About the number of elements of certain real sequences

Zeraoulia Elhadj
Department of Mathematics, University of Tébessa, (12002), Algeria.
E-mail: zelhadj12@yahoo.fr and zeraoulia@mail.univ-tebessa.dz

February 16, 2017

Abstract

This note is concerned with presenting sufficient conditions to proves that the number of elements of certain real sequences is infinite.

Keywords: Real sequences, infinite number of elements.
AMS 2010 Mathematics Subject Classification: Primary 40A05, 11A41, 11A25, Secondary 11A51, 11B34.

1 Introduction

In many situations there is interesting questions about the number of elements of certain real sequences. For example, the twin prime conjecture in number theory: A twin prime is a prime number that has a prime gap of two, i.e., there exist $n > 0$ such that p_n and p_{n+1} are both primes and $p_{n+1}-p_n = 2$. The twin prime conjecture assert that there are infinitely many twin primes. A proof of this conjecture seems to be out of reach by present methods available in the current literature. One step to an important progress was made by Viggo Brun [8] in 1919 where he showed that the sum of the reciprocals of the twin primes converges to a real number called Brun's constant $B_2 = 1.90216054$ calculated in 1976 by using the twin primes up to 100 billion. In 2004 Thomas Nicely gave $B_2 = 1.9021605825820 \pm 000000001620$

based on all twin primes less than 5×10^{15}. Some important progress on small gaps between primes were established by Goldston, et al. [1-2-3-4-5-6-7].

In [9] Zhang showed that for some integer $N < 70 \times 10^6$, there are infinitely many pairs of primes that differ by N. The method of Zhang is a refinement of the work of Goldston, Pintz and Yildirim [4] on the small gaps between consecutive primes along a stronger version of the Bombieri-Vinogradov theorem [12-13-14] that is applicable when the moduli are free from large prime divisors only. Under certain strong assumptions (like the Elliott–Halberstam conjecture and its generalized form), Terence Tao and James Maynard and others reduces this bound to 246, then to 12 and 6 [10].

In this note we give sufficient conditions to prove that the number of elements of certain real sequences is infinite. This note does not claim the proof of any conjecture. It is just a new idea to see how one can solves some problems if some conditions are assumed. Our method is based on the following three steps:

1. The assumption of the **Conditions** (1)-(9) below about the sequences $(u_k)_{k\in\mathbb{N}^*}$ and $(\omega_n)_{n\in\Omega_2}$ and the function f_2.

2. Proving that under **Condition** (7), the iterations of f_2 cannot hit the fixed point B_2 in any finite number of steps.

3. The construction of a strictely increasing and bounded sequence $(c_n)_{n\in\Omega_2}$ and showing that this sequence has an infinite number of elements.

2 Number of elements of certain real sequences

Let $(u_k)_{k\in\mathbb{N}^*}$ be a real sequence verifying the following conditions:

Condition (1) The sequence $(u_k)_{k\in\mathbb{N}^*}$ is strictely positive, i.e., for all $k \in \mathbb{N}^*$, we have $u_k > 0$.

Condition (2) The sequence $(u_k)_{k\in\mathbb{N}^*}$ is strictely increasing, i.e., for all $k \in \mathbb{N}^*$, we have $u_k < u_{k+1}$.

Condition (3) The sequence $(u_k)_{k\in\mathbb{N}^*}$ is not arithmetic, i.e., $(u_k)_{k\in\mathbb{N}^*}$ do not verify $u_{k+1} = u_k + 2$ for all $k \in \mathbb{N}^*$, that is, the difference between the consecutive terms is not constant.

Condition (4) The sequence $(u_k)_{k\in\mathbb{N}^*}$ is unbounded, i.e., $\lim_{k\to+\infty} u_k = +\infty$.

Condition (5) The series $\sum_{k=1}^{+\infty} \frac{1}{u_k}$ diverges, i.e., $\sum_{k=1}^{+\infty} \frac{1}{u_k} = +\infty$.

Condition (6) The sum of the reciprocals of the terms verifying $u_{n+1} = u_n + 2$ (pairs of terms which differ by 2) converges to a finite value B_2, i.e., there exist a non-empty set $\Omega_2 \subset \mathbb{N}$ such that

$$\sum_{n\in\Omega_2} \left(\frac{1}{u_n} + \frac{1}{u_n + 2} \right) = B_2 > 0 \tag{1}$$

In other words, the sum in (1) either has finitely many terms or has infinitely many terms but is convergent.

Condition (7) There exist a real function $f_2 : D_{f_2} \to D_{f_2}$ where $D_{f_2} = \mathbb{R}/\{B_2 + \theta\}$ (for some $0 < \theta < 1$) is its set of definition such that $[0, B_2] \subset [0, B_2 + \theta) \subset D_{f_2}$ with the following properties:

(1) $f_2(x)$ is a continuous and differentiable function with continuous derivative for all $x \in D_{f_2}$.

(2) $f_2(x)$ is a strictly increasing function for all $x \in D_{f_2}$ and $f_2(x) > 0$ for all $x \in [0, B_2 + \theta)$.

(3) $f_2(x)$ is a bijection in every closed interval $[0, \lambda] \subset [0, B_2 + \theta)$ to itself.

(4) The only fixed point of f_2 in D_{f_2} is $x = B_2$.

(5) The interval $[0, B_2]$ is invariant under f_2, i.e., $f_2([0, B_2]) \subset [0, B_2]$.

(6) The equation $f_2'(x) = 1$ has the only real solution $x = B_2$ in D_{f_2}.

(7) The derivative f_2' is a strictly increasing (i.e., $f_2''(x) > 0$) and bijective in every closed interval $[0, \lambda] \subset [0, B_2 + \theta)$ to itself.

(8) For all $x \in D_{f_2}$ we have

$$0 < f_2'(x) \leq 1 \tag{2}$$

i.e., at $x = B_2$, the derivative $f_2'(x)$ gets it maximum value (which is 1) in D_{f_2}.

Condition (8) There exist a strictely increasing real sequence $(\omega_n)_{n\in\Omega_2} \subset [0, B_2]$ satisfying

$$\omega_{n+1} = f_2(\omega_n) \text{ if and only if } u_{n+1} = u_n + 2 \text{ for all } n \in \Omega_2 \qquad (3)$$

Condition (9) For any finite $n \in \Omega_2$, any real solution b of the equation

$$f_2(b) - b = \omega_{n+1} - \omega_n \qquad (4)$$

is located in the open interval $(0, B_2 + \theta)$ for some $0 < \theta < 1$.

Here we keep using the symbol B_2 for this general case. We restrict our investigation to the class of real sequences $(u_k)_{k\in\mathbb{N}^*}$ verifying **Conditions** (5) and (6) in accordance with some known sequences of prime numbers such as the sequence of twin primes. We will use the following mean values theorem:

Theorem 1 *Let $f : [a, b] \to [a, b]$ be a continuous function on the closed interval $[a, b]$, and differentiable on the open interval (a, b), where $a < b$. Then there exists some $c \in (a, b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a} \qquad (5)$$

If f is a strictely increasing function with a strictely increasing derivative f', then c is included strictely in the open interval (a, b), i.e., $c \neq a$ and $c \neq b$, that is, the cases $f'(a) = \frac{f(b)-f(a)}{b-a}$ and $f'(b) = \frac{f(b)-f(a)}{b-a}$ are impossible since f' is at least injective. Theorem 1 is used to construct the strictely increasing and bounded sequence $(c_n)_{n\in\Omega_2}$ defined by (6) below.

2.1 Number of iterations for the convergence to the fixed point B_2 of the function f_2

Let f_2^k denote the k-composition of f_2 with itself, i.e., $f_2^k = f_2 \circ f_2 \circ ... \circ f_2$ (k-times). Denote the only fixed point of f_2 by B_2 (**Condition** (7-(4))), and the initial point of the iteration by $x_0 \in [0, B_2]$. Let x_k be the value of f_2^k at x_0, i.e., $x_k = f_2^k(x_0)$.

Lemma 2 *The sequence $x_{k+1} = f_2(x_k)$ converges to the fixed point B_2 of the function f_2 after an infinite number of steps.*

Proof. Restricting our study to the interval $[0, B_2]$. We have $f_2(x) > 0$ for all $0 \leq x \leq B_2$ by **Condition** (7-(5)). By **Condition** (7-(3)), the function $f_2 : [0, B_2] \to [0, B_2]$ is injective. If $x_0 = B_2$ then we get the fixed point after 0 iterations. If $x_0 \neq B_2$, we now have $f_2^2(x_0) \in [0, B_2]$. By **Condition** (7-(3)), the function f_2 is injective from $[0, B_2]$ to $[0, B_2]$, so if $z \in [0, B_2]$ then $f_2(z) = B_2$ if and only if $z = B_2$ by **Condition** (7-(4)). So in this case we will not hit the fixed point B_2 in any finite number of steps. This fact can be proven by induction. Indeed, taking $x_2 = f_2^2(x_0) \neq B_2$ as the base case and the inductive hypothesis as $x_k = f_2^k(x_0) \neq B_2$, where $k \geq 2$. Since $k \geq 2$, $x_k = f_2^k(x_0) \in [0, B_2]$, on which f_2 is injective. Using the inductive hypothesis and the injectivity of f_2 we conclude that $x_{k+1} = f_2^{k+1}(x_0) = f_2\left(f_2^k(x_0)\right) \neq f_2(B_2) = B_2$. So $x_{k+1} = f_2^{k+1}(x_0) \neq B_2$. By induction $x_k = f_2^k(x_0) \neq B_2$ for all $k \geq 2$. ■

2.2 About the number of elements of Ω_2

Theorem 3 *Assuming* ***Conditions*** *(1)-(9), then the number of elements of Ω_2 is infinite.*

Proof. Condition (1) is necessary to garanty that $B_2 > 0$. **Condition** (3) is necessary to avoid contradiction between (5) and (6).

Let $a = \omega_n$ with $n \geq 2$ and let $b \in D_{f_2}$ such that $b > \omega_n$, then by the mean values theorem 1 applied to the function f_2 there exist $c_n \in (\omega_n, b)$ such that

$$f_2'(c_n) = \frac{f_2(b) - f_2(\omega_n)}{b - \omega_n} = \frac{f_2(b) - \omega_{n+1}}{b - \omega_n} \tag{6}$$

This is allowed by the existence of the function f_2 given by **Condition** (7) and the existence of $(\omega_n)_{n \in \Omega_2}$ by **Condition** (8). By construction, the sequence $(\omega_n)_{n \in \Omega_2}$ is an increasing sequence if and only if f_2 is an increasing real function by **Condition** (7-(2)). By **Condition** (7-(7)) f_2' is a strictly increasing bijective function, then the values of c_n never doubled for all $n \in \Omega_2$ and by this method and **Condition** (7-(7)) and **Condition** (7-(8)) we have constructed a strictly increasing and bounded sequence $(c_n)_{n \in \Omega_2} \subset (\omega_2, B_2)$ such that

$$c_2 < c_3 < c_5 < ... < c_n < ... < B_2 \tag{7}$$

This means that the application $\Psi : \Omega_2 \to (\omega_2, B_2)$ defined by

$$\Psi(n) = c_n \tag{8}$$

is bijective and strictly increasing. To prove that the number of elements of the set Ω_2 is infinite we will looking to the points b verifying (6) such that the corresponding c_m verify $f_2'(c_m) = 1$ for certain $m \in \Omega_2$. We have $f_2'(c_m) = 1$ if and only if

$$c_m = B_2 \tag{9}$$

by **Condition** (7-(6))). At $x = B_2$, the function $f_2'(x)$ gets it maximum value (which is 1) in the interval $[0, B_2]$ by inequality (2). Also, equation (9) holds true for only one value $m \in \Omega_2$. Otherwise, we get $c_m = c_q = B_2$ with $m \neq q$. This is a contradiction since $\Psi(n) = c_n$ is bijective. This value of m must be the biggest element of Ω_2. Indeed, we have $c_m = B_2$, if there exists $q \in \Omega_2 : q > m$ such that $f_2'(c_q) = 1$, then we must have $c_q = B_2$ by **Condition** (7-(6)), then we get $c_m = c_q = B_2$ and this is also a contradiction since $\Psi(n) = c_n$ is bijective.

Assuming by contradiction that m is finite. Then by Lemma 2, $\omega_m = f^m(x_0) \neq B_2$, that is, $\omega_m < B_2$ (here $x_0 \neq B_2$) by **Condition** (7-(5)). Equation (9) and Theorem 1 and **Condition** (9) implies that $B_2 < b < B_2 + \theta$ for some $0 < \theta < 1$. Applying again the mean values theorem 1 to the function f_2 in the interval $[B_2\ , b]$, then there exist $\alpha \in (B_2\ , b)$ such that

$$f_2'(\alpha) = \frac{f_2(b) - f_2(B_2)}{b - B_2} = \frac{f_2(b) - B_2}{b - B_2}$$

with $f_2'(\alpha) > f_2'(B_2) = 1$. But at $x = B_2$, the function $f_2'(x)$ gets it maximum value (which is 1) in the interval $[0, B_2]$ by inequality (2). This implies that $f_2'(\alpha) = 1$ and hence $\alpha = B_2$ and this is a contradiction since the mean values theorem 1 says that $\alpha \in (B_2\ , b)$ with $\alpha \neq B_2$ and $\alpha \neq b$ for any strictly increasing function with a strictly increasing derivative. Hence, m must be inifinite and this is allowed by **Condition** (4). Finally, the number of elements of Ω_2 is infinite. ■

For the case, where we wante to get $u_{n+1} = u_n + d$ for some $d > 0$. We proceed as before and replacing B_2, Ω_2, f_2 by B_d, Ω_d, f_d and replacing $u_{n+1} = u_n + 2$ by $u_{n+1} = u_n + d$ in all the text and replacing $u_{k+1} = u_k + 2$ by $u_{k+1} = u_k + d$ in **Condition** (3). The values of d are admissible according to the set of values of u_n. For the case of positive integer sequences $(u_k)_{k \in \mathbb{N}^*} \subset \mathbb{N}^*$, the difference d must be a positive integer. If we looking now to the set of all prime numbers denoted by $(p_k)_{k \geq 1}$, then we must have $d \geq 2$ and d is an even positive integer since all primes > 2 are odd and the difference between two odd numbers is always even.

Generally, in number theory, the famous Polignac's conjecture states [11]:

Conjecture 4 *For any positive even number d, there are infinitely many prime gaps of size d, i.e., There are infinitely many cases of two consecutive prime numbers with difference d.*

Conjecture 4 is not proven or disproven for a given value of d. For $d = 2$, it is the twin prime conjecture discussed in the introduction. The case $d = 4$ is concerned with the cousin primes $(p_n, p_n + 4)$. The case $d = 6$ is concerned with the sexy primes $(p_n, p_n + 6)$, etc...

After writting this note, we are able now to construct the sequence $(\omega_n)_{n\in\Omega_2}$ and the function f_2 as follow:

$$\omega_n = \sum_{j\in\Omega_2-\{n\}} \left(\frac{1}{p_j} + \frac{1}{p_j+2}\right) = B_2 - \left(\frac{1}{p_n} + \frac{1}{2+p_n}\right) \tag{10}$$

and

$$f_2(x) = \frac{5x - 2x^2 + 2B_2^2 - 2B_2 - 2 + 2(B_2 + 1 - x)\sqrt{(x - B_2)^2 + 1}}{4(B_2 - x) + 3} \tag{11}$$

For the general case, we have

$$\psi_n = \sum_{j\in\Omega_2-\{n\}} \left(\frac{1}{p_j} + \frac{1}{p_k+d}\right) = B_d - \left(\frac{1}{p_n} + \frac{1}{d+p_n}\right) \tag{12}$$

and

$$f_d(x) = \frac{-d^2x^2 + 5dx - 2dB_d - 4 + d^2B_d^2 + (dB_d - dx + 2)\sqrt{d^2(x - B_d)^2 + 4}}{d(-2dx + 2dB_d + 3)} \tag{13}$$

the details will be avalaible soon.

References

[1] Goldston, D. A., Graham, S. W. Pintz, J., & Yildirim, C. Y. (2009) Small gaps between primes or almost primes, Transactions of the American Mathematical Society, 36, 5285–5330.

[2] Goldston, D. A., Graham, S. W. Pintz, J., & Yildirim, C. Y. (2009) Small gaps between products of two primes, Proceedings of London Mathematical Society, 98, 741–774.

[3] Goldston, D. A., Pintz, J., & Yildirim, C. Y. (2006) Primes in tuples III: On the difference $\{p_{n+v} - p_n\}$, Funct. Approx. Comment. Math, 35(1), 79-89.

[4] Goldston, D. A., Pintz, J., & Yildirim, C. Y. (2009) Primes in tuples I, Annals of Mathematics, 170, 819–862.

[5] Goldston, D. A., Pintz, J., & Yildirim, C. Y. (2010) Primes in tuples II, Acta Math, 204, 1–47.

[6] Goldston, D. A., Pintz, J., & Yildirim, C. Y . (2011) Positive proportion of small gaps between consecutive primes, Publ. Math. Debrecen, 79, 433–444.

[7] Goldston, D. A. Pintz, J., & Yildirim, C. Y. (2013) Primes in tuples IV: Density of small gaps between consecutive primes, Acta Arithmetica, 160(1), 37-53.

[8] Brun, V. (1915) Uber das Goldbachsche Gesetz und die Anzahl der Primzahlpaare, Arch. f. Math. og Naturv. (in German), 34(8), 3–19.

[9] Yitang, Z. (2014) Bounded gaps between primes, Annals of Mathematics, 179(3), 1121–1174.

[10] D. H. J. Polymath. 2014 The "bounded gaps between primes" Polymath project - a retrospective, preprint

[11] Dickson, L. E. (2005) History of the Theory of Numbers, Vol. 1: Divisibility and Primality. New York: Dover.

[12] Bombieri, E., Friedlander, J. B., & Iwaniec, H. (1986) Primes in arithmetic progressions to large moduli, Acta Math, 156, 203-251.

[13] Bombieri, E., Friedlander, J. B., & Iwaniec, H. (1987) Primes in arithmetic progressions to large moduli. II, Math. Ann, 277, 361-393.

[14] Bombieri, E., Friedlander, J. B., & Iwaniec, H. (1989) Primes in arithmetic progressions to large moduli. III, J. Amer. Math. Soc, 2, 215-224.