An Elliptic Integral

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Abstract

This note presents some formulas related with the elliptic integrals

Introducción. En la página 264 de la referencia (3) , fórmula 5. aparece la integral:

$$I=\int_0^1 \frac{1}{\sqrt{1-x^3}}\,dx=\frac{1}{2\pi\sqrt{3}\sqrt[3]{2}}\left(\Gamma\left(\frac{1}{3}\right)\right)^3 \tag{1}$$

Recordamos que:

$$\frac{1}{2\pi\sqrt{3}\sqrt[3]{2}}\Gamma\left(\frac{1}{3}\right)^3=\frac{2\pi^2\sqrt[3]{4}}{9}\Gamma\left(\frac{2}{3}\right)^{-3}=\frac{2\pi^{3/2}\sqrt{3}}{9}\Gamma\left(\frac{2}{3}\right)^{-1}\Gamma\left(\frac{5}{6}\right)^{-1} \tag{2}$$

En esta nota mostramos algunas fórmulas relacionadas con la integral (1).

Integral elíptica incompleta de primera especie.

La integral elíptica incompleta de primera especie se define por:

$$F(k,\varphi)=\int_0^{\varphi}\frac{1}{\sqrt{1-k^2\sin^2\theta}}\,d\theta=\int_0^{y}\frac{1}{\sqrt{(1-x^2)(1-k^2x^2)}}\,dx \tag{3}$$

donde $y=\sin\varphi, 0<k<1$.

La transformación de Landen.

Con el cambio de variable:

$$\tan\varphi=\frac{\sin 2\varphi_1}{k+\cos 2\varphi_1} \quad o \quad k\sin\varphi=\sin(2\varphi_1-\varphi) \tag{4}$$

se tiene:

$$F(k,\varphi)=\int_0^{\varphi}\frac{1}{\sqrt{1-k^2\sin^2\theta}}\,d\theta=\frac{2}{1+k}\int_0^{\varphi_1}\frac{1}{\sqrt{1-k_1^2\sin^2\theta_1}}\,d\theta_1 \tag{5}$$

donde

$$k_1 = \frac{2\sqrt{k}}{1+k} \tag{6}$$

La fórmula (5) se puede escribir como:

$$F(k,\varphi) = \frac{2}{1+k} F(k_1,\varphi_1) \tag{7}$$

Por aplicaciones sucesivas de (4), se obtiene:

$$F(k,\varphi) = \sqrt{\frac{k_1 k_2 k_3 \ldots}{k}} \ln \tan\left(\frac{\pi}{4} + \frac{\Phi}{2}\right) \tag{8}$$

donde

$$k_{n+1} = \frac{2\sqrt{k_n}}{1+k_n}, \quad n = 0,1,2,3,\ldots; k_0 = k \tag{9}$$

$$\Phi = \lim_{n\to\infty} \varphi_n \tag{10}$$

$$k < k_1 < k_2 < \ldots < 1 \quad , \quad \lim_{n\to\infty} k_n = 1 \tag{11}$$

Algunas fórmulas relacionadas con la integral (1).

$$I = \int_1^\infty \frac{1}{\sqrt{x^4 - x}}\, dx \tag{12}$$

$$I = \int_0^1 \frac{1}{\sqrt{x\left(3 - 3x + x^2\right)}}\, dx \tag{13}$$

$$I = \int_1^\infty \frac{1}{\sqrt{x\left(3x^2 - 3x + 1\right)}}\, dx \tag{14}$$

$$I = \int_0^\infty \frac{1}{\sqrt{x\left(3 + 6x + 4x^2 + x^3\right)}}\, dx \tag{15}$$

$$I = 2\int_0^1 \frac{1}{\sqrt{3 - 3x^2 + x^4}}\, dx \tag{16}$$

$$I = \frac{1}{3}\int_0^1 x^{-1/2}\left(1-x\right)^{-2/3} dx = \frac{1}{3}\int_0^1 x^{-2/3}\left(1-x\right)^{-1/2} dx \tag{17}$$

$$I = 1 + \int_1^{\infty} \left(1 - \sqrt[3]{1 - x^{-2}}\right) dx \tag{18}$$

$$I = \frac{2}{3} \int_0^1 \frac{x^{-1/3}}{\sqrt{1 - x^2}} \, dx \tag{19}$$

$$I = \frac{2}{3} \int_0^{\pi/2} \left(\sin x\right)^{-1/3} dx = \frac{2}{3} \int_0^{\pi/2} \left(\cos x\right)^{-1/3} dx \tag{20}$$

$$I = \frac{\pi}{3} + \frac{2}{3} \int_1^{\infty} \sin^{-1}\left(x^{-3}\right) dx \tag{21}$$

$$I = \frac{\pi}{3} + \frac{2}{3} \int_0^1 \frac{\sin^{-1}\left(x^3\right)}{x^2} \, dx \tag{22}$$

$$I = \frac{\pi}{3} + \frac{2}{9} \int_0^1 \frac{\sin^{-1} x}{x\sqrt[3]{x}} \, dx \tag{23}$$

$$I = 4\sqrt[6]{2} \int_0^{\infty} \frac{\cosh x}{\left(\cosh(6x)\right)^{5/6}} \, dx \tag{24}$$

$$I = \frac{\sqrt{3}+1}{\sqrt[4]{3}} \sin^{-1}\left(\frac{2\sqrt[4]{3}}{\sqrt{3}+1}\right) - \frac{1}{\sqrt[4]{3}} \int_1^{1+\sqrt{3}} \sin^{-1}\left(\left(\sqrt{6}-\sqrt{2}\right)\sqrt{1-x^{-2}}\right) dx \tag{25}$$

$$I = \frac{1}{\sqrt[4]{3}} \int_0^z \frac{1}{\sqrt{\left(1-x^2\right)\left(1-k^2x^2\right)}} \, dx \tag{26}$$

donde $z = \dfrac{2\sqrt[4]{3}}{\sqrt{3}+1}$, $k = \dfrac{\sqrt{6}+\sqrt{2}}{4}$.

$$I = \frac{1}{\sqrt[4]{3}} \int_0^{\varphi} \frac{1}{\sqrt{1-k^2\sin^2\theta}} \, d\theta \tag{27}$$

donde $\varphi = \sin^{-1}\left(\dfrac{2\sqrt[4]{3}}{1+\sqrt{3}}\right)$, $k = \dfrac{\sqrt{6}+\sqrt{2}}{4}$.

$$I = \frac{1}{\sqrt[4]{3}} \int_0^z \frac{1}{\sqrt{\left(1-x^2\right)\left(k^2-x^2\right)}} \, dx \tag{28}$$

donde $z = \dfrac{\sqrt[4]{3}}{\sqrt{2}}$, $k = \dfrac{\sqrt{6}+\sqrt{2}}{4}$.

$$I=\sqrt{2}\int_{1}^{3}\frac{1}{\sqrt{(3-x)\left(3+x^{2}\right)}}\,dx \tag{29}$$

$$I=\frac{2}{3}\int_{0}^{1}\left(1-x^{2}\right)^{-2/3}dx \tag{30}$$

$$I=\int_{0}^{1}\frac{x^{-1/2}}{\sqrt{\left(x-z_{1}\right)\left(x-z_{2}\right)}}\,dx \quad ,z_{1}=\frac{3+i\sqrt{3}}{2},z_{2}=\frac{3-i\sqrt{3}}{2} \tag{31}$$

$$I=\int_{0}^{1}\frac{1}{\sqrt{(1-x)\left(1+x+x^{2}\right)}}\,dx=\int_{0}^{1}\frac{1}{\sqrt{(1-x)\left(x+w_{1}\right)\left(x+w_{2}\right)}}\,dx \tag{32}$$

donde $w_{1}=\frac{1+i\sqrt{3}}{2},w_{2}=\frac{1-i\sqrt{3}}{2}$.

$$I=\frac{1}{3}\int_{0}^{\infty}\frac{x^{-1/2}}{(1+x)^{5/6}}\,dx \tag{33}$$

$$I=\frac{1}{3}\int_{0}^{\infty}\frac{x^{-2/3}}{(1+x)^{5/6}}\,dx \tag{34}$$

$$I=\frac{1}{3}\int_{0}^{1}\frac{x^{-1/2}+x^{-2/3}}{(1+x)^{5/6}}\,dx \tag{35}$$

$$I=\frac{55}{18}\int_{0}^{1}x^{1/3}(1-x)^{1/2}\,dx \tag{36}$$

$$I=\frac{55}{18}\int_{0}^{1}x^{1/2}(1-x)^{1/3}\,dx \tag{37}$$

$$I=\frac{55}{6}\int_{0}^{1}x^{3}\left(1-x^{3}\right)^{1/2}dx \tag{38}$$

$$I=\frac{55}{9}\int_{0}^{1}x^{2}\left(1-x^{2}\right)^{1/3}dx \tag{39}$$

$$I=2-\int_{0}^{1}\frac{(2x+1)\sqrt{1-x}}{\left(1+x+x^{2}\right)^{3/2}}\,dx \tag{40}$$

$$I = 2 - \int_0^1 \frac{(3-2x)\sqrt{x}}{\left(3-3x+x^2\right)^{3/2}}\, dx \tag{41}$$

$$I = \frac{\pi}{3} + \frac{2}{3}\int_0^1\int_0^1 \frac{y}{\sqrt{1-x^2y^6}}\, dxdy \tag{42}$$

$$I = \int_0^1 \frac{x^{-1/2}}{\sqrt{3+6x+4x^2+x^3}}\, dx + \sum_{n=0}^{\infty} \binom{2n}{n} \frac{2^{-5n-1}}{3n+1} \tag{43}$$

$$I = 2\left(2+\sqrt{3}\right) - \frac{1}{2\sqrt[4]{3}} \int_1^{5+3\sqrt{3}} \sqrt{\alpha - 2\sqrt{\beta + \gamma x^{-2}}}\, dx \tag{44}$$

donde

$$\alpha = 18 - 8\sqrt{3}\,,\ \beta = 97 - 56\sqrt{3}\,,\ \gamma = 8\left(\sqrt{3}-1\right)^2 \tag{45}$$

$$I = \int_1^2 \frac{1}{\sqrt{x^4 - x}}\, dx - \frac{1}{\sqrt{14}} + \int_0^{1/\sqrt{14}} \left(-1 + \sqrt[4]{x^{-2} + \sqrt[4]{x^{-2} + \sqrt[4]{x^{-2} + \ldots}}}\right) dx \tag{46}$$

$$I = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{2^{-2n}}{3n+1} \tag{47}$$

$$I = \frac{2}{\sqrt{3}} \sum_{n=0}^{\infty} \binom{2n}{n} 2^{-2n} \sum_{k=0}^{n} \binom{n}{k} \frac{(-3)^{-k}}{2n+2k+1} \tag{48}$$

$$I = \frac{2}{\sqrt{3}} \sum_{n=0}^{\infty} \binom{2n}{n} \left(-\frac{3}{4}\right)^n \sum_{k=n}^{2n} \binom{n}{k-n} \frac{(-3)^{-k}}{2k+1} \tag{49}$$

$$I = \frac{2}{\sqrt{3}} \sum_{n=0}^{\infty} \frac{12^{-n}}{2n+1} \sum_{k=0}^{n} \binom{2n-2k}{n-k} \binom{n-k}{k} (-1)^k\, 2^{2k} 3^{n-k} \tag{50}$$

$$I = -\sum_{n=0}^{\infty} \frac{(-1/2)_n\,(2n-1)}{n!(3n+1)} \tag{51}$$

$$I = -\frac{2}{3} \sum_{n=0}^{\infty} \frac{(-1/3)_n\,(3n-1)}{n!(2n+1)} \tag{52}$$

$$I = \sqrt{2} \sum_{n=0}^{\infty} \binom{2n}{n} 2^{-3n} (-1)^n \sum_{k=0}^{n} \sum_{m=0}^{k} \binom{n}{k} \binom{k}{m} \frac{(-3)^{k-m}}{2k+2m+1} \tag{53}$$

$$I = \frac{1}{3} \sum_{n=0}^{\infty} \frac{(5/6)_n (-1)^n}{n!} \left(\frac{2}{2n+1} + \frac{3}{3n+1}\right) \tag{54}$$

$$I=\frac{2}{3\sqrt[3]{2}}\sum_{n=0}^{\infty}\frac{(5/6)_n\,2^{-n}}{n!}\sum_{k=0}^{n}\binom{n}{k}\frac{(-2)^k}{2k+1}+\frac{1}{\sqrt[3]{2}}\sum_{n=0}^{\infty}\frac{(5/6)_n\,(-2)^{-n}}{n!(3n+1)} \tag{55}$$

$$I=\frac{1}{\sqrt{2}}\sum_{n=0}^{\infty}\frac{(5/6)_n\,2^{-n}}{n!}\sum_{k=0}^{n}\binom{n}{k}\frac{(-2)^k}{3k+1}+\frac{\sqrt{2}}{3}\sum_{n=0}^{\infty}\frac{(5/6)_n\,(-2)^{-n}}{n!(2n+1)} \tag{56}$$

$$I=\frac{\sqrt{2}}{3}\sum_{n=0}^{\infty}\frac{(1/2)_n\,(2/3)_n\,2^{-n}}{n!(3/2)_n}+\frac{1}{\sqrt[3]{2}}\sum_{n=0}^{\infty}\frac{(1/2)_n\,(1/3)_n\,2^{-n}}{n!(4/3)_n} \tag{57}$$

$$I=\frac{\sqrt[6]{2}}{3}\sum_{n=0}^{\infty}\frac{(5/6)_n\,2^{-n}}{(3/2)_n}+\frac{1}{\sqrt[6]{32}}\sum_{n=0}^{\infty}\frac{(5/6)_n\,2^{-n}}{(4/3)_n} \tag{58}$$

$$I=\frac{\pi}{3}+\frac{1}{3}\sum_{n=0}^{\infty}\binom{2n}{n}\frac{2^{-2n}}{(2n+1)(3n+1)} \tag{59}$$

$$I=2\prod_{n=1}^{\infty}\frac{(n+1)(6n-1)}{(2n+1)(3n+1)} \tag{60}$$

$$I=\frac{\sqrt{\pi}}{6}\sum_{n=0}^{\infty}\Gamma\left(\frac{3n+1}{3}\right)\Gamma\left(\frac{6n+11}{6}\right)^{-1} \tag{61}$$

$$I=2-\frac{2}{3\sqrt{3}}\sum_{n=0}^{\infty}\frac{(3/2)_n}{n!}\sum_{k=0}^{n}\binom{n}{k}(-3)^{-k}\left(\frac{3}{2n+2k+3}-\frac{2}{2n+2k+5}\right) \tag{62}$$

$$\pi=\sqrt{A+B\sqrt{A+B\sqrt{A+\ldots}}}=B+\cfrac{A}{B+\cfrac{A}{B+\ldots}} \tag{63}$$

donde

$$A=\frac{3}{2\sqrt[3]{4}}\Gamma\left(\frac{2}{3}\right)^3\sum_{n=0}^{\infty}\binom{2n}{n}\frac{2^{-2n}}{(2n+1)(3n+1)}\quad,\quad B=\frac{3}{2\sqrt[3]{4}}\Gamma\left(\frac{2}{3}\right)^3 \tag{64}$$

$$\frac{1}{\pi}=\sqrt{C+D\sqrt{C+D\sqrt{C+\ldots}}}=D+\cfrac{C}{D+\cfrac{C}{D+\ldots}} \tag{65}$$

donde

$$C=\frac{2\sqrt[3]{2}}{\sqrt{3}}\Gamma\left(\frac{1}{3}\right)^{-3}\quad,\quad D=\frac{2\sqrt[3]{2}}{\sqrt{3}}\Gamma\left(\frac{1}{3}\right)^{-3}\sum_{n=0}^{\infty}\binom{2n}{n}\frac{2^{-2n}}{(2n+1)(3n+1)} \tag{66}$$

Algunas evaluaciones de la tranformación de Landen.

La integral (1) es equivalente a la integral elíptica (27):

$$I = \int_0^1 \frac{1}{\sqrt{1-x^3}}\,dx = \frac{1}{\sqrt[4]{3}}\int_0^{\varphi} \frac{1}{\sqrt{1-k^2\sin^2\theta}}\,d\theta \tag{67}$$

donde

$$\varphi = \sin^{-1}\left(\frac{2\sqrt[4]{3}}{1+\sqrt{3}}\right) \quad , \quad k = \frac{\sqrt{6}+\sqrt{2}}{4} \tag{68}$$

Aplicando la transformación de Landen se tiene:

$$I = \lim_{n\to\infty} I_n \tag{69}$$

donde

$$I_n = \frac{1}{\sqrt[4]{3}}\sqrt{\frac{k_1k_2k_3\ldots}{k}}\ \ln\tan\left(\frac{\pi}{4}+\frac{\varphi_n}{2}\right) \quad , n\in\mathbb{N} \tag{70}$$

$$k_{n+1} = \frac{2\sqrt{k_n}}{1+k_n} \quad , n = 0,1,2,3,\ldots; k_0 = k \tag{71}$$

$$\varphi_{n+1} = \frac{1}{2}\left(\varphi_n + \sin^{-1}\left(k_n \sin\varphi_n\right)\right) \quad , n = 0,1,2,3,\ldots; \varphi_0 = \varphi \tag{72}$$

n	k_n	φ_n
1	0.9998497830170996188009468...	1.2477972335775161825504830...
2	0.9999999971789334958753430...	1.2475729925820250769812066...
3	0.99999999999999999990051979...	1.2475729883711080546165305...

n	I_n	$\lvert e_n\rvert = \lvert I - I_n\rvert$
1	1.4027280783181030228636184...	$5.45\cdot 10^{-4}$
2	1.4021821155743645546492604...	$1.02\cdot 10^{-8}$
3	1.4021821053254542647891262...	$3.61\cdot 10^{-18}$

Notaciones:

Coeficiente Binomial: $\binom{n}{k} = \frac{n(n-1)...(n-k+1)}{k!} = \binom{n}{n-k}$

Símbolo de Pochhammer: $(a)_n = a(a+1)...(a+n-1) \quad ,(a)_0 = 1$

Función Gamma: $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt \text{ , } x > 0 \quad , \Gamma(x+1) = x\Gamma(x)$

Constante pi: $\pi = 4\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$

Referencias

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