The analysis of Chris Van Den Broeck applied to the Natario warp drive spacetime using the original Alcubierre shape function to generate the Broeck spacetime distortion:The Natario-Broeck warp drive

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Abstract

Warp Drives are solutions of the Einstein Field Equations that allows superluminal travel within the framework of General Relativity. There are at the present moment two known solutions: The Alcubierre warp drive discovered in 1994 and the Natario warp drive discovered in 2001. However the major drawback concerning warp drives is the huge amount of negative energy density able to sustain the warp bubble.In order to perform an interstellar space travel to a "nearby" star at 20 light-years away in a reasonable amount of time a ship must attain a speed of about 200 times faster than light.However the negative energy density at such a speed is directly proportional to the factor 10^{48} which is 1.000.000.000.000.000.000.000.000 times bigger in magnitude than the mass of the planet Earth!!. With the correct form of the shape function the Natario warp drive can overcome this obstacle at least in theory.Other drawbacks that affects the warp drive geometry are the collisions with hazardous interstellar matter(asteroids,comets,interstellar dust etc)that will unavoidably occurs when a ship travels at superluminal speeds and the problem of the Horizons(causally disconnected portions of spacetime).The geometrical features of the Natario warp drive are the required ones to overcome these obstacles also at least in theory.Some years ago in 1999 Chris Van Den Broeck appeared with a very interesting idea.Broeck proposed a warp bubble with a large internal radius able to accommodate a ship inside while having a submicroscopic outer radius and a submicroscopic contact external surface in order to better avoid the collisions against the interstellar matter.The Broeck spacetime distortion have the shape of a bottle with 200 meters of inner diameter able to accommodate a spaceship inside the bottle but the bottleneck possesses a very small outer radius with only 10^{-15} meters 100 billion time smaller than a millimeter therefore reducing the probabilities of collisions against large objects in interstellar space.In this work we apply the Broeck idea to the Natario warp drive spacetime but out bottle have 200 kilometers of inner size 1000 times the size of the original Broeck bottle and we use the original Alcubierre shape function to generate our version of the Broeck bottle with very low energy density requirements.The Broeck idea is more than welcome and solves definitively the problem of the collisions against large objects. Any future development for the Natario warp drive must encompass the Broeck bottle and this approach must be named as the Natario-Broeck warp drive.

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1 Introduction:

The warp drive as a solution of the Einstein field equations of General Relativity that allows superluminal travel appeared first in 1994 due to the work of Alcubierre.([1]) The warp drive as conceived by Alcubierre worked with an expansion of the spacetime behind an object and contraction of the spacetime in front.The departure point is being moved away from the object and the destination point is being moved closer to the object.The object do not moves at all[1].It remains at the rest inside the so called warp bubble but an external observer would see the object passing by him at superluminal speeds(pg 8 in [1])(pg 1 in [2]).

Later on in 2001 another warp drive appeared due to the work of Natario.([2]).This do not expands or contracts spacetime but deals with the spacetime as a "strain" tensor of Fluid Mechanics(pg 5 in [2]). Imagine the object being a fish inside an aquarium and the aquarium is floating in the surface of a river but carried out by the river stream.The warp bubble in this case is the aquarium whose walls do not expand or contract. An observer in the margin of the river would see the aquarium passing by him at a large speed but inside the aquarium the fish is at the rest with respect to his local neighborhoods.

However there are 3 major drawbacks that compromises the warp drive physical integrity as a viable tool for superluminal interstellar travel.

The first drawback is the quest of large negative energy requirements enough to sustain the warp bubble. In order to travel to a "nearby" star at 20 light-years at superluminal speeds in a reasonable amount of time a ship must attain a speed of about 200 times faster than light.However the negative energy density at such a speed is directly proportional to the factor 10^{48} which is 1.000.000.000.000.000.000.000.000 times bigger in magnitude than the mass of the planet Earth!!!(see [7],[8] and [9]).

Another drawback that affects the warp drive is the quest of the interstellar navigation:Interstellar space is not empty and from a real point of view a ship at superluminal speeds would impact asteroids,comets,interstellar space dust and photons.(see [5],[7] and [8]).

The last drawback raised against the warp drive is the fact that inside the warp bubble an astronaut cannot send signals with the speed of the light to control the front of the bubble because an Horizon(causally disconnected portion of spacetime)is established between the astronaut and the warp bubble.(see [5],[7] and [8]).

We can demonstrate that the Natario warp drive can "easily" overcome these obstacles as a valid candidate for superluminal interstellar travel(see [7],[8] and [9]).

In this work we cover only the Natario warp drive and we avoid comparisons between the differences of the models proposed by Alcubierre and Natario since these differences were already deeply covered by the existing available literature.(see [5],[6] and [7])However we use the Alcubierre shape function to define its Natario counterpart.

[1]do not violates Relativity

Alcubierre([12]) used the so-called $3+1$ Arnowitt-Dresner-Misner(ADM) formalism using the approach of Misner-Thorne-Wheeler(MTW)([11]) to develop his warp drive theory.As a matter of fact the first equation in his warp drive paper is derived precisely from the original $3+1$ ADM formalism(see eq 2.2.4 pgs [67(b)],[82(a)] in [12], see also eq 1 pg 3 in [1])[2] and we have strong reasons to believe that Natario which followed the Alcubierre steps also used the original $3+1$ ADM formalism to develop the Natario warp drive The Natario warp drive equation that obeys the $3+1$ ADM formalism is given below:[3]

$$ds^2 = (1 - X_{rs}X^{rs} - X_\theta X^\theta)dt^2 + 2(X_{rs}drs + X_\theta d\theta)dt - drs^2 - rs^2 d\theta^2 \qquad (1)$$

$$ds^2 = dt^2 - [(drs - X^{rs}dt)^2 + (rs^2)(d\theta - X^\theta dt)^2] \qquad (2)$$

From the works in [5],[7] and [8] we can see that impacts of the warp bubble against the particles of the Interstellar Medium IM (eg:asteroids,comets,space debris,supernova remnants,clouds of gas etc) are tremendously hazardous for a spaceship at superluminal velocities. However and according to the cited works we know that the negative energy density in the case of the Natario warp bubble do not vanish in the equatorial plane meaning that the repulsive gravitational behavior of the negative energy density in front of the ship can theoretically deflect the IM particles offering some degree of protection to the ship and crew members.

Although we are counting on the negative energy density in front of the ship in the case of the Natario warp drive to offer protection to the ship and the crew members we know that collisions of the warp bubble walls against IM particles are unavoidable and as large the warp bubble is [4] this means a large bubble surface exposed to heavy bombardment by the IM particles.

The ideal situation for a warp bubble in a real superluminal interstellar spaceflight would the one in which the warp bubble possesses a large internal diameter with the size enough to contains a spaceship inside the bubble but the region of the bubble in contact with the interstellar space and hence with the IM particles remains very small reducing the probabilities of dangerous collisions.

What we need is a warp bubble with a large internal radius able to accommodate a ship inside while having a submicroscopic outer radius and a submicroscopic contact external surface in order to better avoid the collisions against the IM particles.

Some years ago in 1999 Chris Van Den Broeck appeared with this idea. Broeck introduced inside the Alcubierre warp drive metric in 1999 a new mathematical term $B(rs)$ with very interesting features:$B(rs)$ creates inside the Alcubierre warp bubble a spacetime distortion with the shape of a bottle.The bottle have an inner large radius and hence a large diameter with the size enough to contains a spaceship inside the bottle but the part of the bottle in contact with our Universe and hence with the dangerous IM particles is the bottle bottleneck wirh a very small microscopic radius and hence a small microscopic surface exposed to collisions against the IM particles protecting effectively the ship inside the bottle.Although the bottle can have an arbitrarily large size an external observer in our Universe would only see the microscopic bottleneck.

[2]see the Remarks section on our system to quote pages in bibliographic references

[3]see Appendices A and B for details

[4]the warp bubble must possesses size enough to contains a spaceship inside

Broeck created inside an Alcubierre warp bubble with a radius R of 3×10^{-15} meters a bottle with 200 meters of inner diameter and a microscopic bottleneck radius with only 10^{-15} meters. So although a spaceship is contained(or hidden) in the inner space of a bottle with 200 meters of diameter the part of the bottle an external observer in our Universe would see would only be the bottleneck of the bottle with 10^{-15} meters and 10^{-15} meters is 10^{12} times or 100.000.000.000 times or 100 billion times smaller than a millimeter.(see pg 5 in [10]).

Effectively a surface with 10^{-15} square millimeters have less probabilities to suffer a collision than a surface of 100 square meters.And with plenty of room space with 200 meters large enough to accommodate a spaceship and hidden from our Universe and in consequence being kept isolated from the dangerous IM particles.The Broeck idea is more than welcome.[5]

The Broeck bottle provides the ideal scenario for the Natario warp drive and in this work we apply the Broeck mathematical term $B(rs)$ to the Natario warp drive equation in ADM formalism but using the original Alcubierre shape function to generate the term $B(rs)$.

Our successful approach allows ourselves to generate a Broeck bottle inside the Natario warp drive with a bottleneck radius also with 10^{-15} meters but with 200 kilometers of inner diameter.200 kilometers are 1000 times the size of the original Broeck bottle and provides a room of space large enough to contains not only a single spaceship but a large number of spaceships and with very low energy density requirements.

This work is organized as follows:

In section 2 we present the definition of the Natario warp drive equation in the original ADM formalism in order to explain in section 12 how the Natario spacetime geometry can receive in its structure the inclusion of the mathematical term $B(rs)$ that generates the Broeck bottle.

In section 3 we explain how the Alcubierre shape function $f(rs)$ can be used to define the Natario shape function counterpart $N(rs)$ using also the warp factor WF and we calculate the derivatives of the Natario shape function in order to obtain in the formulas of the derivatives the terms $1 - f(rs)$ and $f(rs)$ raised to powers of the warp factor WF.

In sections 4 to 10 we demonstrate that these terms cancel each other in the derivatives of the Natario shape function except in the warp bubble radius giving a very low value for the derivatives of the Natario shape function over the bubble radius and in consequence very low values for the negative energy density.

In section 11 we demonstrate that the negative energy density in the equatorial plane of the Natario warp bubble do not vanish and due to the gravitational repulsive behavior of the negative energy density this can provide protection against collisions with the Interstellar Medium IM that unavoidably would occur in a real superluminal spaceflight.

[5] see Appendices G,H and I

Also in section 11 we discuss the Interstellar Medium IM and we arrive at the conclusion that the negative energy density of the warp bubble walls must be higher in modulus than the positive energy density of the IM in order to allow the gravitational repulsion of the IM particles by the warp bubble walls and we introduce an empirical formula to obtain the desirable amount of negative energy density needed to deflect the IM particles multiplying the modulus of the density of the IM by the Machian coefficient of the fraction $\frac{vs}{c}$ which means to say the multiples of the light speed c in the spaceship velocity vs.The negative energy density of the Natario warp drive must exceed this product in modulus.

Collisions between the walls of the warp bubble and the IM particles would certainly occur and although the negative energy density in front of the Natario warp bubble can theoretically protect the ship we borrow in section 12 the idea of Chris Van Den Broeck proposed some years ago in 1999 in order to increase the degree of protection.

Any future development for the Natario warp drive must encompass the more than welcome idea of the Broeck bottle.As a matter of fact we are so confident in the success of the junction of both ideas that we propose the name of the new combined solution as the Natario-Broeck warp drive spacetime.

In this work we use the Geometrized System of Units in which $c = G = 1$ for geometric purposes and the International System of Units SI or MKS for purposes or energy density calculations.

We also make extensive use of footnotes and Appendices and this may be regarded ad an exhaustive reading for experienced readers already familiarized with the ideas of Alcubierre Broeck or Natario but these Appendices and footnotes are mainly destined to students beginners or readers at an introductory level eager to assimilate these ideas.

Although this work was designed to be an independent self-contained and self-consistent work it may be regarded as a companion work to our works in [5],[7] and [8]

2 The equation of the Natario warp drive spacetime metric in the original $3+1$ ADM formalism

The equation of the Natario warp drive spacetime in the original $3+1$ ADM formalism is given by:[6]

$$ds^2 = (1 - X_{rs}X^{rs} - X_{\theta}X^{\theta})dt^2 + 2(X_{rs}drs + X_{\theta}d\theta)dt - drs^2 - rs^2 d\theta^2 \tag{3}$$

Or by:

$$ds^2 = dt^2 - [(drs - X^{rs}dt)^2 + (rs^2)(d\theta - X^{\theta}dt)^2] \tag{4}$$

The equation of the Natario vector nX(pg 2 and 5 in [2]) is given by:

$$nX = X^{rs}drs + X^{\theta}rsd\theta \tag{5}$$

With the contravariant shift vector components X^{rs} and X^{θ} given by:(see pg 5 in [2])[7]

$$X^{rs} = 2v_s n(rs)\cos\theta \tag{6}$$

$$X^{\theta} = -v_s(2n(rs) + (rs)n'(rs))\sin\theta \tag{7}$$

The covariant shift vector components X_{rs} and X_{θ} are given by:

$$X_{rs} = X^{rs} = 2v_s n(rs)\cos\theta \tag{8}$$

$$X_{\theta} = rs^2 X^{\theta} = -rs^2 v_s(2n(rs) + (rs)n'(rs))\sin\theta \tag{9}$$

Considering a valid $n(rs)$ as a Natario shape function being $n(rs) = \frac{1}{2}$ for large rs(outside the warp bubble) and $n(rs) = 0$ for small rs(inside the warp bubble) while being $0 < n(rs) < \frac{1}{2}$ in the walls of the warp bubble also known as the Natario warped region(pg 5 in [2]):

We can see that the Natario warp drive equation given above satisfies the Natario requirements for a warp bubble defined by:

any Natario vector nX generates a warp drive spacetime if $nX = 0$ and $X = vs = 0$ for a small value of rs defined by Natario as the interior of the warp bubble and $nX = vs(t)dx$ with $X = vs$ for a large value of rs defined by Natario as the exterior of the warp bubble with $vs(t)$ being the speed of the warp bubble.(pg 4 in [2])

Natario in its warp drive uses the spherical coordinates rs and θ.In order to simplify our analysis we consider motion in the $x - axis$ or the equatorial plane rs where $\theta = 0$ $\sin(\theta) = 0$ and $\cos(\theta) = 1$.(see pgs 4,5 and 6 in [2]). In a $1+1$ spacetime the equatorial plane we get¿:

$$ds^2 = (1 - X_{rs}X^{rs})dt^2 + 2(X_{rs}drs)dt - drs^2 \tag{10}$$

[6] see Appendix A for details
[7] see also Appendix B for details

But since $X_{rs} = X^{rs}$ the equation can be written as given below:

$$ds^2 = (1 - X_{rs}X^{rs})dt^2 + 2(X_{rs}drs)dt - drs^2 \tag{11}$$

$$ds^2 = (1 - [X^{rs}]^2)dt^2 + 2(X^{rs}drs)dt - drs^2 \tag{12}$$

$$ds^2 = dt^2 - [(drs - X^{rs}dt)^2] \tag{13}$$

3 The Natario warp drive continuous shape function

Introducing here $f(rs)$ as the Alcubierre shape function that defines the Alcubierre warp drive spacetime we can construct the Natario shape function $N(rs)$ that defines the Natario warp drive spacetime using its Alcubierre counterpart.Below is presented the equation of the Alcubierre shape function.[8].

$$f(rs) = \frac{1}{2}[1 - tanh[@(rs - R)] \tag{14}$$

$$rs = \sqrt{(x - xs)^2 + y^2 + z^2} \tag{15}$$

According with Alcubierre any function $f(rs)$ that gives 1 inside the bubble and 0 outside the bubble while being $1 > f(rs) > 0$ in the Alcubierre warped region is a valid shape function for the Alcubierre warp drive.(see eqs 6 and 7 pg 4 in [1] or top of pg 4 in [2]). In the Alcubierre shape function xs is the center of the warp bubble where the ship resides. R is the radius of the warp bubble and @ is the Alcubierre parameter related to the thickness.According to Alcubierre these can have arbitrary values.We outline here the fact that according to pg 4 in [1] the parameter @ can have arbitrary values.rs is the path of the so-called Eulerian observer that starts at the center of the bubble $xs = rs = 0$ and ends up outside the warp bubble $rs > R$.

The square derivative of the Alcubierre shape function is given by:

$$f'(rs)^2 = \frac{1}{4}[\frac{@^2}{cosh^4[@(rs - R)]}] \tag{16}$$

According with Natario(pg 5 in [2]) any function that gives 0 inside the bubble and $\frac{1}{2}$ outside the bubble while being $0 < N(rs) < \frac{1}{2}$ in the Natario warped region is a valid shape function for the Natario warp drive. The Natario warp drive continuous shape function can be defined by:

$$N(rs) = [\frac{1}{2}][1 - f(rs)^{WF}]^{WF} \tag{17}$$

This shape function gives the result of $N(rs) = 0$ inside the warp bubble and $N(rs) = \frac{1}{2}$ outside the warp bubble while being $0 < N(rs) < \frac{1}{2}$ in the Natario warped region. Note that the Alcubierre shape function is being used to define its Natario shape function counterpart.For the Natario shape function introduced above it is easy to figure out when $f(rs) = 1$(interior of the Alcubierre bubble) then $N(rs) = 0$(interior of the Natario bubble) and when $f(rs) = 0$(exterior of the Alcubierre bubble)then $N(rs) = \frac{1}{2}$(exterior of the Natario bubble).

The derivative square of the Natario shape function is :

$$N'(rs)^2 = [\frac{1}{4}]WF^4[1 - f(rs)^{WF}]^{2(WF-1)}[f(rs)^{2(WF-1)}]f'(rs)^2 \tag{18}$$

The term WF in the Natario shape function is dimensionless too:it is the warp factor.It is important to outline that the warp factor $WF >> |R|$ is much greater than the modulus of the bubble radius.Note that the square derivative of the Alcubierre shape function appears in the expression of the square derivative of the Natario shape function.

[8] $tanh[@(rs + R)] = 1$,$tanh(@R) = 1$ for very high values of the Alcubierre thickness parameter $@ >> |R|$

- Numerical plot for the Alcubierre and Natario shape functions with $@ = 50000$ bubble radius $R = 100$ meters and warp factor with a value $WF = 200$

rs	$f(rs)$	$N(rs)$	$f'(rs)^2$	$N'(rs)^2$
$9,99970000000E+001$	1	0	$2,650396620740E-251$	0
$9,99980000000E+001$	1	0	$1,915169647489E-164$	0
$9,99990000000E+001$	1	0	$1,383896564748E-077$	0
$1,00000000000E+002$	$0,5$	$0,5$	$6,250000000000E+008$	$3,872591914849E-103$
$1,00001000000E+002$	0	$0,5$	$1,383896486082E-077$	0
$1,00002000000E+002$	0	$0,5$	$1,915169538624E-164$	0
$1,00003000000E+002$	0	$0,5$	$2,650396470082E-251$	0

According with the numerical plot above when $@ = 50000$ the square derivative of the Alcubierre shape function is zero[9] from the center of the bubble until $99,996$ meters.At $99,997$ meters the square derivative of the Alcubierre shape function is $2,65 \times 10^{-251}$ and starts to increase reaching the maximum value of $6,25 \times 10^{8}$ at 100 meters from the center of the bubble precisely in the bubble radius decreasing again to the minimum value of $2,65 \times 10^{-251}$ at $100,003$ meters from the center of the bubble.At 100.004 meters from the center of the bubble the square derivative of the Alcubierre shape function is again zero.Note that with respect to the distance of 100 meters from the center of the bubble exactly the bubble radius the powers of the square derivative of the Alcubierre shape function are diametrically symmetrically opposed.We have the values of 10^{-77} at $99,999$ meters and at $100,001$ meters.We have the value of 10^{-164} at $99,998$ meters and at $100,002$ meters.So the thickness of the warped region is limited or defined by the square derivatives of the shape function when these are different than zero.In the case of $@ = 50000$ the warped region starts at $99,997$ meters and ends up at $100,003$ meters.The thickness of the warped region is then $0,006$ meters.

Note that inside the bubble the Alcubierre shape function possesses the value of 1 and the Natario shape function possesses the value of 0 and outside the bubble the Alcubierre shape function possesses the value of 0 and the Natario shape function possesses the value of $\frac{1}{2}$ as requested.

Also while the square derivative of the Alcubierre shape function is not zero inside and outside the bubble however at the neighborhoods of the bubble radius and possesses the maximum value exactly at the bubble radius the square derivative of the Natario shape function is always zero inside and outside the bubble and possesses also a maximum value at the bubble radius however this value is extremely small when compared to its Alcubierre counterpart.

[9] not exactly zero but possesses extremely low values and we are limited by the floating-point precision of our software

- Numerical plot for the Alcubierre and Natario shape functions with $@ = 75000$ bubble radius $R = 100$ meters and warp factor with a value $WF = 200$

rs	$f(rs)$	$N(rs)$	$f'(rs)^2$	$N'(rs)^2$
$9,99980000000E+001$	1	0	$5,963392481410E-251$	0
$9,99990000000E+001$	1	0	$1,158345097767E-120$	0
$1,00000000000E+002$	$0,5$	$0,5$	$1,406250000000E+009$	$8,713331808411E-103$
$1,00001000000E+002$	0	$0,5$	$1,158344999000E-120$	0
$1,00002000000E+002$	0	$0,5$	$5,963391972940E-251$	0

According with the numerical plot above when $@ = 75000$ the square derivative of the Alcubierre shape function is zero from the center of the bubble until $99,997$ meters.At $99,998$ meters the square derivative of the Alcubierre shape function is $5,96 \times 10^{-251}$ and starts to increase reaching the maximum value of $1,4 \times 10^{9}$ at 100 meters from the center of the bubble precisely in the bubble radius decreasing again to the minimum value of $5,96 \times 10^{-251}$ at $100,002$ meters from the center of the bubble.At 100.003 meters from the center of the bubble the square derivative of the Alcubierre shape function is again zero.Note that with respect to the distance of 100 meters from the center of the bubble exactly the bubble radius the powers of the square derivative of the Alcubierre shape function are diametrically symmetrically opposed.We have the values of 10^{-120} at $99,999$ meters and at $100,001$ meters.So the thickness of the warped region is limited or defined by the square derivatives of the shape function when these are different than zero.In the case of $@ = 75000$ the warped region starts at $99,998$ meters and ends up at $100,002$ meters.The thickness of the warped region is then $0,004$ meters.

Note that inside the bubble the Alcubierre shape function possesses the value of 1 and the Natario shape function possesses the value of 0 and outside the bubble the Alcubierre shape function possesses the value of 0 and the Natario shape function possesses the value of $\frac{1}{2}$ as requested.

Also while the square derivative of the Alcubierre shape function is not zero inside and outside the bubble however at the neighborhoods of the bubble radius and possesses the maximum value exactly at the bubble radius the square derivative of the Natario shape function is always zero inside and outside the bubble and possesses also a maximum value at the bubble radius however this value is extremely small when compared to its Alcubierre counterpart.

The previous plots demonstrate the important role of the thickness parameter @ in the warp bubble geometry wether in both Alcubierre or Natario warp drive spacetimes.For a bubble of 100 meters radius $R = 100$ the regions where $1 > f(rs) > 0$(Alcubierre warped region) and $0 < N(rs) < \frac{1}{2}$(Natario warped region) becomes thicker or thinner as @ becomes higher.In the case of $@ = 50000$ the warped region starts at $99,997$ meters and ends up at $100,003$ meters.The thickness of the warped region is then $0,006$ meters and in the case of $@ = 75000$ the warped region starts at $99,998$ meters and ends up at $100,002$ meters.The thickness of the warped region is then $0,004$ meters.

Then the geometric position where both Alcubierre and Natario warped regions begins with respect to R the bubble radius is $rs = R - \epsilon < R$ and the geometric position where both Alcubierre and Natario warped regions ends with respect to R the bubble radius is $rs = R + \epsilon > R$.The thickness of the warp bubble is then $2 \times \epsilon$.As large as @ becomes as smaller ϵ becomes too.

Note from the plots of the previous pages that we really have two warped regions:

- 1)-The geometrized warped region where $1 > f(rs) > 0$(Alcubierre warped region) and $0 < N(rs) < \frac{1}{2}$ (Natario warped region).The geometrized warped region lies precisely in the bubble radius.[10]

- 2)-The energized warped region where the derivative squares of both Alcubierre and Natario shape functions are not zero.

The parameter @ affects both energized warped regions wether in Alcubierre or Natario cases but is more visible for the Alcubierre shape function because the warp factor WF in the Natario shape functions squeezes the energized warped region in a region of very small thickness centered in the bubble radius.

The negative energy density for the Natario warp drive is given by(see pg 5 in [2])

$$\rho = T_{\mu\nu}u^{\mu}u^{\nu} = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[3(N'(rs))^2\cos^2\theta + \left(N'(rs) + \frac{rs}{2}N''(rs)\right)^2\sin^2\theta\right] \tag{19}$$

Converting from the Geometrized System of Units to the International System we should expect for the following expression[11]:

$$\rho = -\frac{c^2}{G}\frac{vs^2}{8\pi}\left[3(N'(rs))^2\cos^2\theta + \left(N'(rs) + \frac{rs}{2}N''(rs)\right)^2\sin^2\theta\right]. \tag{20}$$

Rewriting the Natario negative energy density in cartezian coordinates we should expect for[12]:

$$\rho = T_{\mu\nu}u^{\mu}u^{\nu} = -\frac{c^2}{G}\frac{v_s^2}{8\pi}\left[3(N'(rs))^2(\frac{x}{rs})^2 + \left(N'(rs) + \frac{rs}{2}N''(rs)\right)^2(\frac{y}{rs})^2\right] \tag{21}$$

Considering as a simplified case the equatorial plane($1+1$ dimensional spacetime with $rs = x - xs$,$y = 0$ and center of the bubble $xs = 0$) we have:

$$\rho = T_{\mu\nu}u^{\mu}u^{\nu} = -\frac{c^2}{G}\frac{v_s^2}{8\pi}\left[3(N'(rs))^2\right] \tag{22}$$

Note that in the above expressions for the negative energy density the warp drive speed vs appears raised to a power of 2 and it is being multiplied by the square derivative of the shape function. Considering our Natario warp drive moving with $vs = 200$ which means to say 200 times light speed in order to make a round trip from Earth to a nearby star at 20 light-years away in a reasonable amount of time(in months not in years) we would get in the expression of the negative energy the factor $c^2 = (3\times10^8)^2 = 9\times10^{16}$ being divided by $6,67\times10^{-11}$ giving $1,35\times10^{27}$ and this is multiplied by $(6\times10^{10})^2 = 36\times10^{20}$ coming from the term $vs = 200$ giving $1,35\times10^{27}\times36\times10^{20} = 1,35\times10^{27}\times3,6\times10^{21} = 4,86\times10^{48}$!!!

A number with 48 zeros!!!The planet Earth have a mass[13] of about $6\times10^{24}kg$

[10]In the bubble radius the presented value for the Natario shape function in the numerical plots of the previous pages is 0,5 but actually is a value between $0 < N(rs) < \frac{1}{2}$ but very close to 0,5.Again we are limited by the floating-point precision of our software

[11]see Appendix D

[12]see Appendix C

[13]see Wikipedia:The free Encyclopedia

This term is 1.000.000.000.000.000.000.000.000 times bigger in magnitude than the mass of the planet Earth!!!or better:The amount of negative energy density needed to sustain a warp bubble at a speed of 200 times faster than light requires the magnitude of the masses of 1.000.000.000.000.000.000.000.000 planet Earths!!!

Note that if the negative energy density is proportional to 10^{48} this would render the warp drive impossible but fortunately the term 10^{48} is being multiplied the square derivative of the shape function and in the Natario case the square derivative of the shape function possesses values of 10^{-102} or 10^{-103} completely obliterating the factor 10^{48} making the warp drive negative energy density more "affordable" because $10^{48} \times 10^{-102} = 10^{-54} \frac{Joules}{meters^3}$ a very low and affordable negative energy density.So in order to get a physically feasible Natario warp drive the square derivative of the Natario shape function must obliterate the factor 10^{48} and fortunately this is really happening with our chosen shape function .

Now we need to explain how and why the warp factor WF in the Natario shape functions squeezes the energized warped region in a region of very small thickness centered in the bubble radius.

The Alcubierre shape function and its derivative square are given by:

$$f(rs) = \frac{1}{2}[1 - tanh[@(rs - R)] \tag{23}$$

$$f'(rs)^2 = \frac{1}{4}[\frac{@^2}{cosh^4[@(rs - R)]}] \tag{24}$$

The Natario shape function and its derivative square are given by:

$$N(rs) = [\frac{1}{2}][1 - f(rs)^{WF}]^{WF} \tag{25}$$

$$N'(rs)^2 = [\frac{1}{4}]WF^4[1 - f(rs)^{WF}]^{2(WF-1)}[f(rs)^{2(WF-1)}]f'(rs)^2 \tag{26}$$

$$N'(rs)^2 = [\frac{1}{4}]WF^4[1 - f(rs)^{WF}]^{2(WF-1)}[f(rs)^{2(WF-1)}][\frac{1}{4}[\frac{@^2}{cosh^4[@(rs - R)]}]] \tag{27}$$

$$N'(rs)^2 = [\frac{1}{16}]WF^4[1 - f(rs)^{WF}]^{2(WF-1)}[f(rs)^{2(WF-1)}][\frac{@^2}{cosh^4[@(rs - R)]}] \tag{28}$$

The negative energy density in the $1+1$ Natario warp drive spacetime is given by:

$$\rho = T_{\mu\nu}u^\mu u^\nu = -\frac{c^2}{G}\frac{v_s^2}{8\pi}\left[3(N'(rs))^2\right] \tag{29}$$

Examining now the negative energy density in the $1+1$ spacetime from the Natario shape function with warp factors:

$$N'(rs)^2 = [\frac{1}{16}]WF^4[1 - f(rs)^{WF}]^{2(WF-1)}[f(rs)^{2(WF-1)}][\frac{@^2}{cosh^4[@(rs - R)]}] \tag{30}$$

$$\rho = T_{\mu\nu}u^\mu u^\nu = -\frac{c^2}{G}\frac{v_s^2}{8\pi}\left[[\frac{3}{16}]WF^4[1 - f(rs)^{WF}]^{2(WF-1)}[f(rs)^{2(WF-1)}][\frac{@^2}{cosh^4[@(rs - R)]}]\right] \tag{31}$$

The dominant term here is the term resulting from the warp factor which is:

$$[1 - f(rs)^{WF}]^{2(WF-1)}[f(rs)^{2(WF-1)}] \tag{32}$$

This term is composed by two expressions that complementary neutralizes each other giving values of zero inside and outside the bubble squeezing the Natario warped region in a thin layer geometrically placed around the neighborhoods of the bubble radius.

The first expression that neutralizes the square derivative of the Natario shape function inside the bubble is:

$$[1 - f(rs)^{WF}]^{2(WF-1)} \tag{33}$$

And the second expression that neutralizes the square derivative of the Natario shape function outside the bubble is:

$$[f(rs)^{2(WF-1)}] \tag{34}$$

Inside the bubble $f(rs) = 1$ and $[1 - f(rs)^{WF}]^{2(WF-1)} = 0$ resulting in a $N'(rs)^2 = 0$.This is the reason why the Natario shape function with warp factors do not have numerical values for the derivatives inside the bubble.

Outside the bubble $f(rs) = 0$ and $[f(rs)^{2(WF-1)}] = 0$ resulting also in a $N'(rs)^2 = 0$.This is the reason why the Natario shape function with warp factors do not have numerical values for the derivatives outside the bubble.

Inside the bubble $f(rs) = 1$ and $[f(rs)^{2(WF-1)}] = 1$ however $[1 - f(rs)^{WF}]^{2(WF-1)} = 0$ and hence the warp factor product $[1 - f(rs)^{WF}]^{2(WF-1)}[f(rs)^{2(WF-1)}] = 0$.

Outside the bubble $f(rs) = 0$ and $[f(rs)^{2(WF-1)}] = 0$ however $[1 - f(rs)^{WF}]^{2(WF-1)} = 1$ and hence the warp factor product is also $[1 - f(rs)^{WF}]^{2(WF-1)}[f(rs)^{2(WF-1)}] = 0$.

Note that from the statements pointed above when one of the expressions have the value of 1 the other have the value of 0 and vice-versa.This explains how and why each expression complementary neutralizes each other in the regions inside and outside the bubble

In the Alcubierre warped region[14] $1 > f(rs) > 0$.In this region the derivatives of the Natario shape function do not vanish because if $f(rs) < 1$ then $f(rs)^{WF} << 1$ resulting in an $[1 - f(rs)^{WF}]^{2(WF-1)} << 1$ but greater than zero.Consider for example a warp factor $WF = 200$ and an Alcubierre shape function $f(rs) = \frac{1}{2}$ then $f(rs)^{WF} = f(rs)^{200} = \frac{1}{2^{200}}$.Since $2^{200} = 1,6069380442590E + 060$ then $\frac{1}{2^{200}} = 6,2230152778612E - 061$ and $6,2 \times 10^{-61}$ is very small when compared to $\frac{1}{2}$.

.Also if $f(rs) < 1$ then $[f(rs)^{2(WF-1)}] << 1$ too and using the numbers given above then $f(rs)^{398} = \frac{1}{2^{398}}$.Since $2^{398} = 6,4556246952173E + 119$ then $\frac{1}{2^{398}} = 1,5490367659397E - 120$ and $1,5 \times 10^{-120}$ is also very small when compared to $\frac{1}{2}$.

Note that if $[1 - f(rs)^{WF}]^{2(WF-1)} << 1$ and $[f(rs)^{2(WF-1)}] << 1$ then their product $[1 - f(rs)^{WF}]^{2(WF-1)}[f(rs)^{2(WF-1)}] <<<< 1$ resulting in a very low derivative square for the Natario shape function in the Alcubierre warped region and hence in the Natario warped region with both centered geometrically over the bubble radius.

Note also that inside the Alcubierre warped region $1 > f(rs) > 0$ when $f(rs)$ approaches 1 $N'(rs)^2$ approaches 0 due to the factor $[1 - f(rs)^{WF}]^{2(WF-1)}$ and when $f(rs)$ approaches 0 $N'(rs)^2$ approaches 0 again due to the factor $[f(rs)^{2(WF-1)}]$ .Both expressions complementary neutralizes each other giving a very small product and hence a very small square derivative for the Natario shape function.

We will examine the above statement of the expressions that complementary neutralizes each other in the Natario warp drive $1 + 1$ spacetime with details in the section 4.

Now we must analyze the more sophisticated case of the Natario warp drive in a real $3 + 1$ spacetime where the negative energy density in this case is given by the following expressions(pg 5 in [2])[15]:

- 1)-$3 + 1$ spacetime expression for the negative energy density with trigonometric terms:

$$\rho = -\frac{c^2}{G}\frac{vs^2}{8\pi}\left[3(N'(rs))^2\cos^2\theta + \left(N'(rs) + \frac{rs}{2}N''(rs)\right)^2\sin^2\theta\right]. \tag{35}$$

- 2)-$3 + 1$ spacetime expression for the negative energy density with cartezian coordinates[16]:

$$\rho = T_{\mu\nu}u^{\mu}u^{\nu} = -\frac{c^2}{G}\frac{v_s^2}{8\pi}\left[3(N'(rs))^2(\frac{x}{rs})^2 + \left(N'(rs) + \frac{rs}{2}N''(rs)\right)^2(\frac{y}{rs})^2\right] \tag{36}$$

[14] Remember that the Natario warped region is defined in function of its Alcubierre counterpart

[15] see Appendix D

[16] see Appendix C

Working with the expanded trigonometric $3+1$ spacetime expression for the negative energy density we have:

$$\rho_{3+1} = -\frac{c^2}{G}\frac{vs^2}{8\pi}\left[3(N'(rs))^2\cos^2\theta\right] - \frac{c^2}{G}\frac{vs^2}{8\pi}\left[\left(N'(rs) + \frac{rs}{2}N''(rs)\right)^2 \sin^2\theta\right] \tag{37}$$

$$\rho_{3+1} = \rho_1 + \rho_2 \tag{38}$$

$$\rho_1 = -\frac{c^2}{G}\frac{vs^2}{8\pi}\left[3(N'(rs))^2\cos^2\theta\right] \tag{39}$$

$$\rho_2 = -\frac{c^2}{G}\frac{vs^2}{8\pi}\left[\left(N'(rs) + \frac{rs}{2}N''(rs)\right)^2 \sin^2\theta\right] \tag{40}$$

Comparing the above expressions with the negative energy density in the $1+1$ spacetime :

$$\rho_{1+1} = -\frac{c^2}{G}\frac{v_s^2}{8\pi}\left[3(N'(rs))^2\right] \tag{41}$$

We can see that the term in ρ_1 almost matches the term in the $1+1$ spacetime except for the trigonometric term in $\cos^2\theta$ and this term produces a very low derivative square for the Natario shape function of about 10^{-103} and this will be seen in section 3.So the term ρ_2 is the term that really accounts for the negative energy density in the $3+1$ spacetime.

The dominant expression in ρ_2 is:

$$\left[\left(N'(rs) + \frac{rs}{2}N''(rs)\right)^2\right] \tag{42}$$

The expansion of the square in the binomial expression gives:

$$\left[\left(N'(rs) + \frac{rs}{2}N''(rs)\right)^2\right] = N'(rs)^2 + 2N'(rs)\frac{rs}{2}N''(rs) + N''(rs)^2 \tag{43}$$

$$\left[\left(N'(rs) + \frac{rs}{2}N''(rs)\right)^2\right] = N'(rs)^2 + N'(rs)rsN''(rs) + N''(rs)^2 \tag{44}$$

$$\left[\left(N'(rs) + \frac{rs}{2}N''(rs)\right)^2\right] = N'(rs)^2 + rsN'(rs)N''(rs) + N''(rs)^2 \tag{45}$$

Since the derivative of second order of the Natario shape function $N''(rs)$ is a lengthly expression with many algebraic terms then its square $N''(rs)^2$ results in an even more complicated expression with even more algebraic terms.And the product of both the first and second order derivatives $N'(rs)N''(rs)$ also results in a lengthly expression.Then in order to avoid algebraic complications we must work numerically with the new dominant term which is:

$$N'(rs) + \frac{rs}{2}N''(rs) \tag{46}$$

Raising to the square only the final numerically evaluated result.

The new dominant term in the expression for the negative energy density of the Natario warp drive in a $3+1$ spacetime is:

$$N'(rs) + \frac{rs}{2}N''(rs) \tag{47}$$

The first order derivative of the Natario shape function is given by:

$$N'(rs) = -[\frac{1}{2}]WF^2[1 - f(rs)^{WF}]^{(WF-1)}[f(rs)^{(WF-1)}]f'(rs) \tag{48}$$

With $f'(rs)$ being the first order derivative of the Alcubierre shape function

$$f'(rs) = -\frac{1}{2}[\frac{@}{cosh^2[@(rs-R)]}] \tag{49}$$

The second order derivative of the Natario shape function is given by the lengthly expression:

$$N''(rs) = [\frac{1}{2}]WF^3(WF-1)[1 - f(rs)^{WF}]^{(WF-2)}[f(rs)^{2(WF-1)}]f'(rs)^2 \tag{50}$$

$$-[\frac{1}{2}]WF^2[1 - f(rs)^{WF}]^{(WF-1)}(WF-1)[f(rs)^{(WF-2)}]f'(rs)^2 \tag{51}$$

$$-[\frac{1}{2}]WF^2[1 - f(rs)^{WF}]^{(WF-1)}[f(rs)^{(WF-1)}]f''(rs) \tag{52}$$

With $f''(rs)$ being the second order derivative of the Alcubierre shape function

$$f''(rs) = [\frac{(@^2)sinh[@(rs-R)}{cosh^3[@(rs-R)]}] \tag{53}$$

From above we can see that the square of the second order derivative of the Natario shape function $N''(rs)^2$ would result in a very algebraic complicated expression.In order to simplify our study we decompose the second order derivative of the Natario shape function in separated algebraic expressions as shown below

$$N''(rs) = A + B + C \tag{54}$$

With the expressions for A B and C given respectively by:

$$A = [\frac{1}{2}]WF^3(WF-1)[1 - f(rs)^{WF}]^{(WF-2)}[f(rs)^{2(WF-1)}]f'(rs)^2 \tag{55}$$

$$B = -[\frac{1}{2}]WF^2[1 - f(rs)^{WF}]^{(WF-1)}(WF-1)[f(rs)^{(WF-2)}]f'(rs)^2 \tag{56}$$

$$C = -[\frac{1}{2}]WF^2[1 - f(rs)^{WF}]^{(WF-1)}[f(rs)^{(WF-1)}]f''(rs) \tag{57}$$

Then the expressions that really accounts for a numerical evaluation of the new dominant term in the equation for the negative energy density of the Natario warp drive in a $3+1$ spacetime which is:

$$N'(rs) + \frac{rs}{2}N''(rs) = N'(rs) + \frac{rs}{2}(A+B+C) \tag{58}$$

Are the following ones:

$$N'(rs) = -[\frac{1}{2}]WF^2[1-f(rs)^{WF}]^{(WF-1)}[f(rs)^{(WF-1)}]f'(rs) \tag{59}$$

$$A = [\frac{1}{2}]WF^3(WF-1)[1-f(rs)^{WF}]^{(WF-2)}[f(rs)^{2(WF-1)}]f'(rs)^2 \tag{60}$$

$$B = -[\frac{1}{2}]WF^2[1-f(rs)^{WF}]^{(WF-1)}(WF-1)[f(rs)^{(WF-2)}]f'(rs)^2 \tag{61}$$

$$C = -[\frac{1}{2}]WF^2[1-f(rs)^{WF}]^{(WF-1)}[f(rs)^{(WF-1)}]f''(rs) \tag{62}$$

Since

$$N''(rs) = A+B+C \tag{63}$$

With $f'(rs)$ being the first order derivative of the Alcubierre shape function

$$f'(rs) = -\frac{1}{2}[\frac{@}{cosh^2[@(rs-R)]}] \tag{64}$$

And $f''(rs)$ being the second order derivative of the Alcubierre shape function

$$f''(rs) = [\frac{(@^2)sinh[@(rs-R)}{cosh^3[@(rs-R)]}] \tag{65}$$

Note that the most meaningful term $f(rs)$ that occurs in all these expressions raised to powers of the warp factor WF is the Alcubierre shape function.

$$f(rs) = \frac{1}{2}[1-tanh[@(rs-R)] \tag{66}$$

And we recall that the Natario shape function defined in function of its Alcubierre counterpart is given by:

$$N(rs) = [\frac{1}{2}][1-f(rs)^{WF}]^{WF} \tag{67}$$

Evaluating the first order derivative of Natario shape function which is given by:

$$N'(rs) = -[\frac{1}{2}]WF^2[1 - f(rs)^{WF}]^{(WF-1)}[f(rs)^{(WF-1)}]f'(rs) \tag{68}$$

With $f'(rs)$ being the first order derivative of the Alcubierre shape function

$$f'(rs) = -\frac{1}{2}[\frac{@}{cosh^2[@(rs - R)]}] \tag{69}$$

The dominant term here is the term resulting from the warp factor which is:

$$[1 - f(rs)^{WF}]^{(WF-1)}[f(rs)^{(WF-1)}] \tag{70}$$

This term is composed by two expressions that complementary neutralizes each other giving values of zero inside and outside the bubble squeezing the Natario warped region in a thin layer geometrically placed around the neighborhoods of the bubble radius.

The first expression that neutralizes the derivative of the Natario shape function inside the bubble is:

$$[1 - f(rs)^{WF}]^{(WF-1)} \tag{71}$$

And the second expression that neutralizes the derivative of the Natario shape function outside the bubble is:

$$[f(rs)^{(WF-1)}] \tag{72}$$

Inside the bubble $f(rs) = 1$ and $[1 - f(rs)^{WF}]^{(WF-1)} = 0$ resulting in a $N'(rs) = 0$.This is the reason why the Natario shape function with warp factors do not have numerical values for the derivatives inside the bubble.

Outside the bubble $f(rs) = 0$ and $[f(rs)^{(WF-1)}] = 0$ resulting also in a $N'(rs) = 0$.This is the reason why the Natario shape function with warp factors do not have numerical values for the derivatives outside the bubble.

Inside the bubble $f(rs) = 1$ and $[f(rs)^{(WF-1)}] = 1$ however $[1 - f(rs)^{WF}]^{(WF-1)} = 0$ and hence the warp factor product $[1 - f(rs)^{WF}]^{(WF-1)}[f(rs)^{(WF-1)}] = 0$.

Outside the bubble $f(rs) = 0$ and $[f(rs)^{(WF-1)}] = 0$ however $[1 - f(rs)^{WF}]^{(WF-1)} = 1$ and hence the warp factor product is also $[1 - f(rs)^{WF}]^{(WF-1)}[f(rs)^{(WF-1)}] = 0$.

Note that from the statements pointed above when one of the expressions have the value of 1 the other have the value of 0 and vice-versa.This explains how and why each expression complementary neutralizes each other in the regions inside and outside the bubble

Note that if $[1 - f(rs)^{WF}]^{(WF-1)} << 1$ and $[f(rs)^{(WF-1)}] << 1$ then their product $[1-f(rs)^{WF}]^{(WF-1)}[f(rs)^{(WF-1)}] <<<< 1$ resulting in a very low derivative for the Natario shape function in the Alcubierre warped region and hence in the Natario warped region with both centered geometrically over the bubble radius.

We will examine the above statement of the expressions that complementary neutralizes each other in the first order derivative of Natario shape function with details in the section 5.

Evaluating now the term A in the second order derivative of the Natario shape function given by:

$$A = [\frac{1}{2}]WF^3(WF-1)[1-f(rs)^{WF}]^{(WF-2)}[f(rs)^{2(WF-1)}]f'(rs)^2 \qquad (73)$$

With $f'(rs)^2$ being the derivative square of the Alcubierre shape function which is :

$$f'(rs)^2 = \frac{1}{4}[\frac{@^2}{cosh^4[@(rs-R)]}] \qquad (74)$$

The dominant term here is the term resulting from the warp factor which is:

$$[1-f(rs)^{WF}]^{(WF-2)}[f(rs)^{2(WF-1)}] \qquad (75)$$

This term is composed by two expressions that complementary neutralizes each other giving values of zero inside and outside the bubble squeezing the Natario warped region in a thin layer geometrically placed around the neighborhoods of the bubble radius.

The first expression that neutralizes the term A in the second order derivative of the Natario shape function inside the bubble is:

$$[1-f(rs)^{WF}]^{(WF-2)} \qquad (76)$$

And the second expression that neutralizes the term A in the second order derivative of the Natario shape function outside the bubble is:

$$[f(rs)^{2(WF-1)}] \qquad (77)$$

Inside the bubble $f(rs) = 1$ and $[1-f(rs)^{WF}]^{(WF-2)} = 0$ resulting in a $A = 0$.

Outside the bubble $f(rs) = 0$ and $[f(rs)^{2(WF-1)}] = 0$ resulting also in a $A = 0$.

Inside the bubble $f(rs) = 1$ and $[f(rs)^{2(WF-1)}] = 1$ however $[1-f(rs)^{WF}]^{(WF-2)} = 0$ and hence the warp factor product $[1-f(rs)^{WF}]^{(WF-2)}[f(rs)^{2(WF-1)}] = 0$.

Outside the bubble $f(rs) = 0$ and $[f(rs)^{2WF-1)}] = 0$ however $[1-f(rs)^{WF}]^{(WF-2)} = 1$ and hence the warp factor product is also $[1-f(rs)^{WF}]^{(WF-2)}[f(rs)^{2(WF-1)}] = 0$.

Note that from the statements pointed above when one of the expressions have the value of 1 the other have the value of 0 and vice-versa.This explains how and why each expression complementary neutralizes each other in the regions inside and outside the bubble

Note that if $[1-f(rs)^{WF}]^{(WF-2)} << 1$ and $[f(rs)^{2(WF-1)}] << 1$ then their product $[1-f(rs)^{WF}]^{(WF-2)}[f(rs)^{2(WF-1)}] <<<< 1$ resulting in a very low value for the term A in the Alcubierre warped region and hence in the Natario warped region with both centered geometrically over the bubble radius.

We will examine the above statement of the expressions that complementary neutralizes each other in the term A of the second order derivative of Natario shape function with details in the section 6.

Evaluating now the term B in the second order derivative of the Natario shape function given by:

$$B = -[\frac{1}{2}]WF^2[1 - f(rs)^{WF}]^{(WF-1)}(WF-1)[f(rs)^{(WF-2)}]f'(rs)^2 \tag{78}$$

With $f'(rs)^2$ being the derivative square of the Alcubierre shape function which is :

$$f'(rs)^2 = \frac{1}{4}[\frac{@^2}{cosh^4[@(rs - R)]}] \tag{79}$$

The dominant term here is the term resulting from the warp factor which is:

$$[1 - f(rs)^{WF}]^{(WF-1)}[f(rs)^{(WF-2)}] \tag{80}$$

This term is composed by two expressions that complementary neutralizes each other giving values of zero inside and outside the bubble squeezing the Natario warped region in a thin layer geometrically placed around the neighborhoods of the bubble radius.

The first expression that neutralizes the term B in the second order derivative of the Natario shape function inside the bubble is:

$$[1 - f(rs)^{WF}]^{(WF-1)} \tag{81}$$

And the second expression that neutralizes the term B in the second order derivative of the Natario shape function outside the bubble is:

$$[f(rs)^{(WF-2)}] \tag{82}$$

Inside the bubble $f(rs) = 1$ and $[1 - f(rs)^{WF}]^{(WF-1)} = 0$ resulting in a $B = 0$.

Outside the bubble $f(rs) = 0$ and $[f(rs)^{(WF-2)}] = 0$ resulting also in a $B = 0$.

Inside the bubble $f(rs) = 1$ and $[f(rs)^{(WF-2)}] = 1$ however $[1 - f(rs)^{WF}]^{(WF-1)} = 0$ and hence the warp factor product $[1 - f(rs)^{WF}]^{(WF-1)}[f(rs)^{(WF-2)}] = 0$.

Outside the bubble $f(rs) = 0$ and $[f(rs)^{WF-2)}] = 0$ however $[1 - f(rs)^{WF}]^{(WF-1)} = 1$ and hence the warp factor product is also $[1 - f(rs)^{WF}]^{(WF-1)}[f(rs)^{(WF-2)}] = 0$.

Note that from the statements pointed above when one of the expressions have the value of 1 the other have the value of 0 and vice-versa.This explains how and why each expression complementary neutralizes each other in the regions inside and outside the bubble

Note that if $[1 - f(rs)^{WF}]^{(WF-1)} << 1$ and $[f(rs)^{(WF-2)}] << 1$ then their product $[1 - f(rs)^{WF}]^{(WF-1)}[f(rs)^{(WF-2)}] <<<< 1$ resulting in a very low value for the term B in the Alcubierre warped region and hence in the Natario warped region with both centered geometrically over the bubble radius.

We will examine the above statement of the expressions that complementary neutralizes each other in the term B of the second order derivative of Natario shape function with details in the section 7.

Evaluating now the term C in the second order derivative of the Natario shape function given by:

$$C = -[\frac{1}{2}]WF^2[1 - f(rs)^{WF}]^{(WF-1)}[f(rs)^{(WF-1)}]f''(rs) \tag{83}$$

With $f''(rs)$ being the second order derivative of the Alcubierre shape function

$$f''(rs) = [\frac{(@^2)sinh[@(rs-R)}{cosh^3[@(rs-R)]}] \tag{84}$$

The dominant term here is the term resulting from the warp factor which is:

$$[1 - f(rs)^{WF}]^{(WF-1)}[f(rs)^{(WF-1)}] \tag{85}$$

This term is composed by two expressions that complementary neutralizes each other giving values of zero inside and outside the bubble squeezing the Natario warped region in a thin layer geometrically placed around the neighborhoods of the bubble radius.

The first expression that neutralizes the term C in the second order derivative of the Natario shape function inside the bubble is:

$$[1 - f(rs)^{WF}]^{(WF-1)} \tag{86}$$

And the second expression that neutralizes the term C in the second order derivative of the Natario shape function outside the bubble is:

$$[f(rs)^{(WF-1)}] \tag{87}$$

Inside the bubble $f(rs) = 1$ and $[1 - f(rs)^{WF}]^{(WF-1)} = 0$ resulting in a $C = 0$.

Outside the bubble $f(rs) = 0$ and $[f(rs)^{(WF-1)}] = 0$ resulting also in a $C = 0$.

Inside the bubble $f(rs) = 1$ and $[f(rs)^{(WF-1)}] = 1$ however $[1 - f(rs)^{WF}]^{(WF-1)} = 0$ and hence the warp factor product $[1 - f(rs)^{WF}]^{(WF-1)}[f(rs)^{(WF-1)}] = 0$.

Outside the bubble $f(rs) = 0$ and $[f(rs)^{(WF-1)}] = 0$ however $[1 - f(rs)^{WF}]^{(WF-1)} = 1$ and hence the warp factor product is also $[1 - f(rs)^{WF}]^{(WF-1)}[f(rs)^{(WF-1)}] = 0$.

Note that from the statements pointed above when one of the expressions have the value of 1 the other have the value of 0 and vice-versa.This explains how and why each expression complementary neutralizes each other in the regions inside and outside the bubble

Note that if $[1 - f(rs)^{WF}]^{(WF-1)} << 1$ and $[f(rs)^{(WF-1)}] << 1$ then their product $[1 - f(rs)^{WF}]^{(WF-1)}[f(rs)^{(WF-1)}] <<<< 1$ resulting in a very low value for the term C in the Alcubierre warped region and hence in the Natario warped region with both centered geometrically over the bubble radius.

We will examine the above statement of the expressions that complementary neutralizes each other in the term C of the second order derivative of Natario shape function with details in the section 8.

4 The expressions that complementary neutralizes each other in a Natario warp drive $1+1$ spacetime

The Alcubierre shape function is given by:

$$f(rs) = \frac{1}{2}[1 - tanh[@(rs - R)] \tag{88}$$

And the Natario shape function is given by:

$$N(rs) = [\frac{1}{2}][1 - f(rs)^{WF}]^{WF} \tag{89}$$

- Numerical plot for the Alcubierre and Natario shape functions with $@ = 50000$ bubble radius $R = 100$ meters and warp factor with a value $WF = 200$

rs	$f(rs)$	$N(rs)$
$9,999940000000E+01$	$1,000000000000E+00$	$0,000000000000E+00$
$9,999950000000E+01$	$1,000000000000E+00$	$0,000000000000E+00$
$9,999960000000E+01$	$1,000000000000E+00$	$0,000000000000E+00$
$9,999970000000E+01$	$9,999999999999E-01$	$0,000000000000E+00$
$9,999980000000E+01$	$9,999999979388E-01$	$0,000000000000E+00$
$9,999990000000E+01$	$9,999546021312E-01$	$0,000000000000E+00$
$1,000000000000E+02$	$5,000000000000E-01$	$5,000000000000E-01$
$1,000001000000E+02$	$4,539786855834E-05$	$5,000000000000E-01$
$1,000002000000E+02$	$2,061153636657E-09$	$5,000000000000E-01$
$1,000003000000E+02$	$9,359180097590E-14$	$5,000000000000E-01$
$1,000004000000E+02$	$0,000000000000E+00$	$5,000000000000E-01$
$1,000005000000E+02$	$0,000000000000E+00$	$5,000000000000E-01$
$1,000006000000E+02$	$0,000000000000E+00$	$5,000000000000E-01$

According with the numerical plot above when $@ = 50000$ then from 0 to $99,996$ meters from the center of the bubble we have the region inside the bubble where the Alcubierre shape function $f(rs) = 1$ and the Natario shape function $N(rs) = 0$. At $99,997$ meters from the center of the bubble the Alcubierre shape function starts to decrease and we enter in the Alcubierre geometrized warped region $1 > f(rs) > 0$.The Alcubierre warped region ends at $100,003$ meters from the center of the bubble.At $100,004$ meters from the center of the bubble we reaches the region outside the bubble where the Alcubierre shape function $f(rs) = 0$ and the Natario shape function $N(rs) = \frac{1}{2}$. The thickness or the width of the Alcubierre geometrized warped region is then $0,006$ meters.

The geometrized Natario warped region $0 < N(rs) < \frac{1}{2}$ is centered over the radius of the bubble where the Natario shape function $N(rs)$ possesses a value too much close from $\frac{1}{2}$ although smaller than $\frac{1}{2}$.Then we can see that $N(rs) \simeq \frac{1}{2}$.[17].So we can say that the Natario geometrized warped region starts after $99,999$ meters and ends up before $100,001$ meters.The region inside the bubble for the Natario warp drive goes from 0 to $99,999$ meters and the region outside the bubble starts at $100,001$ meters. The thickness or the width of the Natario geometrized warped region is then smaller than $0,002$ meters.

[17]Remember that we are limited by the floating-point precision of our software

The derivative square of the Natario shape function is :

$$N'(rs)^2 = [\frac{1}{4}]WF^4[1 - f(rs)^{WF}]^{2(WF-1)}[f(rs)^{2(WF-1)}]f'(rs)^2 \tag{90}$$

And the derivative square of the Alcubierre shape function is :

$$f'(rs)^2 = \frac{1}{4}[\frac{@^2}{cosh^4[@(rs - R)]}] \tag{91}$$

- Numerical plot for the square derivative of the Alcubierre shape function and the complementary expressions $[1 - f(rs)^{WF}]^{[2(WF-1)]}$ and $f(rs)^{[2(WF-1)]}$ that neutralizes each other with $@ = 50000$ bubble radius $R = 100$ meters and warp factor with a value $WF = 200$

rs	$f'(rs)^2$	$[1 - f(rs)^{WF}]^{[2(WF-1)]}$	$f(rs)^{[2(WF-1)]}$
$9,999940000000E+01$	$7,667648086763E-043$	$0,0000000000000E+000$	$1,0000000000000E+000$
$9,999950000000E+01$	$3,720075984818E-034$	$0,0000000000000E+000$	$1,0000000000000E+000$
$9,999960000000E+01$	$1,804851393312E-025$	$0,0000000000000E+000$	$1,0000000000000E+000$
$9,999970000000E+01$	$8,756510795027E-017$	$0,0000000000000E+000$	$9,9999999996275E-001$
$9,999980000000E+01$	$4,248354238773E-008$	$0,0000000000000E+000$	$9,9999917966117E-001$
$9,999990000000E+01$	$2,060779370345E+001$	$0,0000000000000E+000$	$9,8209349938841E-001$
$1,000000000000E+02$	$6,250000000000E+008$	$1,0000000000000E+000$	$1,5490367659397E-120$
$1,000001000000E+02$	$2,060779346918E+001$	$1,0000000000000E+000$	$0,0000000000000E+000$
$1,000002000000E+02$	$4,248354190475E-008$	$1,0000000000000E+000$	$0,0000000000000E+000$
$1,000003000000E+02$	$8,756510695477E-017$	$1,0000000000000E+000$	$0,0000000000000E+000$
$1,000004000000E+02$	$1,804851372793E-025$	$1,0000000000000E+000$	$0,0000000000000E+000$
$1,000005000000E+02$	$3,720075942525E-034$	$1,0000000000000E+000$	$0,0000000000000E+000$
$1,000006000000E+02$	$7,667647999592E-043$	$1,0000000000000E+000$	$0,0000000000000E+000$

According with the numerical plot above when $@ = 50000$ at $99,994$ meters from the center of the bubble the square derivative of the Alcubierre shape function is $7,66 \times 10^{-43}$ and starts to increase reaching the maximum value of $6,25 \times 10^{8}$ at 100 meters from the center of the bubble precisely in the bubble radius decreasing again to the value of $7,66 \times 10^{-43}$ at $100,006$ meters from the center of the bubble.Note that with respect to the distance of 100 meters from the center of the bubble exactly the bubble radius the powers of the square derivative of the Alcubierre shape function are diametrically symmetrically opposed.We have the values of 10^1 at $99,999$ meters and at $100,001$ meters.We have the value of 10^{-8} at $99,998$ meters and at $100,002$ meters.

The expression $[1 - f(rs)^{WF}]^{[2(WF-1)]}$ is zero from the center of the bubble to $99,999$ meters and at the radius of the bubble and beyond changes its value to 1.So inside the bubble this expression is 0 and outside the bubble this expression is 1.
The expression $f(rs)^{[2(WF-1)]}$ is 1 from the center of the bubble to $99,996$ meters and at $99,997$ meters starts to decrease reaching its minimum value of $1,54 \times 10^{-120}$ precisely in the bubble radius.At $100,001$ meters its value is zero.

Note that when one of these expressions is zero the other is 1 or possesses values very close to 1.Then one expression neutralizes the other except in the bubble radius but here the value of the product is very low:$1,54 \times 10^{-120}$.This of course obliterates the factor 10^{48}.

The derivative square of the Natario shape function is :

$$N'(rs)^2 = [\frac{1}{4}]WF^4[1 - f(rs)^{WF}]^{2(WF-1)}[f(rs)^{2(WF-1)}]f'(rs)^2 \tag{92}$$

And the derivative square of the Alcubierre shape function is :

$$f'(rs)^2 = \frac{1}{4}[\frac{@^2}{cosh^4[@(rs-R)]}] \tag{93}$$

- Numerical plot for the square derivatives of both the Alcubierre and Natario shape functions with $@ = 50000$ bubble radius $R = 100$ meters and warp factor with a value $WF = 200$

rs	$f'(rs)^2$	$N'(rs)^2$
$9,999940000000E+01$	$7,667648086763E-043$	$0,0000000000000E+000$
$9,999950000000E+01$	$3,720075984818E-034$	$0,0000000000000E+000$
$9,999960000000E+01$	$1,804851393312E-025$	$0,0000000000000E+000$
$9,999970000000E+01$	$8,756510795027E-017$	$0,0000000000000E+000$
$9,999980000000E+01$	$4,248354238773E-008$	$0,0000000000000E+000$
$9,999990000000E+01$	$2,060779370345E+001$	$0,0000000000000E+000$
$1,000000000000E+02$	$6,250000000000E+008$	$3,8725919148493E-103$
$1,000001000000E+02$	$2,060779346918E+001$	$0,0000000000000E+000$
$1,000002000000E+02$	$4,248354190475E-008$	$0,0000000000000E+000$
$1,000003000000E+02$	$8,756510695477E-017$	$0,0000000000000E+000$
$1,000004000000E+02$	$1,804851372793E-025$	$0,0000000000000E+000$
$1,000005000000E+02$	$3,720075942525E-034$	$0,0000000000000E+000$
$1,000006000000E+02$	$7,667647999592E-043$	$0,0000000000000E+000$

From the plots of the previous page we know that the product

$$[1 - f(rs)^{WF}]^{2(WF-1)}[f(rs)^{2(WF-1)}] \tag{94}$$

is always zero except in the bubble radius giving a non-null square derivative of the Natario shape function in the bubble radius but with a very small value.The final value for the square derivative of the Natario shape function is then $3,8 \times 10^{-103}$ layered over the bubble radius and this obliterates the factor 10^{48} in the negative energy density of the Natario warp drive in the $1+1$ spacetime rendering it physically possible.Note also that this value match the value presented in the numerical plot of the previous section for $@ = 50000$. We recall the negative energy density in the $1+1$ Natario warp drive spacetime :

$$\rho = T_{\mu\nu}u^{\mu}u^{\nu} = -\frac{c^2}{G}\frac{v_s^2}{8\pi}\left[3(N'(rs))^2\right] \tag{95}$$

$$\frac{c^2}{G}\frac{v_s^2}{8\pi} \simeq 10^{48} \tag{96}$$

$$(N'(rs))^2 \simeq 3,8 \times 10^{-103} \tag{97}$$

And the product $10^{48} \times 10^{-103} = 10^{-55}$ $\frac{Joules}{Meters^3}$ resulting in a very low negative energy density even for speeds of 200 times faster than light.

5 The expressions that complementary neutralizes each other in the first order derivative of the shape function for a Natario warp drive $3+1$ spacetime metric

Its expression is :

$$N'(rs) = -[\frac{1}{2}]WF^2[1 - f(rs)^{WF}]^{(WF-1)}[f(rs)^{(WF-1)}]f'(rs) \tag{98}$$

With the first order derivative of the Alcubierre shape function being:

$$f'(rs) = -\frac{1}{2}[\frac{@}{cosh^2[@(rs-R)]}] \tag{99}$$

- Numerical plot for the derivative of the Alcubierre shape function and the complementary expressions $[1 - f(rs)^{WF}]^{[(WF-1)]}$ and $f(rs)^{[(WF-1)]}$ that neutralizes each other with $@ = 50000$ bubble radius $R = 100$ meters and warp factor with a value $WF = 200$

rs	$f'(rs)$	$[1 - f(rs)^{WF}]^{[(WF-1)]}$	$f(rs)^{[(WF-1)]}$
$9,999940000000E+01$	$-8,756510770143E-022$	$0,00000000000E+000$	$1,00000000000E+000$
$9,999950000000E+01$	$-1,928749850244E-017$	$0,00000000000E+000$	$1,00000000000E+000$
$9,999960000000E+01$	$-4,248354261725E-013$	$0,00000000000E+000$	$1,00000000000E+000$
$9,999970000000E+01$	$-9,357622986115E-009$	$0,00000000000E+000$	$9,99999999981E-001$
$9,999980000000E+01$	$-2,061153618432E-004$	$0,00000000000E+000$	$9,99999589830E-001$
$9,999990000000E+01$	$-4,539580784989E+000$	$0,00000000000E+000$	$9,91006306432E-001$
$1,000000000000E+02$	$-2,500000000000E+004$	$1,00000000000E+000$	$1,24460305557E-060$
$1,000001000000E+02$	$-4,539580759187E+000$	$1,00000000000E+000$	$0,00000000000E+000$
$1,000002000000E+02$	$-2,061153606715E-004$	$1,00000000000E+000$	$0,00000000000E+000$
$1,000003000000E+02$	$-9,357622932923E-009$	$1,00000000000E+000$	$0,00000000000E+000$
$1,000004000000E+02$	$-4,248354237576E-013$	$1,00000000000E+000$	$0,00000000000E+000$
$1,000005000000E+02$	$-1,928749839281E-017$	$1,00000000000E+000$	$0,00000000000E+000$
$1,000006000000E+02$	$-8,756510720368E-022$	$1,00000000000E+000$	$0,00000000000E+000$

According with the numerical plot above when $@ = 50000$ at $99,994$ meters from the center of the bubble the derivative of the Alcubierre shape function is $-8,75 \times 10^{-22}$ and starts to increase reaching the maximum value of $-2,5 \times 10^4$ at 100 meters from the center of the bubble precisely in the bubble radius decreasing again to the value of $-8,75 \times 10^{-22}$ at $100,006$ meters from the center of the bubble.

The expression $[1 - f(rs)^{WF}]^{[(WF-1)]}$ is zero from the center of the bubble to $99,999$ meters and at the radius of the bubble and beyond changes its value to 1.So inside the bubble this expression is 0 and outside the bubble this expression is 1.
The expression $f(rs)^{[(WF-1)]}$ is 1 from the center of the bubble to $99,996$ meters and at $99,997$ meters starts to decrease reaching its minimum value of $1,24 \times 10^{-60}$ precisely in the bubble radius.At $100,001$ meters its value is zero.

Note that when one of these expressions is zero the other is 1 or possesses values very close to 1.Then one expression neutralizes the other except in the bubble radius but here the value of the product is very low:$1,24 \times 10^{-60}$.

The expression for the first order derivative of the Natario shape function is :

$$N'(rs) = -[\frac{1}{2}]WF^2[1 - f(rs)^{WF}]^{(WF-1)}[f(rs)^{(WF-1)}]f'(rs) \tag{100}$$

With the first order derivative of the Alcubierre shape function being:

$$f'(rs) = -\frac{1}{2}[\frac{@}{cosh^2[@(rs - R)]}] \tag{101}$$

- Numerical plot for the first order derivatives of both the Alcubierre and Natario shape functions with $@ = 50000$ bubble radius $R = 100$ meters and warp factor with a value $WF = 200$

rs	$f'(rs)$	$N'(rs)$
$9,999940000000E+01$	$-8,756510770143E-022$	$0,00000000000E+000$
$9,999950000000E+01$	$-1,928749850244E-017$	$0,00000000000E+000$
$9,999960000000E+01$	$-4,248354261725E-013$	$0,00000000000E+000$
$9,999970000000E+01$	$-9,357622986115E-009$	$0,00000000000E+000$
$9,999980000000E+01$	$-2,061153618432E-004$	$0,00000000000E+000$
$9,999990000000E+01$	$-4,539580784989E+000$	$0,00000000000E+000$
$1,000000000000E+02$	$-2,500000000000E+004$	$3,11150763893E-054$
$1,000001000000E+02$	$-4,539580759187E+000$	$0,00000000000E+000$
$1,000002000000E+02$	$-2,061153606715E-004$	$0,00000000000E+000$
$1,000003000000E+02$	$-9,357622932923E-009$	$0,00000000000E+000$
$1,000004000000E+02$	$-4,248354237576E-013$	$0,00000000000E+000$
$1,000005000000E+02$	$-1,928749839281E-017$	$0,00000000000E+000$
$1,000006000000E+02$	$-8,756510720368E-022$	$0,00000000000E+000$

From the plots of the previous page we know that the product

$$[1 - f(rs)^{WF}]^{(WF-1)}[f(rs)^{(WF-1)}] \tag{102}$$

is always zero except in the bubble radius giving a non-null derivative of the Natario shape function in the bubble radius but with a very small value.The final value for the derivative of the Natario shape function is then $3,11 \times 10^{-54}$ layered over the bubble radius

6 The expressions that complementary neutralizes each other in the term A of the second order derivative of the shape function for a Natario warp drive $3+1$ spacetime metric

$$A = [\frac{1}{2}]WF^3(WF-1)[1-f(rs)^{WF}]^{(WF-2)}[f(rs)^{2(WF-1)}]f'(rs)^2 \tag{103}$$

With $f'(rs)^2$ being the first order derivative square of the Alcubierre shape function which is :

$$f'(rs)^2 = \frac{1}{4}[\frac{@^2}{cosh^4[@(rs-R)]}] \tag{104}$$

- Numerical plot for the square first order derivative of the Alcubierre shape function and the complementary expressions $[1-f(rs)^{WF}]^{[(WF-2)]}$ and $f(rs)^{[2(WF-1)]}$ that neutralizes each other with $@ = 50000$ bubble radius $R = 100$ meters and warp factor with a value $WF = 200$

rs	$f'(rs)^2$	$[1-f(rs)^{WF}]^{[(WF-2)]}$	$f(rs)^{[2(WF-1)]}$
$9,999940000000E+01$	$7,667648086763E-043$	$0,0000000000000E+000$	$1,0000000000000E+000$
$9,999950000000E+01$	$3,720075984818E-034$	$0,0000000000000E+000$	$1,0000000000000E+000$
$9,999960000000E+01$	$1,804851393312E-025$	$0,0000000000000E+000$	$1,0000000000000E+000$
$9,999970000000E+01$	$8,756510795027E-017$	$0,0000000000000E+000$	$9,9999999996275E-001$
$9,999980000000E+01$	$4,248354238773E-008$	$0,0000000000000E+000$	$9,9999917966117E-001$
$9,999990000000E+01$	$2,060779370345E+001$	$0,0000000000000E+000$	$9,8209349938841E-001$
$1,000000000000E+02$	$6,250000000000E+008$	$1,0000000000000E+000$	$1,5490367659397E-120$
$1,000001000000E+02$	$2,060779346918E+001$	$1,0000000000000E+000$	$0,0000000000000E+000$
$1,000002000000E+02$	$4,248354190475E-008$	$1,0000000000000E+000$	$0,0000000000000E+000$
$1,000003000000E+02$	$8,756510695477E-017$	$1,0000000000000E+000$	$0,0000000000000E+000$
$1,000004000000E+02$	$1,804851372793E-025$	$1,0000000000000E+000$	$0,0000000000000E+000$
$1,000005000000E+02$	$3,720075942525E-034$	$1,0000000000000E+000$	$0,0000000000000E+000$
$1,000006000000E+02$	$7,667647999592E-043$	$1,0000000000000E+000$	$0,0000000000000E+000$

According with the numerical plot above when $@ = 50000$ at $99,994$ meters from the center of the bubble the square derivative of the Alcubierre shape function is $7,66 \times 10^{-43}$ and starts to increase reaching the maximum value of $6,25 \times 10^{8}$ at 100 meters from the center of the bubble precisely in the bubble radius decreasing again to the value of $7,66 \times 10^{-43}$ at $100,006$ meters from the center of the bubble.

The expression $[1-f(rs)^{WF}]^{[(WF-2)]}$ is zero from the center of the bubble to $99,999$ meters and at the radius of the bubble and beyond changes its value to 1.So inside the bubble this expression is 0 and outside the bubble this expression is 1.
The expression $f(rs)^{[2(WF-1)]}$ is 1 from the center of the bubble to $99,996$ meters and at $99,997$ meters starts to decrease reaching its minimum value of $1,54 \times 10^{-120}$ precisely in the bubble radius.At $100,001$ meters its value is zero.

Note that when one of these expressions is zero the other is 1 or possesses values very close to 1.Then one expression neutralizes the other except in the bubble radius but here the value of the product is very low:$1,54 \times 10^{-120}$.

The term A of the second order derivative of the shape function for a Natario warp drive 3+1 spacetime metric is given by:

$$A = [\frac{1}{2}]WF^3(WF-1)[1-f(rs)^{WF}]^{(WF-2)}[f(rs)^{2(WF-1)}]f'(rs)^2 \tag{105}$$

With $f'(rs)^2$ being the first order derivative square of the Alcubierre shape function which is :

$$f'(rs)^2 = \frac{1}{4}[\frac{@^2}{cosh^4[@(rs-R)]}] \tag{106}$$

- Numerical plot for the square first order derivative of the Alcubierre shape function and the term A of the second order derivative of the Natario shape function with $@ = 50000$ bubble radius $R = 100$ meters and warp factor with a value $WF = 200$

rs	$f'(rs)^2$	A
$9,999940000000E+01$	$7,667648086763E-043$	$0,00000000000E+000$
$9,999950000000E+01$	$3,720075984818E-034$	$0,00000000000E+000$
$9,999960000000E+01$	$1,804851393312E-025$	$0,00000000000E+000$
$9,999970000000E+01$	$8,756510795027E-017$	$0,00000000000E+000$
$9,999980000000E+01$	$4,248354238773E-008$	$0,00000000000E+000$
$9,999990000000E+01$	$2,060779370345E+001$	$0,00000000000E+000$
$1,000000000000E+02$	$6,250000000000E+008$	$7,70645791055E-103$
$1,000001000000E+02$	$2,060779346918E+001$	$0,00000000000E+000$
$1,000002000000E+02$	$4,248354190475E-008$	$0,00000000000E+000$
$1,000003000000E+02$	$8,756510695477E-017$	$0,00000000000E+000$
$1,000004000000E+02$	$1,804851372793E-025$	$0,00000000000E+000$
$1,000005000000E+02$	$3,720075942525E-034$	$0,00000000000E+000$
$1,000006000000E+02$	$7,667647999592E-043$	$0,00000000000E+000$

From the plots of the previous page we know that the product

$$[1-f(rs)^{WF}]^{(WF-2)}[f(rs)^{2(WF-1)}] \tag{107}$$

is always zero except in the bubble radius giving a non-null value for the term A of the second order derivative of the Natario shape function in the bubble radius but with a very small value.The final value for the term A is then $7,701 \times 10^{-103}$ layered over the bubble radius

7 The expressions that complementary neutralizes each other in the term B of the second order derivative of the shape function for a Natario warp drive $3+1$ spacetime metric

$$B = -[\frac{1}{2}]WF^2[1-f(rs)^{WF}]^{(WF-1)}(WF-1)[f(rs)^{(WF-2)}]f'(rs)^2 \tag{108}$$

With $f'(rs)^2$ being the first order derivative square of the Alcubierre shape function which is :

$$f'(rs)^2 = \frac{1}{4}[\frac{@^2}{cosh^4[@(rs-R)]}] \tag{109}$$

- Numerical plot for the square first order derivative of the Alcubierre shape function and the complementary expressions $[1-f(rs)^{WF}]^{[(WF-1)]}$ and $f(rs)^{[(WF-2)]}$ that neutralizes each other with $@ = 50000$ bubble radius $R = 100$ meters and warp factor with a value $WF = 200$

rs	$f'(rs)^2$	$[1-f(rs)^{WF}]^{[(WF-1)]}$	$f(rs)^{[(WF-2)]}$
$9,999940000000E+01$	$7,667648086763E-043$	$0,0000000000000E+000$	$1,000000000000E+000$
$9,999950000000E+01$	$3,720075984818E-034$	$0,0000000000000E+000$	$1,000000000000E+000$
$9,999960000000E+01$	$1,804851393312E-025$	$0,0000000000000E+000$	$1,000000000000E+000$
$9,999970000000E+01$	$8,756510795027E-017$	$0,0000000000000E+000$	$9,999999999815E-001$
$9,999980000000E+01$	$4,248354238773E-008$	$0,0000000000000E+000$	$9,999995918917E-001$
$9,999990000000E+01$	$2,060779370345E+001$	$0,0000000000000E+000$	$9,910512980490E-001$
$1,000000000000E+02$	$6,250000000000E+008$	$1,0000000000000E+000$	$2,489206111144E-060$
$1,000001000000E+02$	$2,060779346918E+001$	$1,0000000000000E+000$	$0,000000000000E+000$
$1,000002000000E+02$	$4,248354190475E-008$	$1,0000000000000E+000$	$0,000000000000E+000$
$1,000003000000E+02$	$8,756510695477E-017$	$1,0000000000000E+000$	$0,000000000000E+000$
$1,000004000000E+02$	$1,804851372793E-025$	$1,0000000000000E+000$	$0,000000000000E+000$
$1,000005000000E+02$	$3,720075942525E-034$	$1,0000000000000E+000$	$0,000000000000E+000$
$1,000006000000E+02$	$7,667647999592E-043$	$1,0000000000000E+000$	$0,000000000000E+000$

According with the numerical plot above when $@ = 50000$ at $99,994$ meters from the center of the bubble the square derivative of the Alcubierre shape function is $7,66 \times 10^{-43}$ and starts to increase reaching the maximum value of $6,25 \times 10^8$ at 100 meters from the center of the bubble precisely in the bubble radius decreasing again to the value of $7,66 \times 10^{-43}$ at $100,006$ meters from the center of the bubble.

The expression $[1-f(rs)^{WF}]^{[(WF-1)]}$ is zero from the center of the bubble to $99,999$ meters and at the radius of the bubble and beyond changes its value to 1.So inside the bubble this expression is 0 and outside the bubble this expression is 1.
The expression $f(rs)^{[(WF-2)]}$ is 1 from the center of the bubble to $99,996$ meters and at $99,997$ meters starts to decrease reaching its minimum value of $2,48 \times 10^{-60}$ precisely in the bubble radius.At $100,001$ meters its value is zero.

Note that when one of these expressions is zero the other is 1 or possesses values very close to 1.Then one expression neutralizes the other except in the bubble radius but here the value of the product is very low:$2,48 \times 10^{-60}$.

The term B of the second order derivative of the shape function for a Natario warp drive 3+1 spacetime metric is given by:

$$B = -[\frac{1}{2}]WF^2[1 - f(rs)^{WF}]^{(WF-1)}(WF - 1)[f(rs)^{(WF-2)}]f'(rs)^2 \tag{110}$$

With $f'(rs)^2$ being the first order derivative square of the Alcubierre shape function which is :

$$f'(rs)^2 = \frac{1}{4}[\frac{@^2}{cosh^4[@(rs - R)]}] \tag{111}$$

- Numerical plot for the square first order derivative of the Alcubierre shape function and the term B of the second order derivative of the Natario shape function with $@ = 50000$ bubble radius $R = 100$ meters and warp factor with a value $WF = 200$

rs	$f'(rs)^2$	B
$9,999940000000E+01$	$7,667648086763E-043$	$0,000000000000E+000$
$9,999950000000E+01$	$3,720075984818E-034$	$0,000000000000E+000$
$9,999960000000E+01$	$1,804851393312E-025$	$0,000000000000E+000$
$9,999970000000E+01$	$8,756510795027E-017$	$0,000000000000E+000$
$9,999980000000E+01$	$4,248354238773E-008$	$0,000000000000E+000$
$9,999990000000E+01$	$2,060779370345E+001$	$0,000000000000E+000$
$1,000000000000E+02$	$6,250000000000E+008$	$-6,191900201472E-045$
$1,000001000000E+02$	$2,060779346918E+001$	$0,000000000000E+000$
$1,000002000000E+02$	$4,248354190475E-008$	$0,000000000000E+000$
$1,000003000000E+02$	$8,756510695477E-017$	$0,000000000000E+000$
$1,000004000000E+02$	$1,804851372793E-025$	$0,000000000000E+000$
$1,000005000000E+02$	$3,720075942525E-034$	$0,000000000000E+000$
$1,000006000000E+02$	$7,667647999592E-043$	$0,000000000000E+000$

From the plots of the previous page we know that the product

$$[1 - f(rs)^{WF}]^{(WF-1)}[f(rs)^{(WF-2)}] \tag{112}$$

is always zero except in the bubble radius giving a non-null value for the term B of the second order derivative of the Natario shape function in the bubble radius but with a very small value.The final value for the term B is then $-6,191 \times 10^{-45}$ layered over the bubble radius

8 The expressions that complementary neutralizes each other in the term C of the second order derivative of the shape function for a Natario warp drive $3+1$ spacetime metric

$$C = -[\frac{1}{2}]WF^2[1 - f(rs)^{WF}]^{(WF-1)}[f(rs)^{(WF-1)}]f''(rs) \tag{113}$$

With $f''(rs)$ being the second order derivative of the Alcubierre shape function which is:

$$f''(rs) = [\frac{(@^2)sinh[@(rs-R)}{cosh^3[@(rs-R)]}] \tag{114}$$

- Numerical plot for the second order derivative of the Alcubierre shape function and the complementary expressions $[1-f(rs)^{WF}]^{[(WF-1)]}$ and $f(rs)^{[(WF-1)]}$ that neutralizes each other with $@ = 50000$ bubble radius $R = 100$ meters and warp factor with a value $WF = 200$

rs	$f''(rs)$	$[1-f(rs)^{WF}]^{[(WF-1)]}$	$f(rs)^{[(WF-1)]}$
$9,999940000000E+01$	$-8,75651077014E-017$	$0,00000000000E+000$	$1,00000000000E+000$
$9,999950000000E+01$	$-1,92874985024E-012$	$0,00000000000E+000$	$1,00000000000E+000$
$9,999960000000E+01$	$-4,24835426172E-008$	$0,00000000000E+000$	$1,00000000000E+000$
$9,999970000000E+01$	$-9,35762298611E-004$	$0,00000000000E+000$	$9,99999999981E-001$
$9,999980000000E+01$	$-2,06115360993E+001$	$0,00000000000E+000$	$9,99999589830E-001$
$9,999990000000E+01$	$-4,53916861040E+005$	$0,00000000000E+000$	$9,91006306432E-001$
$1,000000000000E+02$	$0,00000000000E+000$	$1,00000000000E+000$	$1,24460305557E-060$
$1,000001000000E+02$	$4,53916858461E+005$	$1,00000000000E+000$	$0,00000000000E+000$
$1,000002000000E+02$	$2,06115359822E+001$	$1,00000000000E+000$	$0,00000000000E+000$
$1,000003000000E+02$	$9,35762293292E-004$	$1,00000000000E+000$	$0,00000000000E+000$
$1,000004000000E+02$	$4,24835423758E-008$	$1,00000000000E+000$	$0,00000000000E+000$
$1,000005000000E+02$	$1,92874983928E-012$	$1,00000000000E+000$	$0,00000000000E+000$
$1,000006000000E+02$	$8,75651072037E-017$	$1,00000000000E+000$	$0,00000000000E+000$

According with the numerical plot above when $@ = 50000$ at $99,994$ meters from the center of the bubble the second order derivative of the Alcubierre shape function is $-8,75 \times 10^{-17}$ however in the bubble radius inverts the signal reaching the value of $8,75 \times 10^{-17}$ at $100,006$ meters from the center of the bubble.In the radius of the bubble the value is 0 due to the term $sinh[@(rs-R)] = 0$ when $rs = R$.

The expression $[1 - f(rs)^{WF}]^{[(WF-1)]}$ is zero from the center of the bubble to $99,999$ meters and at the radius of the bubble and beyond changes its value to 1.So inside the bubble this expression is 0 and outside the bubble this expression is 1.
The expression $f(rs)^{[(WF-1)]}$ is 1 from the center of the bubble to $99,996$ meters and at $99,997$ meters starts to decrease reaching its minimum value of $1,24 \times 10^{-60}$ precisely in the bubble radius.At $100,001$ meters its value is zero.

Note that when one of these expressions is zero the other is 1 or possesses values very close to 1.Then one expression neutralizes the other except in the bubble radius but here the value of the product is very low:$1,24 \times 10^{-60}$.

The term C of the second order derivative of the shape function for a Natario warp drive 3+1 spacetime metric is given by:

$$C = -[\frac{1}{2}]WF^2[1 - f(rs)^{WF}]^{(WF-1)}[f(rs)^{(WF-1)}]f''(rs) \tag{115}$$

With $f''(rs)$ being the second order derivative of the Alcubierre shape function which is:

$$f''(rs) = [\frac{(@^2)sinh[@(rs - R)}{cosh^3[@(rs - R)]}] \tag{116}$$

- Numerical plot for the second order derivative of the Alcubierre shape function and the term C of the second order derivative of the Natario shape function with $@ = 50000$ bubble radius $R = 100$ meters and warp factor with a value $WF = 200$

rs	$f''(rs)$	C
$9,999940000000E+01$	$-8,75651077014E-017$	$0,00000000000E+000$
$9,999950000000E+01$	$-1,92874985024E-012$	$0,00000000000E+000$
$9,999960000000E+01$	$-4,24835426172E-008$	$0,00000000000E+000$
$9,999970000000E+01$	$-9,35762298611E-004$	$0,00000000000E+000$
$9,999980000000E+01$	$-2,06115360993E+001$	$0,00000000000E+000$
$9,999990000000E+01$	$-4,53916861040E+005$	$0,00000000000E+000$
$1,000000000000E+02$	$0,00000000000E+000$	$0,00000000000E+000$
$1,000001000000E+02$	$4,53916858461E+005$	$0,00000000000E+000$
$1,000002000000E+02$	$2,06115359822E+001$	$0,00000000000E+000$
$1,000003000000E+02$	$9,35762293292E-004$	$0,00000000000E+000$
$1,000004000000E+02$	$4,24835423758E-008$	$0,00000000000E+000$
$1,000005000000E+02$	$1,92874983928E-012$	$0,00000000000E+000$
$1,000006000000E+02$	$8,75651072037E-017$	$0,00000000000E+000$

From the plots of the previous page we know that the product

$$[1 - f(rs)^{WF}]^{(WF-1)}[f(rs)^{(WF-1)}] \tag{117}$$

is always zero except in the bubble radius however due to the term $sinh[@(rs - R)] = 0$ when $rs = R$ the value of the term C is always 0.

9 The second order derivative of the shape function for a Natario warp drive $3+1$ spacetime metric

- Numerical plot for the second order derivatives of both the Alcubierre and Natario shape functions with $@ = 50000$ bubble radius $R = 100$ meters and warp factor with a value $WF = 200$

rs	$f''(rs)$	$N''(rs)$
$9,999940000000E+01$	$-8,75651077014E-017$	$0,00000000000E+000$
$9,999950000000E+01$	$-1,92874985024E-012$	$0,00000000000E+000$
$9,999960000000E+01$	$-4,24835426172E-008$	$0,00000000000E+000$
$9,999970000000E+01$	$-9,35762298611E-004$	$0,00000000000E+000$
$9,999980000000E+01$	$-2,06115360993E+001$	$0,00000000000E+000$
$9,999990000000E+01$	$-4,53916861040E+005$	$0,00000000000E+000$
$1,000000000000E+02$	$0,00000000000E+000$	$-6,1919002015E-045$
$1,000001000000E+02$	$4,53916858461E+005$	$0,00000000000E+000$
$1,000002000000E+02$	$2,06115359822E+001$	$0,00000000000E+000$
$1,000003000000E+02$	$9,35762293292E-004$	$0,00000000000E+000$
$1,000004000000E+02$	$4,24835423758E-008$	$0,00000000000E+000$
$1,000005000000E+02$	$1,92874983928E-012$	$0,00000000000E+000$
$1,000006000000E+02$	$8,75651072037E-017$	$0,00000000000E+000$

In order to avoid the mathematical complexities of a lengthly algebraic expression for the second order derivative of the shape function for a Natario warp drive $3+1$ spacetime metric $N''(rs)$ we decomposed in section 2 the expression for $N''(rs)$ in 3 algebraic terms A,B and C and we evaluated numerically and independently each one of the 3 terms that must be added together to provide the final numerical value for $N''(rs)$.

From the previous numerical plots in section 5 we know that the term A is always zero except in the bubble radius possessing a non-null value of $7,70645791055 \times 10^{-103}$ and from the previous numerical plots in section 6 we know that the term B is also always zero except in the bubble radius possessing a value of $-6,191900201472 \times 10^{-45}$ and from the previous numerical plots in section 7 wee know that the term C is always zero even in the bubble radius .

The final value of $N''(rs)$ is also always zero except in the bubble radius.Note that the term B is much larger than the term A so it is the value of B that accounts for the final value of $N''(rs)$ in the bubble radius which is $-6,1919002015 \times 10^{-45}$

10 The negative energy density for a Natario warp drive in a $3+1$ spacetime metric

Reviewing the more sophisticated case of the Natario warp drive in a real $3+1$ spacetime seen in section 3 where the negative energy density in this case is given by the following expression

$$\rho_{3+1} = -\frac{c^2}{G}\frac{vs^2}{8\pi}\left[3(N'(rs))^2\cos^2\theta\right] - \frac{c^2}{G}\frac{vs^2}{8\pi}\left[\left(N'(rs) + \frac{rs}{2}N''(rs)\right)^2\sin^2\theta\right] \tag{118}$$

In section 2 we decomposed the above expression in two terms ρ_1 and ρ_2 given respectively by:

$$\rho_{3+1} = \rho_1 + \rho_2 \tag{119}$$

$$\rho_1 = -\frac{c^2}{G}\frac{vs^2}{8\pi}\left[3(N'(rs))^2\cos^2\theta\right] \tag{120}$$

$$\rho_2 = -\frac{c^2}{G}\frac{vs^2}{8\pi}\left[\left(N'(rs) + \frac{rs}{2}N''(rs)\right)^2\sin^2\theta\right] \tag{121}$$

Comparing the above expressions with the negative energy density for a Natario warp drive in the the $1+1$ spacetime given also in section 2:

$$\rho_{1+1} = -\frac{c^2}{G}\frac{v_s^2}{8\pi}\left[3(N'(rs))^2\right] \tag{122}$$

We can see that the term in ρ_1 almost matches the term in the $1+1$ spacetime except for the trigonometric term in $\cos^2\theta$ and as seen in sections 2 and 3 this term produces a very low derivative square for the Natario shape function of about 10^{-103}.So the term ρ_2 is the term that really accounts for the negative energy density in the $3+1$ spacetime.

The dominant expression in the term ρ_2 is:

$$\left[\left(N'(rs) + \frac{rs}{2}N''(rs)\right)^2\right] \tag{123}$$

Since the derivative of second order of the Natario shape function $N''(rs)$ is a lengthly expression as shown in section 2 with many algebraic terms then its square $N''(rs)^2$ results in an even more complicated expression with even more algebraic terms.And the product of both the first and second order derivatives $N'(rs)N''(rs)$ also results in a lengthly expression.Then in order to avoid algebraic complications we also decided in section 2 to work numerically with the new dominant term which is:

$$N'(rs) + \frac{rs}{2}N''(rs) \tag{124}$$

Raising to the square only the final numerically evaluated result.

- Numerical plot for the terms $N'(rs) + \frac{rs}{2}N''(rs)$ and $\left[\left(N'(rs) + \frac{rs}{2}N''(rs)\right)^2\right]$ with $@ = 50000$ bubble radius $R = 100$ meters and warp factor with a value $WF = 200$

rs	$N'(rs) + [(rs/2)N''(rs)]$	$(N'(rs) + [(rs/2)N''(rs)])^2$
$9,999940000000E+01$	$0,0000000000E+000$	$0,00000000000E+000$
$9,999950000000E+01$	$0,0000000000E+000$	$0,00000000000E+000$
$9,999960000000E+01$	$0,0000000000E+000$	$0,00000000000E+000$
$9,999970000000E+01$	$0,0000000000E+000$	$0,00000000000E+000$
$9,999980000000E+01$	$0,0000000000E+000$	$0,00000000000E+000$
$9,999990000000E+01$	$0,0000000000E+000$	$0,00000000000E+000$
$1,000000000000E+02$	$-3,0959501008E-043$	$9,5849070264E-086$
$1,000001000000E+02$	$0,0000000000E+000$	$0,00000000000E+000$
$1,000002000000E+02$	$0,0000000000E+000$	$0,00000000000E+000$
$1,000003000000E+02$	$0,0000000000E+000$	$0,00000000000E+000$
$1,000004000000E+02$	$0,0000000000E+000$	$0,00000000000E+000$
$1,000005000000E+02$	$0,0000000000E+000$	$0,00000000000E+000$
$1,000006000000E+02$	$0,0000000000E+000$	$0,00000000000E+000$

From the numerical plots in section 4 we know that the derivative of first order of the Natario shape function $N'(rs)$ is always zero except in the bubble radius. Its value in the bubble radius is then $3,11150763893 \times 10^{-54}$. From the numerical plots in section 7 we know that the derivative of second order of the Natario shape function $N''(rs)$ is also always zero except in the bubble radius. Its value in the bubble radius is then $-6,1919002015 \times 10^{-45}$.

Then computing the value of the new dominant term in the expression for ρ_2 which is $[N'(rs)+(\frac{rs}{2})N''(rs)]$ we expect to find values only in the bubble radius. Note that the value of $N''(rs)$ is much bigger in modulus than the value of $N'(rs)$ so it $N''(rs)$ that accounts for the final value of the new dominant term.

The final value of the new dominant term is then $-3,0959501008 \times 10^{-43}$ in the bubble radius. and the final value for the original dominant term in the expression for ρ_2 which is $[N'(rs) + (\frac{rs}{2})N''(rs)]^2$ is then $9,5849070264 \times 10^{-86}$ only in the bubble radius.

This value also obliterates the factor 10^{48} from a speed of 200 times faster than light resulting in a very low negative energy density of $10^{48} \times 10^{-86} = 10^{-38}$.A very low negative energy density of 10^{-38} $\frac{Joules}{meters^3}$ for the term ρ_2 that really accounts for the negative energy density of the Natario warp drive in a $3+1$ spacetime.

Note that the negative energy density in the term ρ_1 have values of 10^{-55} $\frac{Joules}{meters^3}$ according to sections 3 and 4.

11 The average matter density of the Interstellar Medium(IM)

A very serious drawback that affects the warp drive is the quest of the interstellar navigation:Interstellar space is not empty and from a real point of view a ship at superluminal speeds would impact asteroids,comets,interstellar space dust and photons.(see [5],[7] and [8])

In the previous sections we briefly resumed how the negative energy density in the Natario warp drive spacetime can be greatly lowered from 10^{48} to 10^{-55} or 10^{-38} $\frac{Joules}{meters^3}$

The warp factor WF not only squeezes the negative energy density into a very thin region almost centered over the radius of the bubble but also reduces the amount of negative energy density needed to sustain a warp bubble from impossible levels to "affordable" results.

But all we did was only a mathematical demonstration of how far can we go in the reduction of the negative energy density levels by manipulating the warp factor WF. Amounts of 10^{-55} or 10^{-38} $\frac{Joules}{meters^3}$ although desirable are completely unrealistic considering a live scenario for an interstellar travel.

The reason for the statement pointed above is the existence of the so-called Interstellar Medium[18].Interstellar Medium(IM) is mainly composed by 99 percent of gas and 1 percent of dust.[19]

For the gas 91 percent are hydrogen atoms $8,9$ percent are helium atoms and $0,1$ percent are elements heavier than hydrogen or helium.

In dense regions the IM matter is primarily in molecular form and reaches densities of 10^6 molecules per cm^3 while in diffuse regions the density is low by the order of 10^{-4} molecules per cm^3.Compare this with a density of 10^{19} molecules per cm^3 for the air at sea level or 10^{10} molecules per cm^3 for a laboratory vacuum chamber.

This means to say that the IM even in dense regions is 10^{13} times lighter than the air at sea level or better 10.000.000.000.000 times(10 trillion times) lighter than the air at sea level or 10.000 times lighter than the best vacuum chambers.

Working with cubic meters we would have for the IM the numbers of 10^{12} molecules per m^3 in dense regions and 10^2 molecules per m^3 in diffuse regions.

Since 99 percent of the IM is gas and from the gas 91 percent is hydrogen then we can use only the hydrogen atom in the following considerations and from the hydrogen atom we can use only the proton with a mass of about 10^{-27} kilograms neglecting the electron which have a much lighter mass of 10^{-31} kilograms.

Then working with mass densities of kilograms per cubic meters we would have for the IM the numbers of 10^{-15} kilograms per m^3 in dense regions and 10^{-25} kilograms per m^3 in diffuse regions.

[18]see Wikipedia the free Encyclopedia
[19]see Appendices L and M for the composition of the Interstellar Medium IM)

In terms of energy densities of Joules per cubic meters we would have for the IM the numbers of 10 Joules per m^3 in dense regions and 10^{-9} Joules per m^3 in diffuse regions.

By comparison a mass density of 1 kilogram per cubic meter means an energy density of about 10^{16} Joules per cubic meter.

The negative energy density in the Natario warp drive 3 + 1 spacetime is given by the following expressions(pg 5 in [2])[20]:

$$\rho_{3+1} = -\frac{c^2}{G}\frac{vs^2}{8\pi}\left[3(N'(rs))^2\cos^2\theta + \left(N'(rs) + \frac{rs}{2}N''(rs)\right)^2\sin^2\theta\right]. \tag{125}$$

The equation above can be divided in two expressions as shown below:

$$\rho_{3+1} = \rho_1 + \rho_2 \tag{126}$$

$$\rho_1 = -\frac{c^2}{G}\frac{vs^2}{8\pi}\left[3(N'(rs))^2\cos^2\theta\right] \tag{127}$$

$$\rho_2 = -\frac{c^2}{G}\frac{vs^2}{8\pi}\left[\left(N'(rs) + \frac{rs}{2}N''(rs)\right)^2\sin^2\theta\right] \tag{128}$$

From [5],[7] and [8] we know that if a ship travelling at 200 times light speed collides with even a single photon in interstellar space the result would be catastrophic to the physical integrity of the ship and crew members not to mention speeds of 10.000 times faster than light.

Note this as a very important fact:The energy density in the Natario warp drive is being distributed around all the space involving the ship(above the ship $\sin\theta = 1$ and $\cos\theta = 0$ while in front of the ship $\sin\theta = 0$ and $\cos\theta = 1$).The negative energy in front of the ship must "deflect" particles or photons in order to avoid these to reach the ship inside the bubble.[21].

-)-Energy directly above the ship($y - axis$)

$$\rho_2 = -\frac{c^2}{G}\frac{vs^2}{8\pi}\left[\left(N'(rs) + \frac{rs}{2}N''(rs)\right)^2\right] \tag{129}$$

-)-Energy directly in front of the ship($x - axis$)

$$\rho_1 = -\frac{c^2}{G}\frac{vs^2}{8\pi}\left[3(N'(rs))^2\right] \tag{130}$$

[20] see Appendices C D and E
[21] see Appendices E and F

Applying even sample Newtonian concepts we know that positive masses always attract positive masses and negative masses always attracts negative masses[22] but in interactions between positive and negative masses one repels the other.[23]

This repulsive behavior of a negative mass or a negative mass density or a negative energy density useful to deflect hazardous incoming particles from the IM is a key ingredient to protect the ship integrity and the crew members in the scenario of a real superluminal interstellar spaceflight.

The positive energy density of the IM is 10 Joules per m^3 in dense regions and 10^{-9} Joules per m^3 in diffuse regions.However in the previous sections we arrived at following results of 10^{-55} or 10^{-38} $\frac{Joules}{meters^3}$ for the negative energy density of the Natario warp drive spacetime.

From above we can see that the results obtained for the Natario warp drive negative energy density are much lighter when compared to the IM energy density.A Natario warp drive with such negative energy density requirements would never be able to deflect incoming particles from the IM because in such warp drive the negative energy density is less denser or lighter than the energy density of the IM.

But remember again that all we did was only a mathematical demonstration of how far can we go in the reduction of the negative energy density levels by manipulating the warp factor WF.We used a large WF.Of course we dont need a WF of such magnitude.A smaller WF can still obliterate values of 10^{48} while providing a negative energy density denser of heavier than the density of the IM.

A denser of heavier Natario warp drive energy density when compared to the IM density would be able to deflect the incoming hazardous particles protecting the ship and the crew members.We elaborated an empirical formula to do so:

The two key ingredients in a superluminal interstellar spaceflight are the following ones:

- 1)-spaceship velocity
- 2)-IM density

As fast is the spaceship velocity or as denser is the IM the problem of impacts against hazardous particles becomes more and more serious.Considering velocities of about 200 times light speed enough to reach star systems at 20 light-years away from Earth the ideal amount of negative energy density would then be given by the empirical formula shown below:

$$\rho_{3+1} = -1 \times (|\rho_{IM}| \times |\frac{vs}{c}|) \tag{131}$$

In the formula above ρ_{3+1} is the desired negative energy density in the Natario warp drive $|\rho_{IM}|$ is the modulus of the IM density and finally $|\frac{vs}{c}|$ is the modulus of the Machian coefficient for the multiples of the light speed in the spaceship velocity.

[22]the product of two negative masses in the Newton Law of Gravitation is also positive
[23]a minus sign arises in the product of a positive mass by a negative mass in the Newton Law of Gravitation

The positive energy density of the IM is 10 Joules per m^3 in dense regions and 10^{-9} Joules per m^3 in diffuse regions.

Applying the empirical formula of the previous page considering a spaceship velocity of 10.000 times light speed we would get for the desired Natario warp drive negative energy density results the values of -10^6 Joules per m^3 in dense regions of IM and -10^{-4} Joules per m^3 in diffuse regions of IM.

Note that even in dense regions of the IM the corresponding Natario warp drive negative energy density in modulus is 10^{10} times lighter or 10.000.000.000(10 billion) times lighter than the density of 1 kilogram per cubic meter

From the statements pointed above we can take the following important conclusions:

- 1)-A negative energy density lighter or less denser in modulus when compared to the IM density will not have strength enough to deflect hazardous incoming IM particles
- 2)-The modulus of the negative energy density in the Natario warp drive in order to have strength enough to deflect incoming hazardous IM particles must be denser or heavier than the IM density and must exceed the density of the IM by a safe margin because although we used only hydrogen atoms in this study the IM is not only hydrogen but also contains space dust debris etc.[24] The multiplication of the IM density by the multiples of the light speed in the spaceship velocity provides this margin.[25]

[24]see Appendices L and M for the composition of the Interstellar Medium IM)

[25]see Appendix F for a real Natario warp drive in interstellar spaceflight)

12 The analysis of Chris Van Den Broeck applied to the Natario warp drive spacetime using the original Alcubierre shape function to generate the Broeck spacetime distortion

From the previous section we know that the collisions between the outermost layers of the warp bubble and the IM particles is one of the most serious problems a warp drive spaceship must solve in the first place.

Remember that for a warp bubble with a radius R of 100 meters the total surface area is $S = 4\pi R^2$ and the front of the bubble exposed to the collisions against the IM particles have a surface area[26] of $S = 2\pi R^2$.In this case the area exposed to collisions have multiples of 100 square meters approximately 628 square meters considering $S = 2\pi R^2$ and this is a large surface area[27] suited to be heavily bombarded by the dangerous IM particles.

Of course we are counting on the negative energy in front of the spaceship with repulsive gravitational behavior to deflect these incoming IM particles but the ideal result would be the reduction of the surface area of the bubble exposed to collisions.

Our idea is to keep the surface area of the bubble exposed to collisions microscopically small avoiding the collisions with the IM particles while at the same time expanding the spatial volume inside the bubble to a size larger enough to contains a spaceship inside.

Some years ago in 1999 Broeck appeared with exactly this idea.(see pg 3 in [10]).Broeck applied to the Alcubierre original warp drive metric spatial components a new mathematical term $B(rs)$ able to do so as shown below(see eq 3 pg 3 in [10])[28]

$$ds^2 = -dt^2 + B^2(r_s)[(dx - v_s(t)f(r_s)dt)^2 + dy^2 + dz^2]. \tag{132}$$

changing the signature from $(-,+,+,+)$ to $(+,-,-,-)$ we have:

$$ds^2 = dt^2 - B^2(r_s)[(dx - v_s(t)f(r_s)dt)^2 + dy^2 + dz^2]. \tag{133}$$

Broeck created inside the warp bubble of radius R a spatial distortion of radius R_b being R_b microscopically small when seen from outside but inside the sphere generated by this R_b a large internal volume with the size enough to contains a spaceship can easily be accommodated.(see also pg 19 in [15])[29]

Applying the Broeck mathematical term $B(rs)$ to the spatial components of the Natario warp drive equation using the signature $(+,-,-,-)$ we get the following result:[30]

$$ds^2 = dt^2 - B(rs)^2[(drs - X^{rs}dt)^2 + (rs^2)(d\theta - X^{\theta}dt)^2] \tag{134}$$

[26] the front of the bubble is exposed to the IM particles not the rear

[27] in this case we consider $\pi = 3,14$

[28] do not confuses this term $B(rs)$ with the term B used by ourselves to differentiate the Natario shape function.

[29] see Appendix J

[30] see Appendix K

The Broeck spacetime distortion generated by the term $B(rs)$ in which the external circle surface area of the distortion seen by observers in our Universe is microscopically small while at the same time the internal spherical spatial volume inside the distortion is very large able to contains a man or s spaceship is well graphically presented aa a bottle(the Broeck bottle).[31]

According to Broeck this term $B(rs)$ have the following behavior:(see pgs 3 and 4 in [10])[32]

$$B(rs) = \begin{cases} 1+\alpha & rs < R_b \\ 1 < B(rs) \leq 1+\alpha & R_b \leq rs < R_b + \Delta_b \\ 1 & rs \geq R_b + \Delta_b \end{cases} \tag{135}$$

Considering $rs = 0$ the center of the warp bubble with radius R and R_b being the microscopically small outer radius of the Broeck bottle bottleneck circle when seen from outside the bottle but still inside the warp bubble we can analyze the expression above as follows:

In the region where $rs < R_b$ well inside the Broeck bottle the value of $B(rs)$ is very large generating the large spherical internal volume of the bottle and is given by $B(rs) = 1 + \alpha$ being α arbitrarily large.Broeck chooses for α the value of 10^{17}(see pg 5 in in [10]).We choose for α the value of 10^{27} a value 10^{10} or 10 billion times higher than the original Broeck value.Note that $B(rs)$ inside the bottle possesses always the same constant value which means to say that inside the bottle the derivatives of $B(rs)$ are always zero.

In the region where $R_b \leq rs < R_b + \Delta_b$ well exactly over the Broeck bottle bottleneck external circle and its neighborhoods the value of $B(rs)$ is given by $1 < B(rs) \leq 1 + \alpha$.This is the region where $B(rs)$ decreases from $B(rs) = 1 + \alpha$ to $B(rs) = 1$ but never reaching the value of 1 and Δ_b delimitates the thickness of this region as a thin shell in the neighborhoods of the Broeck bottle bottleneck circle.In this region the derivatives of $B(rs)$ are not zero generating an energy density given by the following equation given in Geometrized Units $c = G = 1$ as follows:(see eq 11 pg 6 in [10])

$$T_{\hat{\mu}\hat{\nu}} u^{\hat{\mu}} u^{\hat{\nu}} = T^{\hat{0}\hat{0}} = \frac{1}{8\pi} \left(\frac{1}{B^4} (\partial_r B)^2 - \frac{2}{B^3} \partial_r \partial_r B - \frac{4}{B^3} \partial_r B \frac{1}{r} \right). \tag{136}$$

Finally in the region where $rs \geq R_b + \Delta_b$ well outside the Broeck bottle bottleneck circle we recover the normal space of our Universe where the value of $B(rs)$ is always 1 and hence its derivatives are again zero.

The region where the spacetime geometry is not flat is the region around the Broeck bottle bottleneck (The Broeck bottle bottleneck is the transition region between the large inner space inside the Broeck bottle where $B(rs)$ possesses the value of $B(rs) = 1 + \alpha$ and our Universe where $B(rs)$ always possesses the value of 1) which means to say the region where $R_b \leq rs < R_b + \Delta_b$ with $B(rs)$ possessing the values of $1 < B(rs) \leq 1+\alpha$ but never reaching the value of 1 according to the Broeck criteria shown above.

We are interested in the behavior of $1 \leq B(rs) \leq 1 + \alpha$ decreasing its value from $B(rs) = 1 + \alpha$ to $B(rs) = 1$ using analytical functions[33].Note that in our redefinition of the Broeck bottle bottleneck $B(rs)$ reaches the value of 1 even inside the bottleneck.

[31] see Appendices H and I

[32] see Appendix G

[33] continuous and differentiable in every point of the domain

An elegant way to generate a continuous decrease from $B(rs) = 1 + \alpha$ to $B(rs) = 1$ can be achieved if we consider a second version of the original Alcubierre shape function redefined using the Broeck bottle bottleneck circle radius R_b as follows:[34].

$$f_b(rs) = \frac{1}{2}[1 - tanh[@(rs - R_b)] \tag{137}$$

Note that in this scenario we have two original Alcubierre shape functions:the first function $f(rs)$ was defined in section 3 with a bubble radius R of $R = 100$ meters to generate the Natario shape function $N(rs)$ and also the Natario warp bubble and the second function $f_b(rs)$ defined above generates the Broeck bottle with a radius R_b being R_b the microscopically small outer radius of the Broeck bottle bottleneck circle when seen from outside the bottle but still inside the warp bubble.Remember that the following condition must always be obeyed;$R_b << R$

Broeck chooses for α the value of 10^{17}(see pg 5 in in [10]).We choose for α the value of 10^{27} a value 10^{10} or 10 billion times higher than the original Broeck value.According to Broeck a value of $\alpha = 10^{17}$ for a bottle of bottleneck outer radius $R_b = 10^{-15}$ meters in a warp bubble of radius $R = 3 \times 10^{-15}$ meters can accommodate a bottle with 200 meters of inner diameter.Our $\alpha = 10^{27}$ could perfectly well accommodate a bottle with 200 kilometers of inner diameter in the same circumstances.

For a while and for simplification of the Broeck idea we consider a warp bubble with radius $R = 100$ meters but with a bottleneck radius $R_b = 10$ meters and a large value $\alpha = 10^{27}$ able to generate a bottle with an inner diameter of 200 kilometers with a bottleneck of only 10 meters.

According with Alcubierre any function $f_b(rs)$ that gives 1 inside the bottle and 0 outside the bottle while being $1 > f_b(rs) > 0$ in the bottleneck of the bottle[35] is a valid shape function for the Broeck bottle spacetime distortion.(see eqs 6 and 7 pg 4 in [1] or top of pg 4 in [2]).

The analytical behavior of $1 \leq B(rs) \leq 1 + \alpha$ decreasing its value from $B(rs) = 1 + \alpha$ to $B(rs) = 1$ using analytical functions can easily be achieved if we adopt the following equation for the definition of $B(rs)$ using the second original Alcubierre shape function $f_b(rs)$.

$$B(rs) = 1 + \alpha f_b(rs) \tag{138}$$

Inside the Broeck bottle $f_b(rs) = 1$ and $B(rs) = 1 + \alpha$.Outside the Broeck bottle $f_b(rs) = 0$ and $B(rs) = 1$.In these regions the derivatives of $B(rs)$ are always 0 because the values of $B(rs)$ are always constant.[36]

We must examine the region where the derivatives of $B(rs)$ are not 0 due to the values of a variable $B(rs)$ as being $1 \leq B(rs) \leq 1 + \alpha$ which means to say the region where $1 > f_b(rs) > 0$ in the bottleneck of the Broeck bottle.

[34] $tanh[@(rs + R_b)] = 1$,$tanh(@R_b) = 1$ for very high values of the Alcubierre thickness parameter $@ >> |R_b|$

[35] Remember that in this case the second Alcubierre shape function $f_b(rs)$ is being used to generate the Broeck bottle not the warp bubble.The warp bubble is being generated by the Natario shape function $N(rs)$ using the first Alcubierre shape function $f(rs)$.Note that both $f(rs)$ and $f_b(rs)$ have mathematical structures that resembles each other.One structure gives 1 inside the bubble and 0 outside the bubble while the other structure gives 1 inside the bottle and 0 outside the bottle.

[36] The derivatives of $f_b(rs)$ in these regions are too much close of 0 and can be neglected.

Considering again the definition of the Broeck bottle in the following equation for $B(rs)$ using the second Alcubierre shape function $f_b(rs)$

$$B(rs) = 1 + \alpha f_b(rs) \tag{139}$$

-)-In the following numerical plots[37] we use a bottleneck radius $R_b = 10$ meters a value of $\alpha = 10^{27}$ and a value of the Alcubierre thickness parameter @ as being always @ $= 50000$

rs	$f_b(rs)$	$B(rs)$
$9,999500000000E+00$	$1,000000000000E+00$	$1,0000000000E+27$
$9,999600000000E+00$	$1,000000000000E+00$	$1,0000000000E+27$
$9,999700000000E+00$	$9,999999999999E-01$	$1,0000000000E+27$
$9,999800000000E+00$	$9,999999979388E-01$	$9,9999999794E+26$
$9,999900000000E+00$	$9,999546021313E-01$	$9,9995460213E+26$
$1,000000000000E+01$	$5,000000000000E-01$	$5,0000000000E+26$
$1,000010000000E+01$	$4,539786871155E-05$	$4,5397868712E+22$
$1,000020000000E+01$	$2,061153636657E-09$	$2,0611536367E+18$
$1,000030000000E+01$	$9,359180097590E-14$	$9,3591800976E+13$
$1,000040000000E+01$	$0,000000000000E+00$	$1,0000000000E+00$
$1,000050000000E+01$	$0,000000000000E+00$	$1,0000000000E+00$

In the numerical plot above we can see the bottleneck of the Broeck bottle.From $rs = 0$ to $rs = 9,9996$ meters well inside the bottle the values of $f_b(rs) = 1$ and $B(rs) = 10^{27}$ both values always constant.The bottleneck of the bottle starts up at $9,9997$ meters and ends up at $10,0003$ meters where the value of $f_b(rs)$ is continuously decreasing from 1 to 0 and the value of $B(rs)$ is also continuously decreasing from 10^{27} to 1.From $rs \geq 10,0004$ meters we can see the region outside the bottle where $f_b(rs) = 0$ and $B(rs) = 1$ with both values also always constant

In the region where both $f_b(rs)$ and $B(rs)$ decreases the energy density can be given by the following equation given in Geometrized Units $c = G = 1$ as follows:(see eq 11 pg 6 in [10]):[38]

$$T_{\hat{\mu}\hat{\nu}}u^{\hat{\mu}}u^{\hat{\nu}} = T^{\hat{0}\hat{0}} = \frac{1}{8\pi}\left(\frac{1}{B^4}(\partial_r B)^2 - \frac{2}{B^3}\partial_r\partial_r B - \frac{4}{B^3}\partial_r B\frac{1}{r}\right). \tag{140}$$

We must examine the behavior of the equation above in the bottleneck of the Broeck bottle to determine if the idea of a bottle with a large inner diameter of 200 kilometers and with an external bottleneck of 10 meters remains feasible.

In the equation above a large $B(rs)$ from $1 \leq B(rs) \leq 1 + \alpha$ will generate very small terms $\frac{1}{B(rs)^4}$ $\frac{2}{B(rs)^3}$ and $\frac{4}{B(rs)^3}$ therefore obliterating the values of the derivatives of $B(rs)$ resulting in a very low energy density.

[37] We are using Microsoft Excel and Oracle Open Office and both automatically rounds the calculations

[38] Remember that inside and outside the bottle the derivatives of $f_b(rs)$ in these regions are too much close of 0 and can be neglected

rs	$f_b'(rs)$	$B'(rs)$
$9,999500000000E+00$	$-1,928749847846E-017$	$-1,9287498478E+10$
$9,999600000000E+00$	$-4,248354254933E-013$	$-4,2483542549E+14$
$9,999700000000E+00$	$-9,357622967830E-009$	$-9,3576229678E+18$
$9,999800000000E+00$	$-2,061153613672E-004$	$-2,0611536137E+23$
$9,999900000000E+00$	$-4,539580772895E+000$	$-4,5395807729E+27$
$1,000000000000E+01$	$-2,500000000000E+004$	$-2,5000000000E+31$
$1,000010000000E+01$	$-4,539580774507E+000$	$-4,5395807745E+27$
$1,000020000000E+01$	$-2,061153614404E-004$	$-2,0611536144E+23$
$1,000030000000E+01$	$-9,357622971155E-009$	$-9,3576229712E+18$
$1,000040000000E+01$	$-4,248354256442E-013$	$-4,2483542564E+14$
$1,000050000000E+01$	$-1,928749848531E-017$	$-1,9287498485E+10$

In the numerical plot above we can see the first order derivatives of both $f_b(rs)$ and $B(rs)$.From $rs = 0$ to $rs = 9,9994$ meters the values of both can be neglected.At $rs = 9,9995$ meters the value of $B'(rs)$ is $B'(rs) = -1,9287498478 \times 10^{10}$ and the value of $f_b'(rs)$ is $f_b'(rs) = -1,928749847846 \times 10^{-17}$.Both reaches the maximum value at $rs = R_b = 10$ meters as being $B'(rs) = -2,5 \times 10^{31}$ and $f_b'(rs) = -2,5 \times 10^4$. The minimum values are again reached at $rs = 10,0005$ meters being $B'(rs) = -1,9287498485 \times 10^{10}$ and $f_b'(rs) = -1,928749848531 \times 10^{-17}$.For an $rs > 10,0005$ meters both values can again be neglected.

rs	$f_b''(rs)$	$B''(rs)$
$9,999500000000E+00$	$-1,92874984785E-012$	$-1,92874984785E+015$
$9,999600000000E+00$	$-4,24835425493E-008$	$-4,24835425493E+019$
$9,999700000000E+00$	$-9,35762296783E-004$	$-9,35762296783E+023$
$9,999800000000E+00$	$-2,06115360518E+001$	$-2,06115360518E+028$
$9,999900000000E+00$	$-4,53916859831E+005$	$-4,53916859831E+032$
$1,000000000000E+01$	$0,00000000000E+000$	$0,00000000000E+000$
$1,000010000000E+01$	$4,53916859992E+005$	$4,53916859992E+032$
$1,000020000000E+01$	$2,06115360591E+001$	$2,06115360591E+028$
$1,000030000000E+01$	$9,35762297115E-004$	$9,35762297115E+023$
$1,000040000000E+01$	$4,24835425644E-008$	$4,24835425644E+019$
$1,000050000000E+01$	$1,92874984853E-012$	$1,92874984853E+015$

In the numerical plot above we can see the second order derivatives of both $f_b(rs)$ and $B(rs)$.From $rs = 0$ to $rs = 9,9994$ meters the values of both can be neglected.At $rs = 9,9995$ meters the value of $B''(rs)$ is $B''(rs) = -1,92874984785 \times 10^{15}$ and the value of $f_b''(rs)$ is $f_b''(rs) = -1,92874984785 \times 10^{-12}$.Both reaches the 0 value at $rs = R_b = 10$ meters.After 10 meters the sign is inverted.The minimum values with opposite sign are again reached at $rs = 10,0005$ meters being $B''(rs) = 1,92874984853 \times 10^{15}$ and $f_b''(rs) = 1,92874984853 \times 10^{-12}$.For an $rs > 10,0005$ meters both values can again be neglected.

rs	$f_b'(rs)^2$	$B'(rs)^2$
$9,999500000000E+00$	$3,720075975566E-034$	$3,7200759756E+20$
$9,999600000000E+00$	$1,804851387541E-025$	$1,8048513875E+29$
$9,999700000000E+00$	$8,756510760807E-017$	$8,7565107608E+37$
$9,999800000000E+00$	$4,248354219152E-008$	$4,2483542192E+46$
$9,999900000000E+00$	$2,060779359363E+001$	$2,0607793594E+55$
$1,000000000000E+01$	$6,250000000000E+008$	$6,2500000000E+62$
$1,000010000000E+01$	$2,060779360828E+001$	$2,0607793608E+55$
$1,000020000000E+01$	$4,248354222171E-008$	$4,2483542222E+46$
$1,000030000000E+01$	$8,756510767029E-017$	$8,7565107670E+37$
$1,000040000000E+01$	$1,804851388823E-025$	$1,8048513888E+29$
$1,000050000000E+01$	$3,720075978209E-034$	$3,7200759782E+20$

In the numerical plot above we can see the squares of the first order derivatives of both $f_b(rs)$ and $B(rs)$.From $rs = 0$ to $rs = 9,9994$ meters the values of both can be neglected.At $rs = 9,9995$ meters the value of $B'(rs)^2$ is $B'(rs)^2 = 3,7200759756 \times 10^{20}$ and the value of $f_b'(rs)^2$ is $f_b'(rs)^2 = 3,720075975566 \times 10^{-34}$.Both reaches the maximum value at $rs = R_b = 10$ meters as being $B'(rs)^2 = 6,25 \times 10^{62}$ and $f_b'(rs)^2 = 6,25 \times 10^8$. The minimum values are again reached at $rs = 10,0005$ meters being $B'(rs)^2 = 3,7200759782 \times 10^{20}$ and $f_b'(rs)^2 = 3,720075978209 \times 10^{-34}$.For an $rs > 10,0005$ meters both values can again be neglected.

We defined in this section the second Alcubierre shape function $f_b(rs)$ that generates the Broeck bottle as being:

$$f_b(rs) = \frac{1}{2}[1 - tanh[@(rs - R_b)] \tag{141}$$

And in section 3 the first Alcubierre shape function $f(rs)$ that generates the Natario shape function and the Natario warp bubble as being:

$$f(rs) = \frac{1}{2}[1 - tanh[@(rs - R)] \tag{142}$$

Note that in the numerical plot above when rs reaches the bottleneck radius $R_b = 10$ meters the square of the first order derivative of $f_b(rs)$ becomes equal to $f_b'(rs)^2 = 6,25 \times 10^8$ and from the numerical plots in sections 3 and 4 we know that the square derivative of first order of $f(rs)$ is also $f'(rs)^2 = 6,25 \times 10^8$ when rs reaches the bubble radius $R = 100$ meters.This is not a coincidence and depends on the way we define both Alcubierre shape functions in terms of R_b and R.Having a bottleneck radius $R_b = 10$ meters inside a warp bubble radius of $R = 100$ meters to accommodate a Broeck bottle of 200 kilometers of inner diameter or having a bottleneck radius $R_b = 10^{-15}$ meters inside a warp bubble radius of $R = 3 \times 10^{-15}$ meters to accommodate a Broeck bottle of 200 kilometers of inner diameter the derivatives of both $f_b(rs)$ and $f(rs)$ retains the same values when rs reaches the value of R_b or R.But a warp bubble with a radius of $R = 3 \times 10^{-15}$ meters would have a surface area with a magnitude order of about 10^{-15} square meters many times smaller than the area of 628 square meters thereby reducing the area exposed to collisions against the dangerous IM particles effectively protecting the ship and the crew members.

We already know that the energy density in the Broeck bottle bottleneck is given by the following equation given in Geometrized Units $c = G = 1$ as follows:(see eq 11 pg 6 in [10]):

$$T_{\hat{\mu}\hat{\nu}}u^{\hat{\mu}}u^{\hat{\nu}} = T^{\hat{0}\hat{0}} = \frac{1}{8\pi}\left(\frac{1}{B^4}(\partial_r B)^2 - \frac{2}{B^3}\partial_r\partial_r B - \frac{4}{B^3}\partial_r B\frac{1}{r}\right). \tag{143}$$

In the equation above a large $B(rs)$ from $1 \leq B(rs) \leq 1+\alpha$ will generate very small terms $\frac{1}{B(rs)^4}$ $\frac{2}{B(rs)^3}$ and $\frac{4}{B(rs)^3}$ therefore obliterating the values of the derivatives of $B(rs)$ resulting in a very low energy density.Also the term r above is our term rs

rs	$\frac{1}{B^4}(\partial_r B)^2$	$\frac{2}{B^3}\partial_r\partial_r B$	$\frac{4}{B^3}\partial_r B\frac{1}{r}$
$9,999500000000E+00$	$3,7200759756E-88$	$-3,85749969569E-066$	$-7,7153851606E-72$
$9,999600000000E+00$	$1,8048513875E-79$	$-8,49670850987E-062$	$-1,6994096784E-67$
$9,999700000000E+00$	$8,7565107608E-71$	$-1,87152459357E-057$	$-3,7431614820E-63$
$9,999800000000E+00$	$4,2483542192E-62$	$-4,12230721035E-053$	$-8,2447793503E-59$
$9,999900000000E+00$	$2,0607793594E-53$	$-9,07833719662E-049$	$-1,8158504677E-54$
$1,000000000000E+01$	$6,2500000000E-46$	$0,00000000000E+000$	$-1,0000000000E-50$
$1,000010000000E+01$	$2,0607793608E-53$	$9,07833719985E-049$	$-1,8158141517E-54$
$1,000020000000E+01$	$4,2483542222E-62$	$4,12230721181E-053$	$-8,2444495686E-59$
$1,000030000000E+01$	$8,7565107670E-71$	$1,87152459423E-057$	$-3,7429369004E-63$
$1,000040000000E+01$	$1,8048513888E-79$	$8,49670851288E-062$	$-1,6992737316E-67$
$1,000050000000E+01$	$3,7200759782E-88$	$3,85749969706E-066$	$-7,7146136634E-72$

In the numerical plot above the terms $\frac{1}{B^4}(\partial_r B)^2$ $\frac{2}{B^3}\partial_r\partial_r B$ and $\frac{4}{B^3}\partial_r B\frac{1}{r}$ are being shown individually in the Broeck bottle bottleneck.All the values are too low as expected due to the fractions with powers of $B(rs)$.The highest values are $6,25 \times 10^{-46}$ for the term $\frac{1}{B^4}(\partial_r B)^2$ and -1×10^{-50} for the term $\frac{4}{B^3}\partial_r B\frac{1}{r}$ over the bottleneck radius $R_b = 10$ meters when $rs = R_b$.

Considering a bottleneck radius of $R_b = 10^{-15}$ meters when $rs = R_b$ the term $\frac{1}{B^4}(\partial_r B)^2$ retains the same value but the term $\frac{4}{B^3}\partial_r B\frac{1}{r}$ achieves a new value of -1×10^{-34} due to the fraction $\frac{1}{rs} = 1 \times 10^{15}$ when $rs = R_b = 10^{-15}$ being multiplied by the term $\frac{4}{B^3}\partial_r B$ increasing of course the value of $\frac{4}{B^3}\partial_r B\frac{1}{r}$ while for a bottleneck radius $R_b = 10$ meters when $rs = R_b$ the fraction $\frac{1}{rs} = 1 \times 10^{-1}$ is being multiplied by the term $\frac{4}{B^3}\partial_r B$ decreasing of course the value of $\frac{4}{B^3}\partial_r B\frac{1}{r}$

Since the energy density equation uses the term $-\frac{4}{B^3}\partial_r B\frac{1}{r}$ then when $rs = R_b = 10$ meters the value of this term is 1×10^{-50} and the dominant term in the equation becomes the term $\frac{1}{B^4}(\partial_r B)^2$ with a value of $6,25 \times 10^{-46}$ because in this case $\frac{1}{B^4}(\partial_r B)^2$ is greater than $-\frac{4}{B^3}\partial_r B\frac{1}{r}$ but when $rs = R_b = 10^{-15}$ meters the term 1×10^{-34} becomes the dominant term because in this case $\frac{1}{B^4}(\partial_r B)^2$ becomes smaller than $-\frac{4}{B^3}\partial_r B\frac{1}{r}$.By changing the values of the bottleneck radius R_b from $R_b = 10$ meters to $R_b = 10^{-15}$ meters we change the dominant terms in the equation from $\frac{1}{B^4}(\partial_r B)^2$ to $-\frac{4}{B^3}\partial_r B\frac{1}{r}$.

The term $\frac{2}{B^3}\partial_r\partial_r B$ have very low negative values for $rs < 9,9995$ meters(and very low positive values for $rs > 10,0005$ meters).It starts to grow as rs when $rs < R_b$ approaches R_b becoming 0 when $rs = R_b$ and inverts the signal for $rs > R_b$

In order to compute the value of the energy density in the Broeck bottle bottleneck we must evaluate numerically the following expression:

$$arg = \left(\frac{1}{B^4} (\partial_r B)^2 - \frac{2}{B^3} \partial_r \partial_r B - \frac{4}{B^3} \partial_r B \frac{1}{r} \right). \tag{144}$$

rs	arg
$9,999500000000E+00$	$3,85750741108E-066$
$9,999600000000E+00$	$8,49672550396E-062$
$9,999700000000E+00$	$1,87152833673E-057$
$9,999800000000E+00$	$4,12231545938E-053$
$9,999900000000E+00$	$9,07856143306E-049$
$1,000000000000E+01$	$6,25010000000E-046$
$1,000010000000E+01$	$-9,07811296377E-049$
$1,000020000000E+01$	$-4,12229896312E-053$
$1,000030000000E+01$	$-1,87152085129E-057$
$1,000040000000E+01$	$-8,49669152015E-062$
$1,000050000000E+01$	$-3,85749198245E-066$

In the numerical plot above the term arg that allows ourselves to compute the energy density in the Broeck bottle bottleneck have a set of numerical values plotted around a bottleneck radius $R_b = 10$ meters.Considering the terms $\frac{1}{B^4}(\partial_r B)^2$ $\frac{2}{B^3}\partial_r\partial_r B$ and $\frac{4}{B^3}\partial_r B\frac{1}{r}$ from the plot of the previous page and considering the following powers of 10 also from the plot of the previous page as being 10^{-66} 10^{-62} 10^{-57} 10^{-53} and 10^{-49} when $rs < R_b$ we can see that the dominant term in the expression for arg is $\frac{2}{B^3}\partial_r\partial_r B$ because in this region $\frac{1}{B^4}(\partial_r B)^2$ and $\frac{4}{B^3}\partial_r B\frac{1}{r}$ have smaller values.Remember that in the plots of the previous page both $\frac{2}{B^3}\partial_r\partial_r B$ and $\frac{4}{B^3}\partial_r B\frac{1}{r}$ have negative values when $rs < R_b$ but the expression for arg uses the terms $-\frac{2}{B^3}\partial_r\partial_r B$ and $-\frac{4}{B^3}\partial_r B\frac{1}{r}$ so the negative values for $\frac{2}{B^3}\partial_r\partial_r B$ and $\frac{4}{B^3}\partial_r B\frac{1}{r}$ from the numerical plot of the previous page becomes positive in the expression for arg being the value of $-\frac{2}{B^3}\partial_r\partial_r B$ the largest of all.

On the other hand when $rs > R_b$ and considering again the powers of 10 as 10^{-66} 10^{-62} 10^{-57} 10^{-53} and 10^{-49} also from the plot in the previous page we can see that even in this region the dominant term for the expression of arg is still $\frac{2}{B^3}\partial_r\partial_r B$ because in this region $\frac{2}{B^3}\partial_r\partial_r B$ have positive values in the plot of the previous page larger than the values of $\frac{1}{B^4}(\partial_r B)^2$ and the expression for arg uses the term $-\frac{4}{B^3}\partial_r B\frac{1}{r}$ so again the negative values of $\frac{4}{B^3}\partial_r B\frac{1}{r}$ becomes again positive.Also and again following the plots from the previous page in the region where $rs > R_b$ the values of the term $\frac{2}{B^3}\partial_r\partial_r B$ becomes the largest of all and since the expression for arg. uses the term $-\frac{2}{B^3}\partial_r\partial_r B$ this is the reason why in the region where $rs > R_b$ the term arg possesses negative values.

When $rs = R_b$ from the numerical plot of the previous page the term $\frac{2}{B^3}\partial_r\partial_r B$ is 0 and the dominant term becomes $\frac{1}{B^4}(\partial_r B)^2$ with a value of $6,25 \times 10^{-46}$.The term $\frac{4}{B^3}\partial_r B\frac{1}{r}$ have a value of -1×10^{-50} but the expression for arg uses the term $-\frac{4}{B^3}\partial_r B\frac{1}{r}$ so the value becomes $-\frac{4}{B^3}\partial_r B\frac{1}{r} = 1 \times 10^{-50}$ and this value is added to $6,25 \times 10^{-46}$ giving the final result of $6,2501 \times 10^{-46}$ for the numerical plot of arg shown above as the highest value of the plot.

Note that from the numerical plots of the two previous pages we have two regions one with positive energy density $rs \leq R_b$ and another with negative energy density $rs > R_b$. This result was of course expected since Broeck in abs and pg 6 of [10] mentions positive and negative energy densities.

However we used the original Alcubierre shape function to generate our version of the Broeck bottle while Broeck himself used a different function to generate the original Broeck bottle so our results cannot be exactly equal to the Broeck ones because the bottles are different.We borrowed the Broeck idea of the Broeck bottle but we redefined the definition of the Broeck bottle using the original Alcubierre shape function in order to get better results..

In the top of pg 7 in [10] Broeck reinstates the factor $\frac{c^2}{G}$ to get the total amount of energy in SI units and Broeck arrived in eq 16 at a result of $4,9 \times 10^{30}$ kilograms a value in magnitude comparable to the mass of the Sun[39] which of course is impossible to be artificially generated.

Considering the factor $\frac{c^2}{G}$ as being $\frac{9\times10^{16}}{6,67\times10^{-11}}$ and working only with the powers of 10 we would get $\frac{10^{16}}{10^{-11}}$ giving the final result of 10^{27} a value 1000 times bigger in magnitude than the mass of the Earth which is about 10^{24} kilograms [40] and a factor of 10^{27} in an energy density equation is of course impossible to be generated artificially.

Fortunately our results looks better and promising.We know that the energy density equation of the Broeck bottle is given by the following equation given in Geometrized Units $c = G = 1$ as follows:(see eq 11 pg 6 in [10]):

$$T_{\hat{\mu}\hat{\nu}}u^{\hat{\mu}}u^{\hat{\nu}} = T^{\hat{0}\hat{0}} = \frac{1}{8\pi}\left(\frac{1}{B^4}(\partial_r B)^2 - \frac{2}{B^3}\partial_r\partial_r B - \frac{4}{B^3}\partial_r B\frac{1}{r}\right). \tag{145}$$

Or better:

$$T_{\hat{\mu}\hat{\nu}}u^{\hat{\mu}}u^{\hat{\nu}} = T^{\hat{0}\hat{0}} = \frac{1}{8\pi}arg \tag{146}$$

With the arg term being:

$$arg = \left(\frac{1}{B^4}(\partial_r B)^2 - \frac{2}{B^3}\partial_r\partial_r B - \frac{4}{B^3}\partial_r B\frac{1}{r}\right). \tag{147}$$

The equation with the factor $\frac{c^2}{G}$ would then be:

$$T_{\hat{\mu}\hat{\nu}}u^{\hat{\mu}}u^{\hat{\nu}} = T^{\hat{0}\hat{0}} = \frac{1}{8\pi}\frac{c^2}{G}arg \tag{148}$$

For the sake of simplicity we neglect also the factor $\frac{1}{8\pi}$ and work only with powers of 10.The maximum value of the term arg in the numerical plot of the previous page lies over the bottleneck radius R_b and have a value of about 10^{-46} which can of course obliterate the factor 10^{27} and $10^{-46} \times 10^{27} = 10^{-19}\ \frac{Joules}{meter^3}$ an extremely low value for the energy density in the Broeck bottle bottleneck considering that a density of one kilogram per cubic meter of space would mean a density of $9 \times 10^{16}\ \frac{Joules}{meter^3}$.

[39]see Wikipedia the Free Encyclopedia
[40]see Wikipedia the Free Encyclopedia

Considering also the already mentioned powers of 10 from the last numerical plot 10^{-66} 10^{-62} 10^{-57} 10^{-53} and 10^{-49} we can see that each one of these powers can also obliterate the factor 10^{27} giving even lower values when compared to 9×10^{16} $\frac{Joules}{meter^3}$.

In pg 553(a) or pg 543(b) in [14] we can see that the conversion factor from Geometrized Units to SI Units is actually $\frac{c^4}{G}$ and not $\frac{c^2}{G}$ being $\frac{c^4}{G}$ in powers of 10 equal to $\frac{10^{32}}{10^{-11}} = 10^{43}$ and $10^{-46} \times 10^{43} = 10^{-3}$ $\frac{Joules}{meter^3}$ still an extremely low value for the energy density in the Broeck bottle bottleneck.

Remember that we presented these numerical plots for a Broexk bottle bottleneck radius $R_b = 10$ meters.Considering a bottleneck radius of $R_b = 10^{-15}$ meters we already know that the dominant term in the expression for arg becomes $-\frac{4}{B^3}\partial_r B \frac{1}{r}$ with a maximum value of $1,00000000001 \times 10^{-34}$ when $rs = R_b$ and $10^{-34} \times 10^{43} = 10^9$ $\frac{Joules}{meter^3}$ still an extremely low value for the energy density in the Broeck bottle bottleneck.

We have seen so far that a Broeck bottle with a very small bottleneck outer radius $R_b = 10^{-15}$ meters with a parameter $\alpha = 10^{27}$ can easily accommodate a bottle with a large inner radius of 200 kilometers with an extremely low energy density needed to sustain the bottle.

Reviewing the case of the Natario warp drive in a real $3+1$ spacetime seen in section 10 where the negative energy density in SI Units is given by the following expression

$$\rho_{3+1} = -\frac{c^2}{G}\frac{vs^2}{8\pi}\left[3(N'(rs))^2\cos^2\theta\right] - \frac{c^2}{G}\frac{vs^2}{8\pi}\left[\left(N'(rs) + \frac{rs}{2}N''(rs)\right)^2 \sin^2\theta\right] \tag{149}$$

For a warp bubble radius $R = 100$ meters the value of $N'(rs))^2$ is $3,8725919148493 \times 10^{-103}$ as seen in section 4 giving a negative energy density of $10^{-103} \times 10^{48} = 10^{-55}$ $\frac{Joules}{meter^3}$ for 200 times light speed and the value of $\left(N'(rs) + \frac{rs}{2}N''(rs)\right)^2$ is $9,5849070261 \times 10^{-86}$ as seen in section 10 giving a negative energy density of $10^{-86} \times 10^{48} = 10^{-38}$ $\frac{Joules}{meter^3}$ also for 200 times light speed.

Broeck in pg 5 in [10] used an Alcubierre warp bubble with a radius of $R = 3 \times 10^{-15}$ meters and a bottle bottleneck radius $R_b = 10^{-15}$ meters.Considering a Natario warp bubble with a radius $R = 3 \times 10^{-15}$ meters the negative energy density still remains layered over the bubble radius R and when $rs = R$ the value of $N'(rs))^2$ is still the same value[41] of $3,8725919148493 \times 10^{-103}$ but the value of the term $\left(N'(rs) + \frac{rs}{2}N''(rs)\right)^2$ now becomes $9,6814219888 \times 10^{-108}$ because the term $\frac{rs}{2}N''(rs)$ now being multiplied by $rs = R = 3 \times 10^{-15}$ have lower values when compared to the same term multiplied by $rs = R = 100$ giving a negative energy of $10^{-108} \times 10^{48} = 10^{-60}$ $\frac{Joules}{meter^3}$ also for 200 times light speed. The scenario of the Broeck bottle in the case of the Natario warp drive provides two advantages:The first one is the reduction of the warp bubble radius from 100 meters to 3×10^{-15} meters and in consequence the reduction of the surface area exposed to collisions against the dangerous IM particles which is extremely useful considering large objects(eg:asteroids comets supernova remnants or debris,space dust etc).An area of 10^{-15} square meters is 10^{12} times or 100 billion times smaller than an area of a square millimeter thereby reducing the probabilities of collisions against the dangerous IM particles. The second one is the fact that a submicroscopic bubble radius reduces the amount of negative energy needed to sustain the bubble due to the term $\frac{rs}{2}N''(rs)$.Therefore any future development of the Natario warp drive should include the Broeck bottle.

[41] Because as we already have seen before the derivatives of the original Alcubierre shape function do not change its values when we switch from 10 to 10^{-15} or 100 and the Natario shape function being defined using the Alcubierre shape function retains the same behavior

13 Conclusion:

In this work we applied the geometry of the Broeck spacetime distortion(Broeck bottle) to the Natario warp drive spacetime.

We started this work with the definition of the Natario warp drive equation in the original ADM formalism and this equation is needed to be presented in this work in order to explain how the Natario spacetime geometry can receive in its structure the inclusion of the mathematical term $B(rs)$ that generates the Broeck bottle.

We used the Alcubierre shape function $f(rs)$ to define the Natario shape function counterpart $N(rs)$ using also the warp factor WF and we calculated the derivatives of the Natario shape function in order to obtain in the formulas of the derivatives the terms $1 - f(rs)$ and $f(rs)$ raised to powers of the warp factor WF.

These terms cancel each other in the derivatives of the Natario shape function except in the warp bubble radius giving a very low value for the derivatives of the Natario shape function over the bubble radius and in consequence very low values for the negative energy density.

Also we demonstrated that the negative energy density in the equatorial plane of the Natario warp bubble do not vanish and due to the gravitational repulsive behavior of the negative energy density this can provide protection against collisions with the Interstellar Medium IM that unavoidably would occur in a real superluminal spaceflight.

We discussed the Interstellar Medium IM and we arrived at the conclusion that the negative energy density of the warp bubble walls must be higher in modulus than the positive energy density of the IM in order to allow the gravitational repulsion of the IM particles by the warp bubble walls and we introduced the empirical formula to obtain the desirable amount of negative energy density needed to deflect the IM particles multiplying the modulus of the density of the IM by the Machian coefficient of the fraction $\frac{vs}{c}$ which means to say the multiples of the light speed c in the spaceship velocity vs.The negative energy density of the Natario warp drive must exceed this product in modulus.

Collisions between the walls of the warp bubble and the IM particles would certainly occur and although the negative energy density in front of the Natario warp bubble can theoretically protect the ship we borrowed the idea of Chris Van Den Broeck proposed some years ago in 1999 in order to increase the degree of protection.

Our idea is to keep the surface area of the bubble exposed to collisions microscopically small avoiding the collisions with the IM particles while at the same time expanding the spatial volume inside the bubble to a size larger enough to contains a spaceship inside.

Some years ago in 1999 Broeck appeared with exactly this idea.Broeck applied to the Alcubierre original warp drive metric spatial components a new mathematical term $B(rs)$ able to do so.

This term $B(rs)$ creates inside the Alcubierre or Natario warp bubble a spacetime distortion with the shape of a bottle in which the large inner space of the bottle volume with a large inner radius that can contains a spaceship inside the bottle is maintained isolated from the rest of the Universe and the only contact point between the bottle and the Universe is the bottle bottleneck with a microscopically small outer radius.Broeck created a bottle with 200 meters.We redefined the Broeck mathematical term $B(rs)$ using the original Alcubierre shape function in order to create a Broeck bottle with 200 kilometers of inner diameter maintaining the submicroscopic outer radius of the bottle bottleneck and a low energy density needed to create the bottle.

A submicroscopic outer radius of the bottle bottleneck being the only part in contact with our Universe would mean a submicroscopic surface exposed to the collisions against the IM particles thereby reducing the probabilities of dangerous impacts against large objects (comets asteroids etc) enhancing the protection level of the spaceship and hence the survivability of the crew members.

Any future development for the Natario warp drive must encompass the more than welcome idea of the Broeck bottle.

But unfortunately although we can discuss mathematically how to reduce the negative energy density requirements to sustain a warp drive we dont know how to generate the shape function that distorts the spacetime geometry creating the warp drive effect.So unfortunately all the discussions about warp drives are still under the domain of the mathematical conjectures.

However we are confident to affirm that the Natario-Broeck warp drive will survive the passage of the Century XXI and will arrive to the Future.The Natario-Broeck warp drive as a valid candidate for faster than light interstellar space travel will arrive to the the Century $XXIV$ on-board the future starships up there in the middle of the stars helping the human race to give his first steps in the exploration of our Galaxy

Live Long And Prosper

14 Appendix A:mathematical demonstration of the Natario warp drive equation for a constant speed vs in the original $3+1$ ADM Formalism according to MTW and Alcubierre

General Relativity describes the gravitational field in a fully covariant way using the geometrical line element of a given generic spacetime metric $ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu}$ where do not exists a clear difference between space and time.This generical form of the equations using tensor algebra is useful for differential geometry where we can handle the spacetime metric tensor $g_{\mu\nu}$ in a way that keeps both space and time integrated in the same mathematical entity (the metric tensor) and all the mathematical operations do not distinguish space from time under the context of tensor algebra handling mathematically space and time exactly in the same way.

However there are situations in which we need to recover the difference between space and time as for example the evolution in time of an astrophysical system given its initial conditions.

The $3+1$ ADM formalism allows ourselves to separate from the generic equation $ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu}$ of a given spacetime the 3 dimensions of space and the time dimension.(see pg [64(b)] [79(a)] in [12])

Consider a 3 dimensional hypersurface Σ_1 in an initial time $t1$ that evolves to a hypersurface Σ_2 in a later time $t2$ and hence evolves again to a hypersurface Σ_3 in an even later time $t3$ according to fig 2.1 pg [65(b)] [80(a)] in [12].

The hypersurface Σ_2 is considered and adjacent hypersurface with respect to the hypersurface Σ_1 that evolved in a differential amount of time dt from the hypersurface Σ_1 with respect to the initial time $t1$. Then both hypersurfeces Σ_1 and Σ_2 are the same hypersurface Σ in two different moments of time Σ_t and Σ_{t+dt}.(see bottom of pg [65(b)] [80(a)] in [12])

The geometry of the spacetime region contained between these hypersurfaces Σ_t and Σ_{t+dt} can be determined from 3 basic ingredients:(see fig 2.2 pg [66(b)] [81(a)] in [12])
(see also fig 21.2 pg [506(b)] [533(a)] in [11] where $dx^i + \beta^i dt$ appears to illustrate the equation 21.40 $g_{\mu\nu}\, dx^{\mu}\, dx^{\nu} = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$ at pg [507(b)] [534(a)] in [11])[42]

- 1)-the 3 dimensional metric $dl^2 = \gamma_{ij}dx^i dx^j$ with $i,j = 1,2,3$ that measures the proper distance between two points inside each hypersurface
- 2)-the lapse of proper time $d\tau$ between both hypersurfaces Σ_t and Σ_{t+dt} measured by observers moving in a trajectory normal to the hypersurfaces(Eulerian obsxervers) $d\tau = \alpha dt$ where α is known as the lapse function.
- 3)-the relative velocity β^i between Eulerian observers and the lines of constant spatial coordinates $(dx^i + \beta^i dt)$.β^i is known as the shift vector.

[42]we adopt the Alcubierre notation here

Combining the eqs (21.40),(21.42) and (21.44) pgs $[507, 508(b)]$ $[534, 535(a)]$ in [11] with the eqs (2.2.5) and (2.2.6) pgs $[67(b)]$ $[82(a)]$ in [12] using the signature $(-,+,+,+)$ we get the original equations of the $3+1$ ADM formalism given by the following expressions:

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0j} \\ g_{i0} & g_{ij} \end{pmatrix} = \begin{pmatrix} -\alpha^2 + \beta_k\beta^k & \beta_j \\ \beta_i & \gamma_{ij} \end{pmatrix} \tag{150}$$

$$g_{\mu\nu}\, dx^\mu\, dx^\nu = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \tag{151}$$

The components of the inverse metric are given by the matrix inverse :

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0j} \\ g^{i0} & g^{ij} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\alpha^2} & \frac{\beta^j}{\alpha^2} \\ \frac{\beta^i}{\alpha^2} & \gamma^{ij} - \frac{\beta^i\beta^j}{\alpha^2} \end{pmatrix} \tag{152}$$

The spacetime metric in $3+1$ is given by:

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \tag{153}$$

But since $dl^2 = \gamma_{ij} dx^i dx^j$ must be a diagonalized metric then $dl^2 = \gamma_{ii} dx^i dx^i$ and we have:

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ii}(dx^i + \beta^i dt)^2 \tag{154}$$

$$(dx^i + \beta^i dt)^2 = (dx^i)^2 + 2\beta^i dx^i dt + (\beta^i dt)^2 \tag{155}$$

$$\gamma_{ii}(dx^i + \beta^i dt)^2 = \gamma_{ii}(dx^i)^2 + 2\gamma_{ii}\beta^i dx^i dt + \gamma_{ii}(\beta^i dt)^2 \tag{156}$$

$$\beta_i = \gamma_{ii}\beta^i \tag{157}$$

$$\gamma_{ii}(\beta^i dt)^2 = \gamma_{ii}\beta^i\beta^i dt^2 = \beta_i\beta^i dt^2 \tag{158}$$

$$(dx^i)^2 = dx^i dx^i \tag{159}$$

$$\gamma_{ii}(dx^i + \beta^i dt)^2 = \gamma_{ii} dx^i dx^i + 2\beta_i dx^i dt + \beta_i\beta^i dt^2 \tag{160}$$

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ii} dx^i dx^i + 2\beta_i dx^i dt + \beta_i\beta^i dt^2 \tag{161}$$

$$ds^2 = (-\alpha^2 + \beta_i\beta^i)dt^2 + 2\beta_i dx^i dt + \gamma_{ii} dx^i dx^i \tag{162}$$

Note that the expression above is exactly the eq (2.2.4) pgs $[67(b)]$ $[82(a)]$ in [12].It also appears as eq 1 pg 3 in [1].

With the original equations of the $3+1$ ADM formalism given below:

$$ds^2 = (-\alpha^2 + \beta_i \beta^i)dt^2 + 2\beta_i dx^i dt + \gamma_{ii} dx^i dx^i \tag{163}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} -\alpha^2 + \beta_i \beta^i & \beta_i \\ \beta_i & \gamma_{ii} \end{pmatrix} \tag{164}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\alpha^2} & \frac{\beta^i}{\alpha^2} \\ \frac{\beta^i}{\alpha^2} & \gamma^{ii} - \frac{\beta^i \beta^i}{\alpha^2} \end{pmatrix} \tag{165}$$

and suppressing the lapse function making $\alpha = 1$ we have:

$$ds^2 = (-1 + \beta_i \beta^i)dt^2 + 2\beta_i dx^i dt + \gamma_{ii} dx^i dx^i \tag{166}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} -1 + \beta_i \beta^i & \beta_i \\ \beta_i & \gamma_{ii} \end{pmatrix} \tag{167}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \begin{pmatrix} -1 & \beta^i \\ \beta^i & \gamma^{ii} - \beta^i \beta^i \end{pmatrix} \tag{168}$$

changing the signature from $(-,+,+,+)$ to signature $(+,-,-,-)$ we have:

$$ds^2 = -(-1 + \beta_i \beta^i)dt^2 - 2\beta_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{169}$$

$$ds^2 = (1 - \beta_i \beta^i)dt^2 - 2\beta_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{170}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 - \beta_i \beta^i & -\beta_i \\ -\beta_i & -\gamma_{ii} \end{pmatrix} \tag{171}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \begin{pmatrix} 1 & -\beta^i \\ -\beta^i & -\gamma^{ii} + \beta^i \beta^i \end{pmatrix} \tag{172}$$

Remember that the equations given above corresponds to the generic warp drive metric given below:

$$ds^2 = dt^2 - \gamma_{ii}(dx^i + \beta^i dt)^2 \tag{173}$$

The warp drive spacetime according to Natario is defined by the following equation but we changed the metric signature from $(-,+,+,+)$ to $(+,-,-,-)$(pg 2 in [2])

$$ds^2 = dt^2 - \sum_{i=1}^{3}(dx^i - X^i dt)^2 \tag{174}$$

The Natario equation given above is valid only in cartezian coordinates.For a generic coordinates system we must employ the equation that obeys the $3+1$ ADM formalism:

$$ds^2 = dt^2 - \sum_{i=1}^{3}\gamma_{ii}(dx^i - X^i dt)^2 \tag{175}$$

Comparing all these equations

$$ds^2 = (1 - \beta_i \beta^i)dt^2 - 2\beta_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{176}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 - \beta_i \beta^i & -\beta_i \\ -\beta_i & -\gamma_{ii} \end{pmatrix} \tag{177}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \begin{pmatrix} 1 & -\beta^i \\ -\beta^i & -\gamma^{ii} + \beta^i \beta^i \end{pmatrix} \tag{178}$$

$$ds^2 = dt^2 - \gamma_{ii}(dx^i + \beta^i dt)^2 \tag{179}$$

With

$$ds^2 = dt^2 - \sum_{i=1}^{3} \gamma_{ii}(dx^i - X^i dt)^2 \tag{180}$$

We can see that $\beta^i = -X^i$,$\beta_i = -X_i$ and $\beta_i \beta^i = X_i X^i$ with X^i as being the contravariant form of the Natario shift vector and X_i being the covariant form of the Natario shift vector.Hence we have:

$$ds^2 = (1 - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{181}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 - X_i X^i & X_i \\ X_i & -\gamma_{ii} \end{pmatrix} \tag{182}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \begin{pmatrix} 1 & X^i \\ X^i & -\gamma^{ii} + X^i X^i \end{pmatrix} \tag{183}$$

Looking to the equation of the Natario vector nX(pg 2 and 5 in [2]):

$$nX = X^{rs} drs + X^{\theta} rs d\theta \tag{184}$$

With the contravariant shift vector components X^{rs} and X^{θ} given by:(see pg 5 in [2]):

$$X^{rs} = 2v_s n(rs) \cos\theta \tag{185}$$

$$X^{\theta} = -v_s(2n(rs) + (rs)n'(rs)) \sin\theta \tag{186}$$

But remember that $dl^2 = \gamma_{ii} dx^i dx^i = dr^2 + r^2 d\theta^2$ with $\gamma_{rr} = 1$ and $\gamma_{\theta\theta} = r^2$. Then the covariant shift vector components X_{rs} and X_{θ} with $r = rs$ are given by:

$$X_i = \gamma_{ii} X^i \tag{187}$$

$$X_r = \gamma_{rr} X^r = X_{rs} = \gamma_{rsrs} X^{rs} = 2v_s n(rs) \cos\theta = X^r = X^{rs} \tag{188}$$

$$X_{\theta} = \gamma_{\theta\theta} X^{\theta} = rs^2 X^{\theta} = -rs^2 v_s(2n(rs) + (rs)n'(rs)) \sin\theta \tag{189}$$

The equations of the Natario warp drive in the $3+1$ ADM formalism are given by:

$$ds^2 = (1 - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{190}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 - X_i X^i & X_i \\ X_i & -\gamma_{ii} \end{pmatrix} \tag{191}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \begin{pmatrix} 1 & X^i \\ X^i & -\gamma^{ii} + X^i X^i \end{pmatrix} \tag{192}$$

The matrix components 2×2 evaluated separately for rs and θ gives the following results:[43]

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0r} \\ g_{r0} & g_{rr} \end{pmatrix} = \begin{pmatrix} 1 - X_r X^r & X_r \\ X_r & -\gamma_{rr} \end{pmatrix} \tag{193}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0r} \\ g^{r0} & g^{rr} \end{pmatrix} = \begin{pmatrix} 1 & X^r \\ X^r & -\gamma^{rr} + X^r X^r \end{pmatrix} \tag{194}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0\theta} \\ g_{\theta 0} & g_{\theta\theta} \end{pmatrix} = \begin{pmatrix} 1 - X_\theta X^\theta & X_\theta \\ X_\theta & -\gamma_{\theta\theta} \end{pmatrix} \tag{195}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0\theta} \\ g^{\theta 0} & g^{\theta\theta} \end{pmatrix} = \begin{pmatrix} 1 & X^\theta \\ X^\theta & -\gamma^{\theta\theta} + X^\theta X^\theta \end{pmatrix} \tag{196}$$

Then the equation of the Natario warp drive spacetime in the original $3+1$ ADM formalism is given by:

$$ds^2 = (1 - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{197}$$

$$ds^2 = (1 - X_{rs} X^{rs} - X_\theta X^\theta)dt^2 + 2(X_{rs} drs dt + X_\theta d\theta dt) - drs^2 - rs^2 d\theta^2 \tag{198}$$

$$ds^2 = (1 - X_{rs} X^{rs} - X_\theta X^\theta)dt^2 + 2(X_{rs} drs + X_\theta d\theta)dt - drs^2 - rs^2 d\theta^2 \tag{199}$$

[43] Actually we know that the real matrix is a 3×3 matrix with dimensions t rs and θ.Our 2×2 approach is a simplification

We already know that for the Natario warp drive in a generic coordinates system we must employ the equation that obeys the $3+1$ ADM formalism:

$$ds^2 = dt^2 - \sum_{i=1}^{3} \gamma_{ii}(dx^i - X^i dt)^2 \tag{200}$$

With the contravariant shift vector components X^{rs} and X^{θ} given by:(see pg 5 in [2]):

$$X^{rs} = 2v_s n(rs) \cos\theta \tag{201}$$

$$X^{\theta} = -v_s(2n(rs) + (rs)n'(rs)) \sin\theta \tag{202}$$

But remember that $\gamma_{rr} = 1$ and $\gamma_{\theta\theta} = r^2$.Therefore the Natario warp drive equation in the original ADM formalism can be written as:

$$ds^2 = dt^2 - [(drs - X^{rs}dt)^2 + (rs^2)(d\theta - X^{\theta}dt)^2] \tag{203}$$

15 Appendix B:differential forms,Hodge star and the mathematical demonstration of the Natario vectors $nX = -vsdx$ and $nX = vsdx$ for a constant speed vs

This appendix is being written for novice or newcomer students on Warp Drive theory still not acquainted with the methods Natario used to arrive at the final expression of the Natario Vector nX

The Canonical Basis of the Hodge Star in spherical coordinates can be defined as follows(pg 4 in [2]):

$$e_r \equiv \frac{\partial}{\partial r} \sim dr \sim (rd\theta) \wedge (r\sin\theta d\varphi) \sim r^2 \sin\theta (d\theta \wedge d\varphi) \tag{204}$$

$$e_\theta \equiv \frac{1}{r}\frac{\partial}{\partial \theta} \sim rd\theta \sim (r\sin\theta d\varphi) \wedge dr \sim r\sin\theta (d\varphi \wedge dr) \tag{205}$$

$$e_\varphi \equiv \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi} \sim r\sin\theta d\varphi \sim dr \wedge (rd\theta) \sim r(dr \wedge d\theta) \tag{206}$$

From above we get the following results

$$dr \sim r^2 \sin\theta (d\theta \wedge d\varphi) \tag{207}$$

$$rd\theta \sim r\sin\theta (d\varphi \wedge dr) \tag{208}$$

$$r\sin\theta d\varphi \sim r(dr \wedge d\theta) \tag{209}$$

Note that this expression matches the common definition of the Hodge Star operator * applied to the spherical coordinates as given by(pg 8 in [4]):

$$*dr = r^2 \sin\theta (d\theta \wedge d\varphi) \tag{210}$$

$$*rd\theta = r\sin\theta (d\varphi \wedge dr) \tag{211}$$

$$*r\sin\theta d\varphi = r(dr \wedge d\theta) \tag{212}$$

Back again to the Natario equivalence between spherical and cartezian coordinates(pg 5 in [2]):

$$\frac{\partial}{\partial x} \sim dx = d(r\cos\theta) = \cos\theta dr - r\sin\theta d\theta \sim r^2 \sin\theta\cos\theta d\theta \wedge d\varphi + r\sin^2\theta dr \wedge d\varphi = d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \tag{213}$$

Look that

$$dx = d(r\cos\theta) = \cos\theta dr - r\sin\theta d\theta \tag{214}$$

Or

$$dx = d(r\cos\theta) = \cos\theta dr - \sin\theta rd\theta \tag{215}$$

Applying the Hodge Star operator * to the above expression:

$$*dx = *d(r\cos\theta) = \cos\theta(*dr) - \sin\theta(*rd\theta) \tag{216}$$

$$*dx = *d(r\cos\theta) = \cos\theta[r^2\sin\theta(d\theta\wedge d\varphi)] - \sin\theta[r\sin\theta(d\varphi\wedge dr)] \tag{217}$$

$$*dx = *d(r\cos\theta) = [r^2\sin\theta\cos\theta(d\theta\wedge d\varphi)] - [r\sin^2\theta(d\varphi\wedge dr)] \tag{218}$$

We know that the following expression holds true(see pg 9 in [3]):

$$d\varphi\wedge dr = -dr\wedge d\varphi \tag{219}$$

Then we have

$$*dx = *d(r\cos\theta) = [r^2\sin\theta\cos\theta(d\theta\wedge d\varphi)] + [r\sin^2\theta(dr\wedge d\varphi)] \tag{220}$$

And the above expression matches exactly the term obtained by Natario using the Hodge Star operator applied to the equivalence between cartezian and spherical coordinates(pg 5 in [2]).

Now examining the expression:

$$d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \tag{221}$$

We must also apply the Hodge Star operator to the expression above

And then we have:

$$*d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \tag{222}$$

$$*d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \sim \frac{1}{2}r^2 * d[(\sin^2\theta)d\varphi] + \frac{1}{2}\sin^2\theta * [d(r^2)d\varphi] + \frac{1}{2}r^2\sin^2\theta * d[(d\varphi)] \tag{223}$$

According to pg 10 in [3] the term $\frac{1}{2}r^2\sin^2\theta * d[(d\varphi)] = 0$

This leaves us with:

$$\frac{1}{2}r^2 * d[(\sin^2\theta)d\varphi] + \frac{1}{2}\sin^2\theta * [d(r^2)d\varphi] \sim \frac{1}{2}r^2 2\sin\theta\cos\theta(d\theta\wedge d\varphi) + \frac{1}{2}\sin^2\theta 2r(dr\wedge d\varphi) \tag{224}$$

Because and according to pg 10 in [3]:

$$d(\alpha+\beta) = d\alpha + d\beta \tag{225}$$

$$d(f\alpha) = df\wedge\alpha + f\wedge d\alpha \tag{226}$$

$$d(dx) = d(dy) = d(dz) = 0 \tag{227}$$

From above we can see for example that

$$*d[(\sin^2\theta)d\varphi] = d(\sin^2\theta)\wedge d\varphi + \sin^2\theta\wedge dd\varphi = 2sin\theta\cos\theta(d\theta\wedge d\varphi) \tag{228}$$

$$*[d(r^2)d\varphi] = 2rdr\wedge d\varphi + r^2\wedge dd\varphi = 2r(dr\wedge d\varphi) \tag{229}$$

And then we derived again the Natario result of pg 5 in [2]

$$r^2\sin\theta\cos\theta(d\theta\wedge d\varphi) + r\sin^2\theta(dr\wedge d\varphi) \tag{230}$$

Now we will examine the following expression equivalent to the one of Natario pg 5 in [2] except that we replaced $\frac{1}{2}$ by the function $f(r)$:

$$*d[f(r)r^2\sin^2\theta d\varphi] \tag{231}$$

From above we can obtain the next expressions

$$f(r)r^2 * d[(\sin^2\theta)d\varphi] + f(r)\sin^2\theta * [d(r^2)d\varphi] + r^2\sin^2\theta * d[f(r)d\varphi] \tag{232}$$

$$f(r)r^2 2sin\theta\cos\theta(d\theta\wedge d\varphi) + f(r)\sin^2\theta 2r(dr\wedge d\varphi) + r^2\sin^2\theta f'(r)(dr\wedge d\varphi) \tag{233}$$

$$2f(r)r^2 sin\theta\cos\theta(d\theta\wedge d\varphi) + 2f(r)r\sin^2\theta(dr\wedge d\varphi) + r^2\sin^2\theta f'(r)(dr\wedge d\varphi) \tag{234}$$

Comparing the above expressions with the Natario definitions of pg 4 in [2]):

$$e_r \equiv \frac{\partial}{\partial r} \sim dr \sim (rd\theta)\wedge(r\sin\theta d\varphi) \sim r^2\sin\theta(d\theta\wedge d\varphi) \tag{235}$$

$$e_\theta \equiv \frac{1}{r}\frac{\partial}{\partial\theta} \sim rd\theta \sim (r\sin\theta d\varphi)\wedge dr \sim r\sin\theta(d\varphi\wedge dr) \sim -r\sin\theta(dr\wedge d\varphi) \tag{236}$$

$$e_\varphi \equiv \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi} \sim r\sin\theta d\varphi \sim dr\wedge(rd\theta) \sim r(dr\wedge d\theta) \tag{237}$$

We can obtain the following result:

$$2f(r)\ cos\theta[r^2 sin\theta(d\theta\wedge d\varphi)] + 2f(r)\ sin\theta[r\sin\theta(dr\wedge d\varphi)] + f'(r)r\sin\theta[r\sin\theta(dr\wedge d\varphi)] \tag{238}$$

$$2f(r)\ cos\theta e_r - 2f(r)\ sin\theta e_\theta - rf'(r)\sin\theta e_\theta \tag{239}$$

$$*d[f(r)r^2\sin^2\theta d\varphi] = 2f(r)\ cos\theta e_r - [2f(r) + rf'(r)]\sin\theta e_\theta \tag{240}$$

Defining the Natario Vector as in pg 5 in [2] with the Hodge Star operator * explicitly written :

$$nX = vs(t) * d\left(f(r)r^2\sin^2\theta d\varphi\right) \tag{241}$$

$$nX = -vs(t) * d\left(f(r)r^2\sin^2\theta d\varphi\right) \tag{242}$$

We can get finally the latest expressions for the Natario Vector nX also shown in pg 5 in [2]

$$nX = 2vs(t)f(r)\ cos\theta e_r - vs(t)[2f(r) + rf'(r)]\sin\theta e_\theta \tag{243}$$

$$nX = -2vs(t)f(r)\ cos\theta e_r + vs(t)[2f(r) + rf'(r)]\sin\theta e_\theta \tag{244}$$

With our pedagogical approaches

$$nX = 2vs(t)f(r)\ cos\theta dr - vs(t)[2f(r) + rf'(r)]r\sin\theta d\theta \tag{245}$$

$$nX = -2vs(t)f(r)\ cos\theta dr + vs(t)[2f(r) + rf'(r)]r\sin\theta d\theta \tag{246}$$

The term r in all these equations is our term rs and the function $f(r)$ in all these equations is our Natario shape function $n(r)$ or $n(rs)$ or $N(rs)$

16 Appendix C:The Natario warp drive negative energy density in Cartezian coordinates

The negative energy density according to Natario is given by(see pg 5 in [2])[44]:

$$\rho = T_{\mu\nu}u^{\mu}u^{\nu} = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[3(n'(rs))^2\cos^2\theta + \left(n'(rs) + \frac{r}{2}n''(rs)\right)^2\sin^2\theta\right] \tag{247}$$

In the bottom of pg 4 in [2] Natario defined the x-axis as the polar axis.In the top of page 5 we can see that $x = rs\cos(\theta)$ implying in $\cos(\theta) = \frac{x}{rs}$ and in $\sin(\theta) = \frac{y}{rs}$

Rewriting the Natario negative energy density in cartezian coordinates we should expect for:

$$\rho = T_{\mu\nu}u^{\mu}u^{\nu} = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[3(n'(rs))^2(\frac{x}{rs})^2 + \left(n'(rs) + \frac{r}{2}n''(rs)\right)^2(\frac{y}{rs})^2\right] \tag{248}$$

Considering motion in the equatorial plane of the Natario warp bubble (x-axis only) then $[y^2 + z^2] = 0$ and $rs^2 = [(x - xs)^2]$ and making $xs = 0$ the center of the bubble as the origin of the coordinate frame for the motion of the Eulerian observer then $rs^2 = x^2$ because in the equatorial plane $y = z = 0$.

Rewriting the Natario negative energy density in cartezian coordinates in the equatorial plane we should expect for:

$$\rho = T_{\mu\nu}u^{\mu}u^{\nu} = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[3(n'(rs))^2\right] \tag{249}$$

[44] $n(rs)$ is the Natario shape function.Equation written in the Geometrized System of Units $c = G = 1$

17 Appendix D:Dimensional Reduction from $\frac{c^4}{G}$ to $\frac{c^2}{G}$

The Alcubierre expressions for the Negative Energy Density in Geometrized Units $c = G = 1$ are given by(pg 4 in [2])(pg 8 in [1]):[45]:

$$\rho = -\frac{1}{32\pi} vs^2 \left[f'(rs)\right]^2 [\frac{y^2+z^2}{rs^2}] \tag{250}$$

$$\rho = -\frac{1}{32\pi} vs^2 \left[\frac{df(rs)}{drs}\right]^2 [\frac{y^2+z^2}{rs^2}] \tag{251}$$

In this system all physical quantities are identified with geometrical entities such as lengths,areas or dimensionless factors.Even time is interpreted as the distance travelled by a pulse of light during that time interval,so even time is given in lengths.Energy,Momentum and Mass also have the dimensions of lengths.We can multiply a mass in kilograms by the conversion factor $\frac{G}{c^2}$ to obtain the mass equivalent in meters.On the other hand we can multiply meters by $\frac{c^2}{G}$ to obtain kilograms.The Energy Density($\frac{Joules}{meters^3}$)in Geometrized Units have a dimension of $\frac{1}{length^2}$ and the conversion factor for Energy Density is $\frac{G}{c^4}$.Again on the other hand by multiplying $\frac{1}{length^2}$ by $\frac{c^4}{G}$ we retrieve again ($\frac{Joules}{meters^3}$). [46].

This is the reason why in Geometrized Units the Einstein Tensor have the same dimension of the Stress Energy Momentum Tensor(in this case the Negative Energy Density)and since the Einstein Tensor is associated to the Curvature of Spacetime both have the dimension of $\frac{1}{length^2}$.

$$G_{00} = 8\pi T_{00} \tag{252}$$

Passing to normal units and computing the Negative Energy Density we multiply the Einstein Tensor (dimension $\frac{1}{length^2}$) by the conversion factor $\frac{c^4}{G}$ in order to retrieve the normal unit for the Negative Energy Density ($\frac{Joules}{meters^3}$).

$$T_{00} = \frac{c^4}{8\pi G} G_{00} \tag{253}$$

Examine now the Alcubierre equations:

$vs = \frac{dxs}{dt}$ is dimensionless since time is also in lengths.$\frac{y^2+z^2}{rs^2}$ is dimensionless since both are given also in lengths. $f(rs)$ is dimensionless but its derivative $\frac{df(rs)}{drs}$ is not because rs is in meters. So the dimensional factor in Geometrized Units for the Alcubierre Energy Density comes from the square of the derivative and is also $\frac{1}{length^2}$.Remember that the speed of the Warp Bubble vs is dimensionless in Geometrized Units and when we multiply directly $\frac{1}{length^2}$ from the Negative Energy Density in Geometrized Units by $\frac{c^4}{G}$ to obtain the Negative Energy Density in normal units $\frac{Joules}{meters^3}$ the first attempt would be to make the following:

$$\rho = -\frac{c^4}{G}\frac{1}{32\pi} vs^2 \left[f'(rs)\right]^2 [\frac{y^2+z^2}{rs^2}] \tag{254}$$

$$\rho = -\frac{c^4}{G}\frac{1}{32\pi} vs^2 \left[\frac{df(rs)}{drs}\right]^2 [\frac{y^2+z^2}{rs^2}] \tag{255}$$

[45]See Geometrized Units in Wikipedia

[46]See Conversion Factors for Geometrized Units in Wikipedia

But note that in normal units vs is not dimensionless and the equations above do not lead to the correct dimensionality of the Negative Energy Density because the equations above in normal units are being affected by the dimensionality of vs.

In order to make vs dimensionless again,the Negative Energy Density is written as follows:

$$\rho = -\frac{c^4}{G}\frac{1}{32\pi}(\frac{vs}{c})^2\left[f'(rs)\right]^2[\frac{y^2+z^2}{rs^2}] \tag{256}$$

$$\rho = -\frac{c^4}{G}\frac{1}{32\pi}(\frac{vs}{c})^2\left[\frac{df(rs)}{drs}\right]^2[\frac{y^2+z^2}{rs^2}] \tag{257}$$

Giving:

$$\rho = -\frac{c^2}{G}\frac{1}{32\pi}vs^2\left[f'(rs)\right]^2[\frac{y^2+z^2}{rs^2}] \tag{258}$$

$$\rho = -\frac{c^2}{G}\frac{1}{32\pi}vs^2\left[\frac{df(rs)}{drs}\right]^2[\frac{y^2+z^2}{rs^2}] \tag{259}$$

As already seen.The same results are valid for the Natario Energy Density

Note that from

$$\rho = -\frac{c^4}{G}\frac{1}{32\pi}(\frac{vs}{c})^2\left[f'(rs)\right]^2[\frac{y^2+z^2}{rs^2}] \tag{260}$$

$$\rho = -\frac{c^4}{G}\frac{1}{32\pi}(\frac{vs}{c})^2\left[\frac{df(rs)}{drs}\right]^2[\frac{y^2+z^2}{rs^2}] \tag{261}$$

Making $c = G = 1$ we retrieve again

$$\rho = -\frac{1}{32\pi}vs^2\left[f'(rs)\right]^2[\frac{y^2+z^2}{rs^2}] \tag{262}$$

$$\rho = -\frac{1}{32\pi}vs^2\left[\frac{df(rs)}{drs}\right]^2[\frac{y^2+z^2}{rs^2}] \tag{263}$$

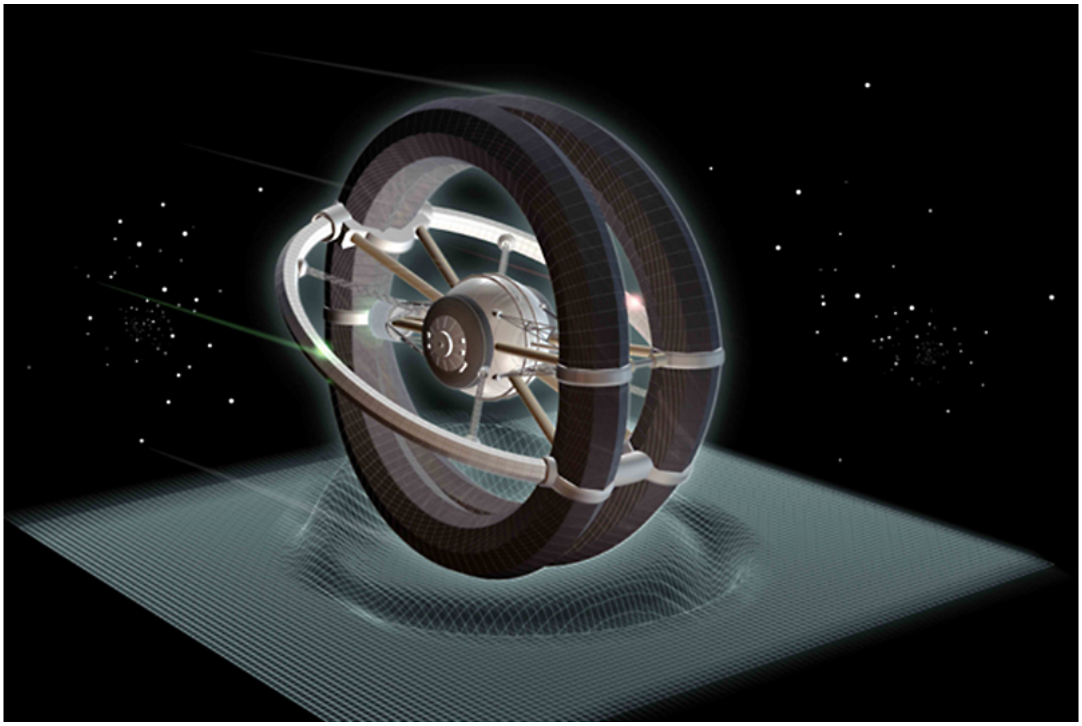

Figure 1: Artistic representation of the Natario warp drive .Note in the bottom of the figure the Alcubierre expansion of the normal volume elements .(Source:Internet)

18 Appendix E:Artistic Presentation of the Natario warp drive

According to the geometry of the Natario warp drive the spacetime contraction in one direction(radial) is balanced by the spacetime expansion in the remaining direction(perpendicular).(pg 5 in [2]).

The expansion of the normal volume elements in the Natario warp drive is given by the following expressions(pg 5 in [2]).

$$K_{rr} = \frac{\partial X^r}{\partial r} = -2v_s n'(r)\cos\theta \tag{264}$$

$$K_{\theta\theta} = \frac{1}{r}\frac{\partial X^\theta}{\partial\theta} + \frac{X^r}{r} = v_s n'(r)\cos\theta; \tag{265}$$

$$K_{\varphi\varphi} = \frac{1}{r\sin\theta}\frac{\partial X^\varphi}{\partial\varphi} + \frac{X^r}{r} + \frac{X^\theta\cot\theta}{r} = v_s n'(r)\cos\theta \tag{266}$$

$$\theta = K_{rr} + K_{\theta\theta} + K_{\varphi\varphi} = 0 \tag{267}$$

If we expand the radial direction the perpendicular direction contracts to keep the expansion of the normal volume elements equal to zero.

This figure is a pedagogical example of the graphical presentarion of the Natario warp drive.

The "bars" in the figure were included to illustrate how the expansion in one direction can be counter-balanced by the contraction in the other directions.These "bars" keeps the expansion of the normal volume elements in the Natario warp drive equal to zero.

Note also that the graphical presentation of the Alcubierre warp drive expansion of the normal volume elements according to fig 1 pg 10 in [1] is also included

Note also that the energy density in the Natario warp drive $3+1$ spacetime being given by the following expressions(pg 5 in [2]):

$$\rho = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[3(n'(r))^2\cos^2\theta + \left(n'(r) + \frac{r}{2}n''(r)\right)^2\sin^2\theta\right]. \tag{268}$$

$$\rho = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[3(\frac{dn(r)}{dr})^2\cos^2\theta + \left(\frac{dn(r)}{dr} + \frac{r}{2}\frac{d^2n(r)}{dr^2}\right)^2\sin^2\theta\right]. \tag{269}$$

Is being distributed around all the space involving the ship(above the ship $\sin\theta = 1$ and $\cos\theta = 0$ while in front of the ship $\sin\theta = 0$ and $\cos\theta = 1$).The negative energy in front of the ship "deflect" photons or other particles so these will not reach the ship inside the bubble.The illustrated "bars" are the obstacles that deflects photons or incoming particles from outside the bubble never allowing these to reach the interior of the bubble.[47]

-)-Energy directly above the ship($y - axis$)

$$\rho = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[\left(\frac{dn(r)}{dr} + \frac{r}{2}\frac{d^2n(r)}{dr^2}\right)^2\sin^2\theta\right]. \tag{270}$$

-)-Energy directly in front of the ship($x - axis$)

$$\rho = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[3(\frac{dn(r)}{dr})^2\cos^2\theta\right]. \tag{271}$$

Note also that even in a $1+1$ dimensional spacetime the Natario warp drive retains the zero expansion behavior:

$$K_{rr} = \frac{\partial X^r}{\partial r} = -2v_s n'(r)\cos\theta \tag{272}$$

$$K_{\theta\theta} = \frac{X^r}{r} = v_s n'(r)\cos\theta; \tag{273}$$

$$K_{\varphi\varphi} = \frac{X^r}{r} = v_s n'(r)\cos\theta \tag{274}$$

$$\theta = K_{rr} + K_{\theta\theta} + K_{\varphi\varphi} = 0 \tag{275}$$

In all these equations the term r is our term rs

[47]See also Appendix F

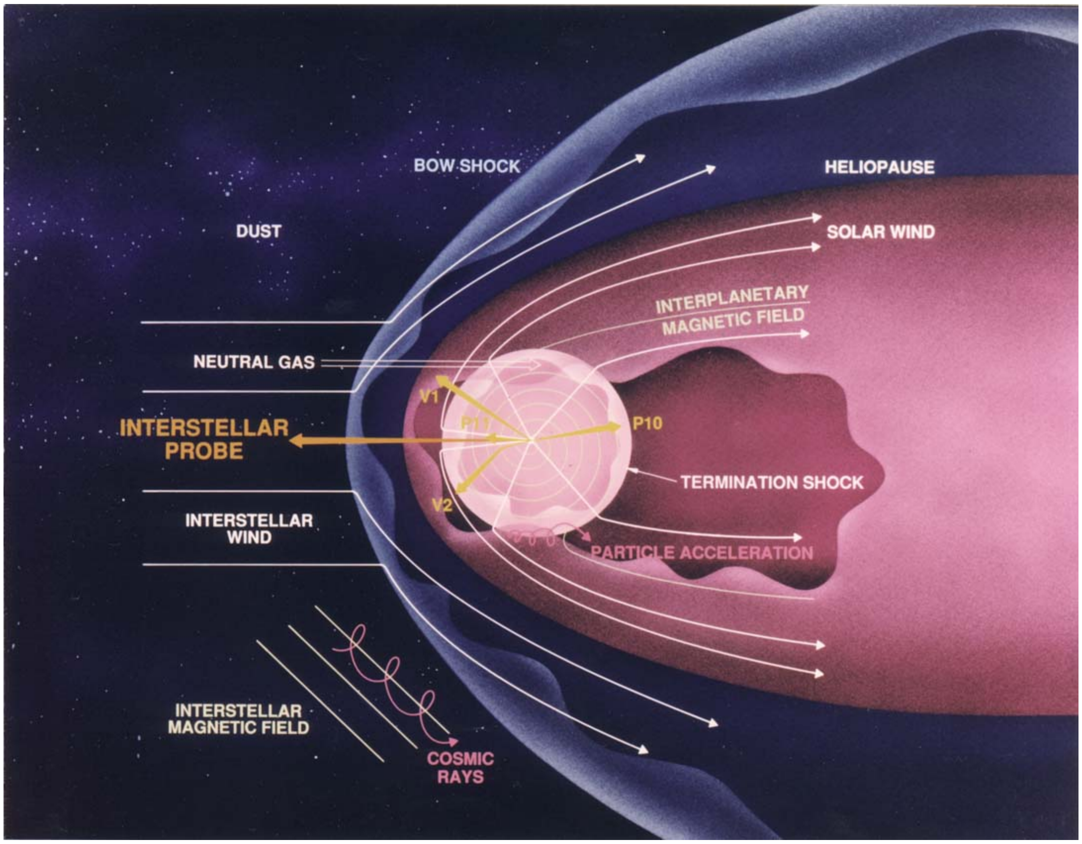

Figure 2: Artistic representation of a Natario warp wrive in a real superluminal space travel .Note the negative energy in front of the ship deflecting incoming hazardous interstellar matter(brown arrows).(Source:Internet)

19 Appendix F:Artistic Presentation of a Natario warp drive in a real faster than light interstellar spaceflight

Above is being presented the artistic presentation of a Natario warp drive in a real interstellar superluminal travel.The "ball" or the spherical shape is the Natario warp bubble with the negative energy surrounding the ship in all directions and mainly protecting the front of the bubble.[48]

The brown arrows in the front of the Natario bubble are a graphical presentation of the negative energy in front of the ship deflecting interstellar dust,neutral gases,hydrogen atoms,interstellar wind photons etc.[49]

The spaceship is at the rest and in complete safety inside the Natario bubble.

In order to allow to the negative energy density of the Natario warp drive the deflection of incoming hazardous particles from the Interstellar Medium(IM) the Natario warp drive energy density must be heavier or denser when compared to the IM density.

[48] See Appendix E

[49] see Appendices L and M for the composition of the Interstellar Medium IM)

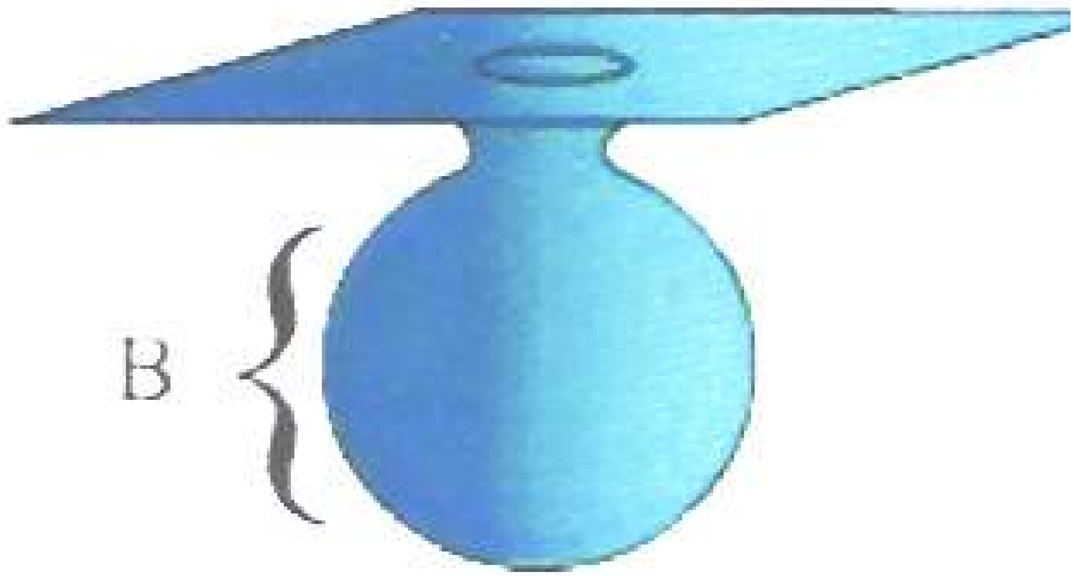

Figure 3: Artistic representation of the Broeck "pocket" or "bottle" with the Broeck coefficient $B(rs)$ shown.(Source:Internet)

20 Appendix G:Artistic Presentation of the Broeck "pocket" or "bottle"

Broeck proposed the idea to keep the surface area of the bubble microscopically small while at the same time expanding the spatial volume inside the bubble to a size larger enough to contains a spaceship inside. (see pg 3 in [10]).The "ball" in the figure above with a large internal volume is the Broeck bottle and the circle of the intersection point between the "ball" and the plane also shown in the figure is the circle of the small surface area(Broeck bottle bottleneck).Broeck created the term $B(rs)$ in order to accomplish this task. According to Broeck this term $B(rs)$ have the following behavior:(see pgs 3 and 4 in [10])

$$B(rs) = \begin{cases} 1+\alpha & rs < R_b \\ 1 < B(rs) \leq 1+\alpha & R_b \leq rs < R_b + \Delta_b \\ 1 & rs \geq R_b + \Delta_b \end{cases} \tag{276}$$

In the equation above the small outer radius R_b is the radius of the shown circle of the Broeck bottle bottleneck.This circle intersects the plane above the Broeck bottle and the plane represents our Universe.The term α according to Broeck have a large value of 10^{17}(pg 5 in [10]).We consider in this work a value of 10^{27}.

Considering the center $rs = 0$ of the bottleneck circle delimitated by the small outer radius R_b any point placed at a distance $rs < R_b$ is a point inside the Broeck bottle $B(rs) = 1+\alpha$ being α the term that generates the large internal volume of the Broeck bottle.

In the region $R_b \leq rs < R_b + \Delta_b$ the value of $B(rs)$ becomes $1 < B(rs) \leq 1 + \alpha$.This region is in the neighborhoods of the small outer radius R_b and is the region where $B(rs)$ decreases from the large value of $B(rs) = 1 + \alpha$ approaching the value of $B(rs) = 1$ but never reaching it.

The term Δ_b delimitates the thickness of the region where $B(rs)$ decreases.This region is a thin shell around the Broeck bottle bottleneck.

Finally in the region where $rs \geq R_b + \Delta_b$ far outside the Broeck bottle bottleneck we recover the normal space of our Universe (the plane above the Broeck bottle) in which $B(rs)$ always possesses the value of $B(rs) = 1$.

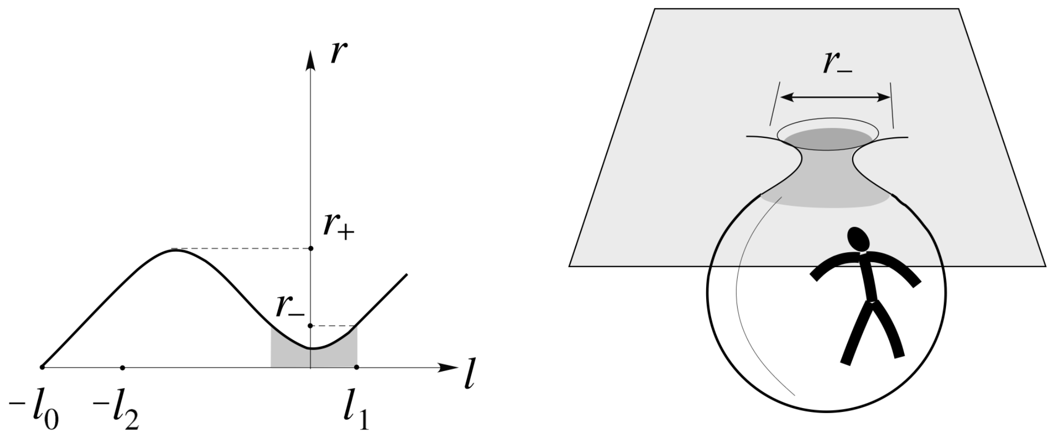

Figure 4: Alternative artistic representation of the Broeck "pocket" or "bottle" with a man shown inside the bottle. (Source:fig 5 pg 19 in [15])

21 Appendix H:Alternative Artistic Presentation of the Broeck "pocket" or "bottle"

The figure shown above represents exactly the point of view we are defending concerning the whole Broeck idea applied to the Natario warp drive in order to reduce the surface area exposed to collisions against the IM particles.

A Broeck bottle with a large internal radius r_+ large enough to contains a man os a spaceship is being graphically depicted.

This bottle intersects the bidimensional plane in the circle delimited by the outer radius r_- being this radius microscopically small.This circle is the bottleneck of the Broeck bottle.

The bidimensional plane represents our Universe and all the dangerous IM particles are contained only in this plane.

Therefore a Broeck bottle a sphere of a large internal radius r_+ able to accommodate a man or a spaceship would be seen by outside observers placed in the bidimensional plane representing our Universe as a circle with a microscopically small outer radius r_- being this circle the bottleneck of the Broeck bottle.(see pg 19 in [15]).

A microscopically small outer radius r_- the R_b in our equations delimitates a very small microscopically surface area therefore reducing the probability of collisions against the dangerous IM particles

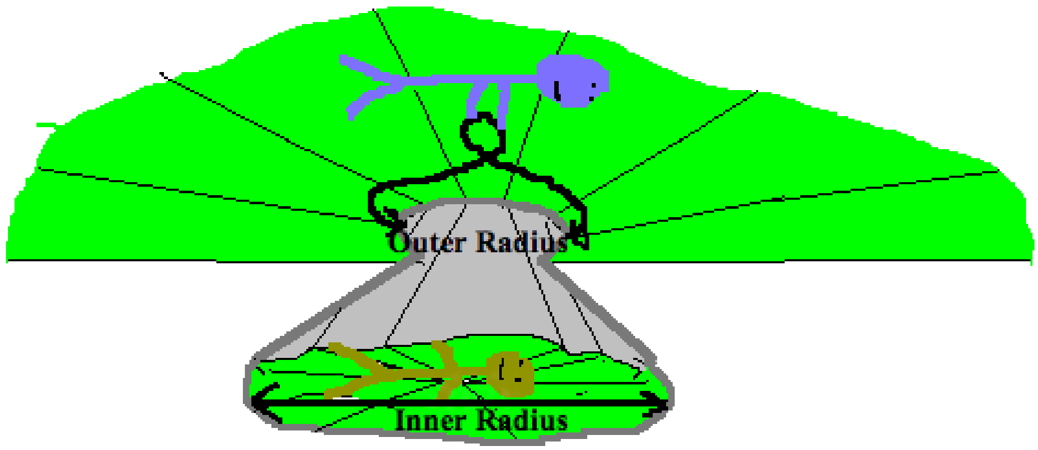

Figure 5: Alternative artistic representation of the Broeck "pocket" or "bottle" with both the inner and outer radius of the bottle shown .(Source:Internet)

22 Appendix I:Alternative Artistic Presentation of the Broeck "pocket" or "bottle"

The figure shown above also represents exactly the point of view we are defending concerning the whole Broeck idea applied to the Natario warp drive in order to reduce the surface area exposed to collisions against the IM particles.

The Broeck bottle with a large internal radius(inner radius) r_+ large enough to contains a man(the brown man) inside the bottle is depicted.

The microscopically small outer radius r_- delimitates the circle surface(bottleneck of the bottle) of the intersection points between the Btoeck bottle and out external Universe(the plane above the bottle where the blue man is placed)

The internal radius(inner radius) r_+ is much larger than the microscopically small outer radius r_-.

Therefore although the Broeck bottle can possesses a large internal volume delimitated by a large internal radius(inner radius) r_+ able to accommodate the brown man inside the Broeck bottle then the blue man in the plane representing our Universe would only see a microscopically small surface circle(bottleneck bottle) delimited by the microscopically small outer radius r_- as in pg 19 in [15].

A microscopically small outer radius r_- the R_b in our equations delimitates a very small microscopically circle surface area therefore reducing the probability of collisions against the dangerous IM particles

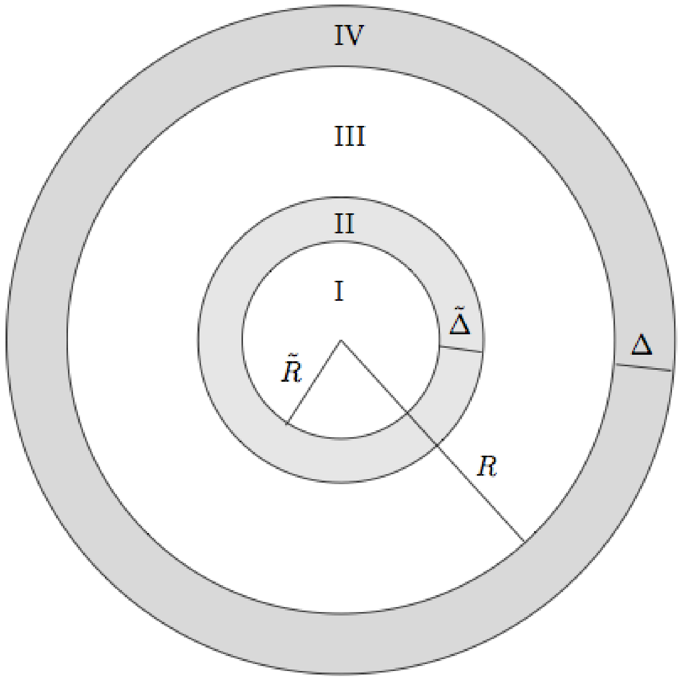

Figure 1: Region I is the 'pocket', which has a large inner metric diameter. II is the transition region from the blown–up part of space to the 'normal' part. It is the region where B varies. From region III outward we have the original Alcubierre metric. Region IV is the wall of the warp bubble; this is the region where f varies. Spacetime is flat, except in the shaded regions.

Figure 6: Artistic Presentation of the Broeck "pocket" or "bottle" inside the Alcubierre warp drive spacetime. (Source:fig 1 pg 4 in [10])

23 Appendix J:Artistic Presentation of the Broeck "pocket" or "bottle" inside the Alcubierre warp drive spacetime

Broeck applied to the Alcubierre original warp drive metric spatial components a new mathematical term $B(rs)$ as shown below(see eq 3 pg 3 in [10]) changing the signature from $(-,+,+,+)$ to $(+,-,-,-)$:

$$ds^2 = dt^2 - B^2(r_s)[(dx - v_s(t)f(r_s)dt)^2 + dy^2 + dz^2]. \tag{277}$$

Broeck created inside the warp bubble of radius R a spatial distortion of radius R_b being R_b microscopically small when seen from outside but inside the sphere generated by this R_b a large internal volume with the size enough to contains a spaceship can easily be accommodated.(see also pg 19 in [15])[50]

In the figure shown above the term $\tilde{R}$ is our small outer radius R_b and the term $\tilde{\Delta}$ is our Δ_b.

[50]see Appendices H and I

According to Broeck this term $B(rs)$ have the following behavior:(see pgs 3 and 4 in [10])[51]

$$B(rs) = \begin{cases} 1+\alpha & rs < R_b \\ 1 < B(rs) \leq 1+\alpha & R_b \leq rs < R_b + \Delta_b \\ 1 & rs \geq R_b + \Delta_b \end{cases} \tag{278}$$

-)-Considering the picture shown in the previous page:

Region 1 is the Broeck bottle or "pocket" with a large inner metric defined by the region where $rs < R_b$ and $B(rs) = 1+\alpha$ being α the term that generates the large internal volume of the Broeck bottle.

Region 2 is the region where the bottleneck of the Broeck bottle is placed.This region is the transition region between the "blown-up" space to the "normal" space.This is the region where $R_b \leq rs < R_b + \Delta_b$ being R_b the radius of the Broeck bottle bottleneck.In this region the value of $B(rs)$ becomes $1 < B(rs) \leq 1+\alpha$ never reaching 1.The term Δ_b delimitates the thickness of the region 2 where $B(rs)$ decreases.This region is a thin shell around the Broeck bottle bottleneck.

Region 3 is the region where $rs \geq R_b + \Delta_b$ far outside the Broeck bottle bottleneck we recover the normal space of our Universe in which $B(rs)$ always possesses the value of $B(rs) = 1$.We also recover the original Alcubierre metric.

Region 4 is the Alcubierre warped region where the Alcubierre shape function $f(rs)$ is varying from 1 to 0.($0 < f(rs) \leq 1$).According with Alcubierre any function $f(rs)$ that gives 1 inside the bubble and 0 outside the bubble while being $1 > f(rs) > 0$ in the Alcubierre warped region is a valid shape function for the Alcubierre warp drive.(see eqs 6 and 7 pg 4 in [1] or top of pg 4 in [2]). Broeck defined the Alcubierre shape function as being:(see pg 4 in [10])

$$f(rs) = \begin{cases} 1 & rs < R \\ 0 < f(rs) \leq 1 & R \leq rs < R + \Delta \\ 0 & rs \geq R + \Delta \end{cases} \tag{279}$$

In the equation above R is the radius of the warp bubble and Δ is the thickness of the Alcubierre warped region which means to say the thin shell region where $0 < f(rs) \leq 1$. Remember that $R >> R_b + \Delta_b$ or $R + \Delta >> R_b + \Delta_b$

Regions from 1 to 3 are completely contained inside the Alcubierre warp bubble.Note that in the regions 1 and 3 the value of $B(rs)$ is constant which means to say that the derivatives of $B(rs)$ are zero.Also in these regions the value of $f(rs)$ is always constant hence the derivatives of $f(rs)$ are also zero.

In the region 2 delimitated by $R_b \leq rs < R_b + \Delta_b$ the value of $B(rs)$ is given by $1 < B(rs) \leq 1+\alpha$ and since $B(rs)$ is varying in this region then the derivatives of $B(rs)$ are different than zero.

In the region 4 delimitated by $R \leq rs < R + \Delta$ the value of $f(rs)$ is given by $0 < f(rs) \leq 1$ and since $f(rs)$ is varying in this region then the derivatives of $f(rs)$ are different than zero.

[51] see Appendix G

Due to the terms $R >> R_b+\Delta_b$ or $R+\Delta >> R_b+\Delta_b$ the regions 2 and 4 do not "overlap" themselves.In the region 2 the derivatives of $B(rs)$ are non-zero but the derivatives of $f(rs)$ are zero and in the region 4 the derivatives of $B(rs)$ are zero but the derivatives of $f(rs)$ are non-zero.This is very important the fact that we can study both regions 2 and 4 completely separated from each other.Otherwise we would need to compute "all-the-way-round" the Christoffel symbols Riemann and Ricci tensors and the Ricci scalar in order to obtain the Einstein tensor and hence the stress-energy-momentum tensor in a long and tedious process of tensor analysis liable of occurrence of calculation errors.

Or we can use computers with programs like $Maple$ or $Mathematica$ (see pgs $[342(b)]$ or $[369(a)]$ in [11], pgs $[276(b)]$ or $[294(a)]$ in [13],pgs $[454, 457, 560(b)]$ or $[465, 468, 567(a)]$ in [14]).

Appendix C pgs $[551-555(b)]$ or $[559-563(a)]$ in [14] shows how to calculate everything until the Einstein tensor from the basic input of the covariant components of the $3+1$ spacetime metric using $Mathematica$.

The energy density for the Broeck region 2 in Geometrized Units $c = G = 1$ is given by the following equation:(see eq 11 pg 6 in [10])

$$T_{\hat{\mu}\hat{\nu}}u^{\hat{\mu}}u^{\hat{\nu}} = T^{\hat{0}\hat{0}} = \frac{1}{8\pi}\left(\frac{1}{B^4}(\partial_r B)^2 - \frac{2}{B^3}\partial_r\partial_r B - \frac{4}{B^3}\partial_r B\frac{1}{r}\right). \tag{280}$$

In the equation above a large $B(rs)$ from $1 < B(rs) \leq 1+\alpha$ where $R_b \leq rs < R_b+\Delta_b$ will generate very small terms $\frac{1}{B(rs)^4}$ $\frac{2}{B(rs)^3}$ and $\frac{4}{B(rs)^3}$ therefore obliterating the values of the derivatives of $B(rs)$ resulting in a very low energy density.

The Alcubierre expressions for the negative energy density of the region 4 in Geometrized Units $c = G = 1$ are given by(pg 4 in [2])(pg 8 in [1]):[52]:

$$\rho = -\frac{1}{32\pi}vs^2\left[f'(rs)\right]^2\left[\frac{y^2+z^2}{rs^2}\right] \tag{281}$$

$$\rho = -\frac{1}{32\pi}vs^2\left[\frac{df(rs)}{drs}\right]^2\left[\frac{y^2+z^2}{rs^2}\right] \tag{282}$$

Note that in the equatorial plane $y = z = 0$ the negative energy density vanishes leaving the ship and therefore the region 2 both unprotected against collisions with the dangerous IM particles.(see the works in [5],[7] and [8])

[52]See Geometrized Units in Wikipedia

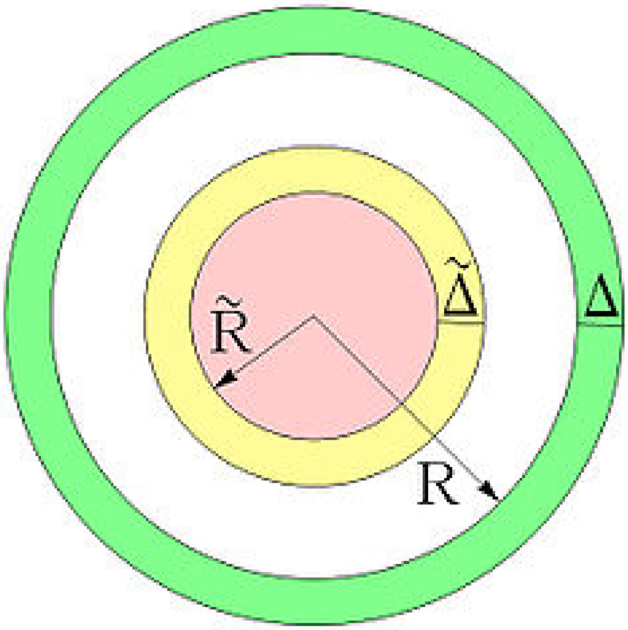

Figure 7: Artistic Presentation of the Broeck "pocket" or "bottle" inside the Natario warp drive spacetime.(Source:Internet)

24 Appendix K:Artistic Presentation of the Broeck "pocket" or "bottle" inside the Natario warp drive spacetime

Applying the Broeck mathematical term $B(rs)$ to the spatial components of the Natario warp drive equation using the signature $(+,-,-,-)$ we get the following result:

$$ds^2 = dt^2 - B(rs)^2[(drs - X^{rs}dt)^2 + (rs^2)(d\theta - X^{\theta}dt)^2] \tag{283}$$

With the contravariant shift vector components X^{rs} and X^{θ} given by:(see pg 5 in [2])[53]

$$X^{rs} = 2v_s n(rs)\cos\theta \tag{284}$$

$$X^{\theta} = -v_s(2n(rs) + (rs)n'(rs))\sin\theta \tag{285}$$

The term $B(rs)$ according to Broeck creates inside the Natario warp bubble of radius R a spatial distortion of radius R_b being R_b microscopically small when seen from outside but inside the sphere generated by this R_b a large internal volume with the size enough to contains a spaceship can easily be accommodated.(see also pg 19 in [15])[54]

In the figure shown above the term $\tilde{R}$ is our small outer radius R_b and the term $\tilde{\Delta}$ is our Δ_b.

[53]see also Appendices A and B for details
[54]see Appendices H and I

According to Broeck this term $B(rs)$ have the following behavior:(see pgs 3 and 4 in [10])[55]

$$B(rs) = \begin{cases} 1+\alpha & rs < R_b \\ 1 < B(rs) \leq 1+\alpha & R_b \leq rs < R_b + \Delta_b \\ 1 & rs \geq R_b + \Delta_b \end{cases} \tag{286}$$

-)-Considering the picture shown in the previous page:

The pink region is the Broeck bottle or "pocket" with a large inner metric defined by the region where $rs < R_b$ and $B(rs) = 1+\alpha$ being α the term that generates the large internal volume of the Broeck bottle.

The faded yellow region is the region where the bottleneck of the Broeck bottle is placed.This region is the transition region between the "blown-up" space to the "normal" space.This is the region where $R_b \leq rs < R_b + \Delta_b$ being R_b the radius of the Broeck bottle bottleneck.In this region the value of $B(rs)$ becomes $1 < B(rs) \leq 1+\alpha$ never reaching 1.The term Δ_b delimitates the thickness of the faded yellow region where $B(rs)$ decreases.This region is a thin shell around the Broeck bottle bottleneck.

The white region is the region where $rs \geq R_b + \Delta_b$ far outside the Broeck bottle bottleneck we recover the normal space of our Universe in which $B(rs)$ always possesses the value of $B(rs) = 1$.We also recover the original Natario metric.

The green region is the Natario warped region where the Natario shape function $n(rs)$ is varying from 0 to $\frac{1}{2}$.($0 < n(rs) \leq \frac{1}{2}$).According with Natario any function $n(rs)$ that gives 0 inside the bubble and $\frac{1}{2}$ outside the bubble while being $\frac{1}{2} > n(rs) > 0$ in the Natario warped region is a valid shape function for the Natario warp drive.(see pg 5 in [2]). We define the Natario shape function as being

$$n(rs) = \begin{cases} 0 & rs < R \\ 0 < n(rs) \leq \frac{1}{2} & R \leq rs < R+\Delta \\ \frac{1}{2} & rs \geq R+\Delta \end{cases} \tag{287}$$

In the equation above R is the radius of the warp bubble and Δ is the thickness of the Natario warped region which means to say the thin shell region where $0 < n(rs) \leq \frac{1}{2}$. Remember that $R >> R_b + \Delta_b$ or $R + \Delta >> R_b + \Delta_b$

The pink,faded yellow and white regions are completely contained inside the Natario warp bubble.Note that in the pink and white regions the value of $B(rs)$ is constant which means to say that the derivatives of $B(rs)$ are zero.Also in these regions the value of $n(rs)$ is always constant hence the derivatives of $n(rs)$ are also zero.

In the faded yellow region delimitated by $R_b \leq rs < R_b+\Delta_b$ the value of $B(rs)$ is given by $1 < B(rs) \leq 1+\alpha$ and since $B(rs)$ is varying in this region then the derivatives of $B(rs)$ are different than zero.

In the green region delimitated by $R \leq rs < R + \Delta$ the value of $n(rs)$ is given by $0 < n(rs) \leq \frac{1}{2}$ and since $n(rs)$ is varying in this region then the derivatives of $n(rs)$ are different than zero.

[55] see Appendix G

Due to the terms $R >> R_b + \Delta_b$ or $R + \Delta >> R_b + \Delta_b$ the regions faded yellow and green do not "overlap" themselves.In the faded yellow region the derivatives of $B(rs)$ are non-zero but the derivatives of $n(rs)$ are zero and in the green region the derivatives of $B(rs)$ are zero but the derivatives of $n(rs)$ are non-zero.This is very important the fact that we can study both regions faded yellow and green completely separated from each other.Otherwise we would need to compute "all-the-way-round" the Christoffel symbols Riemann and Ricci tensors and the Ricci scalar in order to obtain the Einstein tensor and hence the stress-energy-momentum tensor in a long and tedious process of tensor analysis liable of occurrence of calculation errors.

Or we can use computers with programs like $Maple$ or $Mathematica$ (see pgs $[342(b)]$ or $[369(a)]$ in [11], pgs $[276(b)]$ or $[294(a)]$ in [13],pgs $[454, 457, 560(b)]$ or $[465, 468, 567(a)]$ in [14]).

Appendix C pgs $[551-555(b)]$ or $[559-563(a)]$ in [14] shows how to calculate everything until the Einstein tensor from the basic input of the covariant components of the $3+1$ spacetime metric using $Mathematica$.

The energy density for the Broeck faded yellow region in Geometrized Units $c = G = 1$ is given by the following equation:(see eq 11 pg 6 in [10])

$$T_{\hat{\mu}\hat{\nu}}u^{\hat{\mu}}u^{\hat{\nu}} = T^{\hat{0}\hat{0}} = \frac{1}{8\pi}\left(\frac{1}{B^4}(\partial_r B)^2 - \frac{2}{B^3}\partial_r\partial_r B - \frac{4}{B^3}\partial_r B\frac{1}{r}\right). \tag{288}$$

In the equation above a large $B(rs)$ from $1 < B(rs) \leq 1+\alpha$ where $R_b \leq rs < R_b+\Delta_b$ will generate very small terms $\frac{1}{B(rs)^4}$ $\frac{2}{B(rs)^3}$ and $\frac{4}{B(rs)^3}$ therefore obliterating the values of the derivatives of $B(rs)$ resulting in a very low energy density.

The Natario expressions for the negative energy density of the green region in Geometrized Units $c = G = 1$ are given by(pg 5 in [2])

$$\rho = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[3(n'(rs))^2\cos^2\theta + \left(n'(rs) + \frac{rs}{2}n''(rs)\right)^2\sin^2\theta\right]. \tag{289}$$

$$\rho = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[3(\frac{dn(rs)}{drs})^2\cos^2\theta + \left(\frac{dn(rs)}{drs} + \frac{r}{2}\frac{d^2n(rs)}{drs^2}\right)^2\sin^2\theta\right]. \tag{290}$$

Note that in the equatorial plane $\theta = 0$ $\sin(\theta) = 0$,$\cos(\theta) = 1$ the negative energy density do not vanishes protecting the ship and therefore the faded yellow region against collisions with the dangerous IM particles.(see the works in [5],[7] and [8])[56]

[56]see also Appendices E and F

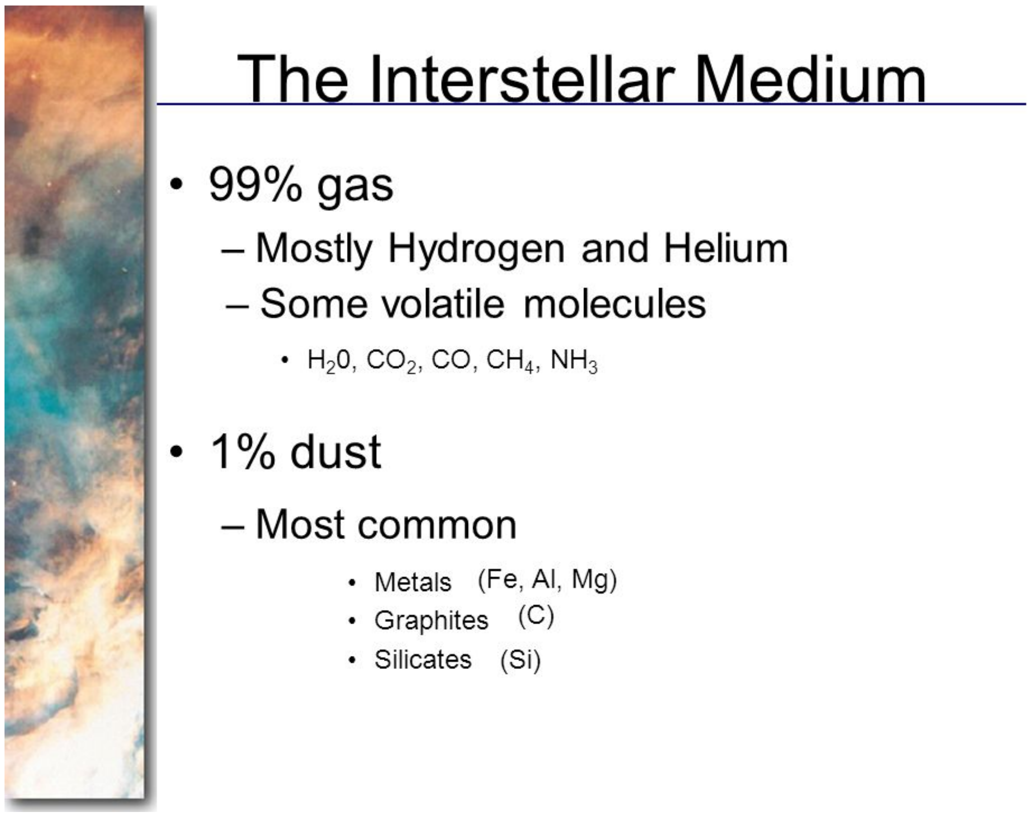

Figure 8: Composition of the Interstellar Medium IM(Source:Internet)

25 Appendix L:Composition of the Interstellar Medium IM

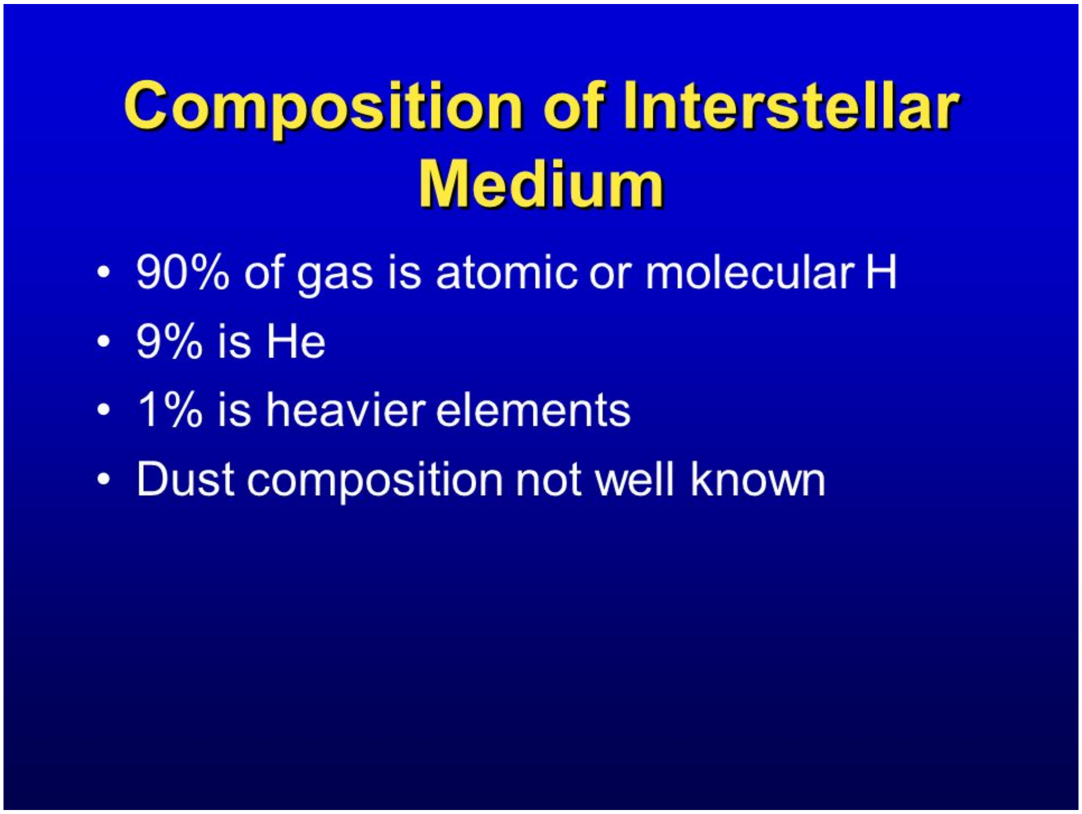

Figure 9: Composition of the Interstellar Medium IM(Source:Internet)

26 Appendix M:Composition of the Interstellar Medium IM

27 Remarks

References [11],[12],[13] and [14] are standard textbooks used to study General Relativity and these books are available or in paper editions or in electronic editions all in Adobe PDF Acrobat Reader.

We have the electronic editions of all these books

In order to make easy the reference cross-check of pages or equations specially for the readers of the paper version of the books we adopt the following convention:when we refer for example the pages $[507, 508(b)]$ or the pages $[534, 535(a)]$ in [11] the (b) stands for the number of the pages in the paper edition while the (a) stands for the number of the same pages in the electronic edition displayed in the bottom line of the Adobe PDF Acrobat Reader

Our numerical plots were made using Microsoft Excel and Oracle Open Office.We can provide all the files for those interested.

28 Epilogue

- "The only way of discovering the limits of the possible is to venture a little way past them into the impossible."-Arthur C.Clarke[57]
- "The supreme task of the physicist is to arrive at those universal elementary laws from which the cosmos can be built up by pure deduction. There is no logical path to these laws; only intuition, resting on sympathetic understanding of experience, can reach them"-Albert Einstein[58][59]

[57] special thanks to Maria Matreno from Residencia de Estudantes Universitas Lisboa Portugal for providing the Second Law Of Arthur C.Clarke

[58] "Ideas And Opinions" Einstein compilation, ISBN $0-517-88440-2$, on page 226."Principles of Research" ([Ideas and Opinions],pp.224-227), described as "Address delivered in celebration of Max Planck's sixtieth birthday (1918) before the Physical Society in Berlin"

[59] appears also in the Eric Baird book Relativity in Curved Spacetime ISBN $978-0-9557068-0-6$

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