let us make a transform in the plane of the satellite's orbit. More precisely a rotation of the system over the angle π+δ (since δ is negative). The coordinate system $X^*\ Y^*\ Z^*$ is given by a counter-clockwise rotation over the angle α :

Fig. 3.3. rotation of the coordinate system.

$$\left(X^*,Y^*,Z^*\right)=\left(-X'\sin\delta+Y'\cos\delta,-Y'\sin\delta-X'\cos\delta,Z'\right) \tag{4.4}$$

Or, for the accelerations:

$$\left(a_x^*,a_y^*,a_z^*\right)=\left(-a_x'\sin\delta+a_y'\cos\delta\,,\,-a_y'\sin\delta-a_x'\cos\delta\,,\,a_z'\right)$$

Wherein we find the radial and the tangential accelerations (see fig.3.2):

$$\left(a_x^*,\,a_y^*,\,a_z^*\right)=\left(a_r^*\,,\,a_t^*\,,\,a_z'\right) \tag{4.5}$$

When using the equations (3.7), the equation (4.5) can be found.

5. Graphical solutions.

The figures 5.1 and 5.2 show the values of the accelerations that satellites undergo by the equation (4.5), written in full by the equation (D.3.a). The tangential acceleration a_t^*, along the satellite's path is zero, as confirmed by the equations in the Appendix D.

In fig. 5.1 we show the radial gyrotational acceleration a_r^* , which points to the Earth's center, for the values of i and α between -π and π.

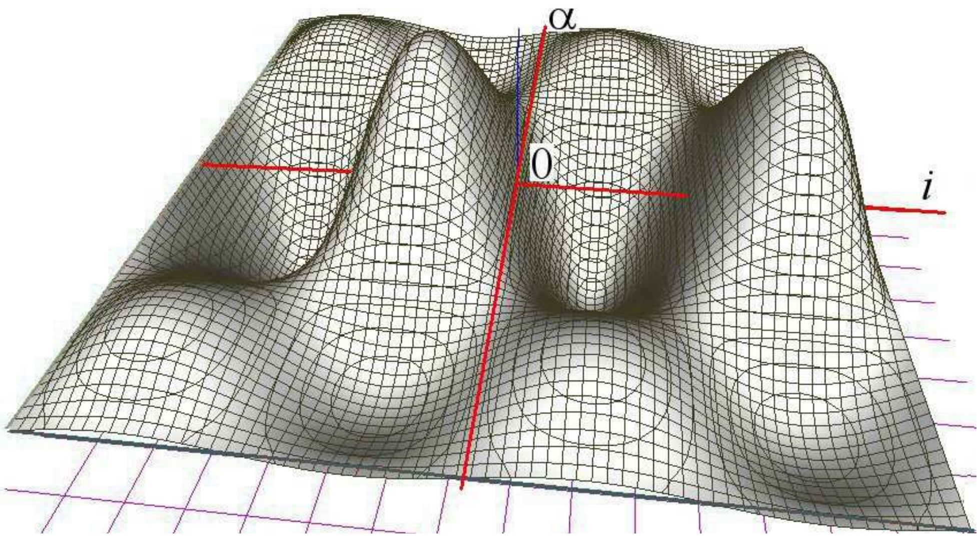


Fig.5.1: The radial gyrotational acceleration a_r^* of satellites about the Earth, in relation to the orbital inclination i and the orbital position α of the satellite. We took $a/b = 2$. The red line are the zero values for α and i. The values of i and α are between -π and π.

The values of a_r^* are zero for the orbital inclinations i that are multiples of π/4. The highest absolute values are found for inclinations i between-in these values, especially for α equal to 0. For α equal to π or -π, there is an attenuation due to the orbit's eccentricity. For circular orbits, there is no attenuation.

In fig. 2.2 we show the gyrotational acceleration a_z^*, scaled at 25%, which is perpendicular to the satellite's orbital plane, for the values of i and α between -π and π.

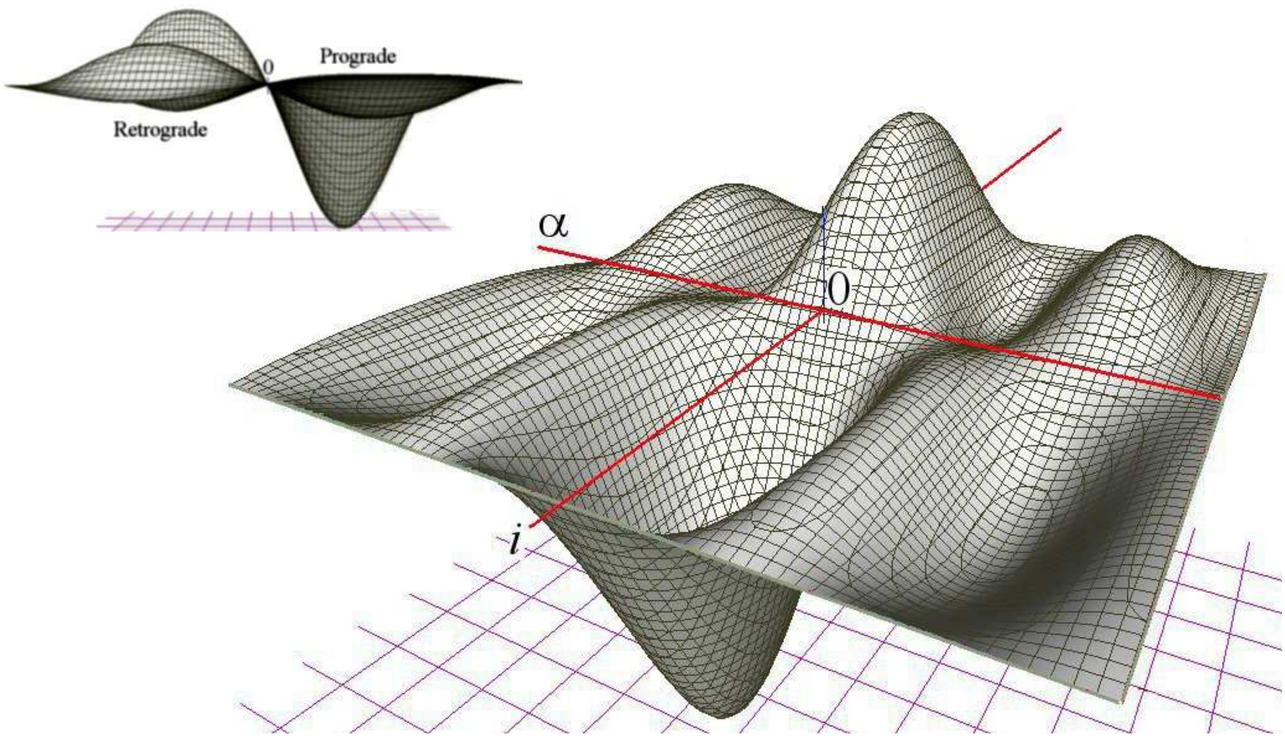


Fig.2.2: The gyrotational acceleration a_z^* , perpendicular to the orbital plane, of satellites about the Earth, in relation to the orbital inclination i and the orbital position α of the satellite. We took $a/b = 2$ and we had to scale a_z^* to 25% compared with a_r^*. The red line are the zero values for α and i. The values of i and α are between -π and π. A side view is also shown.

The scales of the orbital inclination and the orbital position of the satellite are taken the same for both graphs. Here as well, there is an attenuation at $\alpha = \pi$.

The highest values of a_z^* are obtained when the orbital inclination is at $\pi/2$, when the orbit is perpendicular to the Earth's equator. The prograde value is double as large (in absolute value) as the retrograde one, and the width (action radius) is also larger. Prograde orbits always swivel towards equatorial orbits (i = 0) and retrograde orbits first swivel towards $i = \pi/2$, then towards the planet's equator (i = 0). The retrograde value is smaller due to the elliptic shape, which causes an attenuation. This is caused by the choice of an elliptic orbit whereof the pericenter is situated according to fig.1.2.

6. Discussion and conclusions : the swiveling process of inclined orbits.

We have calculated the satellite accelerations due to the Earth's rotation. It is found that the values of a_r^* (perpendicular to the orbital path) are zero for an orbital inclination i equal to $\pi/2$ and its multiples. The highest absolute values are found for an inclination i of $\pi/4$ and $3\pi/4$, for α equal to 0. For α equal to π there is an attenuation due to the orbit's eccentricity. For circular orbits, the value α at π equals that of $\alpha = 0$ (in absolute values).

With the least satellite's orbit inclination, away from the planet's equator, an important radial acceleration occurs upon the satellites. At $i = \pi/4$ already, the acceleration a_r^* comes to an absolute maximum around the pericenter. This explains why significant alterations of the satellites' paths occurred near Saturn.

For specific fly-bys, the double integration of $a_x{}'$ and $a_y{}'$ over time gives the satellite's extra displacement due to the planet's spin. The energy increase can be found from it as well.

There is no gyrotational acceleration along the satellite's path, since a_t^* is found to be zero. A vector product indeed cannot be oriented the same as one of the product's components.

The strongest values for the acceleration a_z^* (acceleration that is perpendicular to the orbital plane) are obtained for the inclinations i that are perpendicular to the planet's equatorial plane, at $\pi/2$. The orbital positions α where the highest absolute values are obtained, are zero. The maximal absolute values of a_z^* are significantly larger than those of a_r^*. Prograde orbits always swivel towards equatorial orbits and retrograde orbits first swivel towards the poles first, then towards the planet's equator.

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Appendix A : Gyrotational field equations written in full.

The values of the velocity v are given in (2.1) and the values of the gyrotation Ω are given in (A.2) below, based upon the equations (1.3.d) and (1.3.e).

$$\left(\Omega_x,\Omega_y,\Omega_z\right)=\frac{G\,m\,R^2}{5r^3c^2}\left[\left(0,0,\omega\right)-3\omega\cos\alpha\sin i\left(\cos\alpha\cos i,\sin\alpha,\cos\alpha\sin i\right)\right]$$

or

$$\left(\Omega_x,\Omega_y,\Omega_z\right)=\frac{G\,m\,R^2}{5r^3c^2}\left[\omega\cos\alpha\sin i\left(-3\cos\alpha\cos i,-3\sin\alpha,\left(1-3\cos\alpha\sin i\right)\right)\right]$$

or

$$\left(\Omega_x,\Omega_y,\Omega_z\right)=\frac{G\,m\,R^2\omega}{10\,r^3c^2}\left(-3\cos^2\alpha\sin 2i,-3\sin 2\alpha\sin i,2\cos\alpha\sin i\left(1-3\cos\alpha\sin i\right)\right) \tag{A.1}$$

Then, we can solve the equation (2.2):

$$\Omega_x{}'=\frac{G\,m\,R^2\omega}{5r^3c^2}\left(\sin i-3\cos\alpha\right)\cos\alpha\sin i$$

$$\Omega_y{}'=-3\frac{G\,m\,R^2\omega}{10\,r^3c^2}\sin 2\alpha\sin i$$

$$\Omega_z{}'=\frac{G\,m\,R^2\omega}{5r^3c^2}\cos\alpha\sin i\cos i$$

(A.2.a) (A.2.b) (A.2.c)

Appendix B : Gyrotational acceleration equations written in full (Cartesian).

Written in full, the accelerations due to the Earth's (planet's) rotation, exerted on a satellite are, (3.1) and (3.2):

$$a_x{}' = -\frac{G\,m\,R^2\omega}{10\,r^3c^2}\sqrt{\frac{GM}{a}\left(\frac{2\left(1+\varepsilon\cos\alpha\right)}{\left(1-\varepsilon^2\right)}-1\right)}\frac{\left(1-\varepsilon-\varepsilon^2\left(1+\cos\alpha\right)\right)\cos\alpha\sin 2i}{\sqrt{\sin^2\alpha+\left(1-\varepsilon-\varepsilon^2\left(1+\cos\alpha\right)\right)^2}}$$

$$a_y{}' = -\frac{G\,m\,R^2\omega}{20\,r^3c^2}\sqrt{\frac{GM}{a}\left(\frac{2\left(1+\varepsilon\cos\alpha\right)}{\left(1-\varepsilon^2\right)}-1\right)}\frac{\sin 2\alpha\sin 2i}{\sqrt{\sin^2\alpha+\left(1-\varepsilon-\varepsilon^2\left(1+\cos\alpha\right)\right)^2}}$$

$$a_z{}' = \frac{G\,m\,R^2\omega}{5\,r^3c^2}\sqrt{\frac{GM}{a}\left(\frac{2\left(1+\varepsilon\cos\alpha\right)}{\left(1-\varepsilon^2\right)}-1\right)}\frac{\left(1-\varepsilon-\varepsilon^2\left(1+\cos\alpha\right)\right)\left(\sin i-3\cos\alpha\right)-3\sin^2\alpha}{\sqrt{\sin^2\alpha+\left(1-\varepsilon-\varepsilon^2\left(1+\cos\alpha\right)\right)^2}}\cos\alpha\sin i$$

(B.1.a) (B.1.b) (B.1.c)

Appendix C : Gyrotational acceleration equations written in full (polar versus the planet).

Written in full, the accelerations due to the Earth's (planet's) rotation, exerted to a satellite are:

$$a_x{}'' = a_r = \frac{G\,m\,R^2\omega}{10\,r^3c^2}\sqrt{\frac{GM}{a}\left(\frac{2\left(1+\varepsilon\cos\alpha\right)}{\left(1-\varepsilon^2\right)}-1\right)}\frac{\sin^2\alpha-\left(1-\varepsilon-\varepsilon^2\left(1+\cos\alpha\right)\right)\cos\alpha}{\sqrt{\sin^2\alpha+\left(1-\varepsilon-\varepsilon^2\left(1+\cos\alpha\right)\right)^2}}\sin 2i\cos\alpha$$

$$a_y{}'' = a_t = \frac{G\,m\,R^2\omega}{20\,r^3c^2}\sqrt{\frac{GM}{a}\left(\frac{2\left(1+\varepsilon\cos\alpha\right)}{\left(1-\varepsilon^2\right)}-1\right)}\frac{\cos\alpha-\left(1-\varepsilon-\varepsilon^2\left(1+\cos\alpha\right)\right)}{\sqrt{\sin^2\alpha+\left(1-\varepsilon-\varepsilon^2\left(1+\cos\alpha\right)\right)^2}}\sin 2i\sin 2\alpha$$

$$a_z{}'' = a_z{}' = \frac{G\,m\,R^2\omega}{5\,r^3c^2}\sqrt{\frac{GM}{a}\left(\frac{2\left(1+\varepsilon\cos\alpha\right)}{\left(1-\varepsilon^2\right)}-1\right)}\frac{\left(1-\varepsilon-\varepsilon^2\left(1+\cos\alpha\right)\right)\left(\sin i-3\cos\alpha\right)-3\sin^2\alpha}{\sqrt{\sin^2\alpha+\left(1-\varepsilon-\varepsilon^2\left(1+\cos\alpha\right)\right)^2}}\cos\alpha\sin i$$

(C.1.a) (C.1.b) (C.1.c)

Appendix D : Gyrotational acceleration equations written in full (polar versus the orbit).

Written in full, the accelerations due to the Earth's (planet's) rotation, exerted to a satellite are:

$$a_x{}^* = a_r{}^* = -\frac{G\,m\,R^2\omega}{10\,r^3c^2}\sqrt{\frac{GM}{a}\left(\frac{2\left(1+\varepsilon\cos\alpha\right)}{\left(1-\varepsilon^2\right)}-1\right)}\sin 2i\cos\alpha$$

$$a_y{}^* = a_t = 0$$

$$a_z{}^* = a_z{}' = \frac{G\,m\,R^2\omega}{5\,r^3c^2}\sqrt{\frac{GM}{a}\left(\frac{2\left(1+\varepsilon\cos\alpha\right)}{\left(1-\varepsilon^2\right)}-1\right)}\frac{\left(1-\varepsilon-\varepsilon^2\left(1+\cos\alpha\right)\right)\left(\sin i-3\cos\alpha\right)-3\sin^2\alpha}{\sqrt{\sin^2\alpha+\left(1-\varepsilon-\varepsilon^2\left(1+\cos\alpha\right)\right)^2}}\cos\alpha\sin i$$

(D.1.a) (D.1.b) (D.1.c)