Infinite Tetration of Euler's Number and the Z-Exponential

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Abstract

It is shown that the infinite tetration of Euler's number is equal to any complex number. It is also found that starting with any complex number except 0 and 1, we can convert the complex number into an exponential with a complex exponent. If this is done recursively for each successive exponent, we find that the complex exponent converges to a constant number, which is named the Z-Exponential (Z_e). Derivatives for the Z-Exponential function are derived as well as its relationship to the exponential and natural logarithm.

Proof

Any complex number can be written in the form:

$$r_0 e^{i\theta_0} = e^{\ln(r_0)} e^{i\theta_0} \tag{1}$$

Where $r_0 > 0$ and $\theta_0 > 0$. This can be written as:

$$e^{\ln(r_0)} e^{i\theta_0} = e^{\ln(r_0)+i\theta_0} \tag{2}$$

If we say $\ln(r_0) + i\theta_0 = r_1 e^{i\theta_1} = e^{\ln(r_1)+i\theta_1}$ we get:

$$e^{\ln(r_0)+i\theta_0} = e^{r_1 e^{i\theta_1}} = e^{e^{\ln(r_1)+i\theta_1}} \tag{3}$$

The calculation of r_n and θ_n was carried out for various orders of magnitude of r_0 and θ_0 and it was found empirically that for all complex numbers evaluated with the exception of 0, 1, and ${}^{m}e$ (for any finite natural number m), the value of r_n converges to 1.3745570107… and θ_n converges to 1.3372357014… as the number of iterations n increases. This complex number has the property that its imaginary part is equal to its angle in radians and its real part is the natural logarithm of its magnitude.

The value of $r_n$ws calculated using the recursive equation:

$$r_n = \sqrt{\ln(r_{n-1})^2 + (\theta_{n-1})^2} \tag{3}$$

And θ_n is the argument of the complex number whose real part is $\ln(r_{n-1})$ and whose imaginary part is θ_{n-1}. A plot of the typical behavior of r_n and θ_n is given in Figure 1 for $r_0 = 3$ and $\theta_0 = 2$:

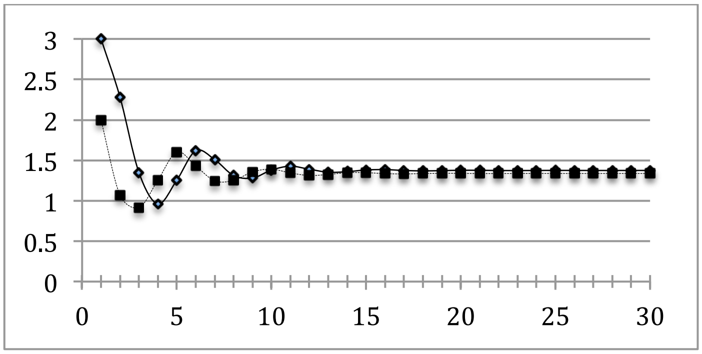

Figure 1 - r_n (Solid) and θ_n (Dashed) vs. Number of Iterations

Therefore, repeating this process of tetrating *e* an infinite number of times gives:

$$^{\infty}e = r_0 e^{i\theta_0} \tag{4}$$

For any $r_0 > 0,\ \theta_0 \geq 0$ and $r_0 \neq 1$ or ^{m}e when $\theta_0 = 0$. Furthermore, the numbers corresponding to the converging r_n and θ_n form a complex number that we will call Z_e (the Z-exponential). From the above analysis, it follows that:

$$Z_e = e^{Z_e} \tag{5}$$

We can define the function:

$${Z_e}^{x} = e^{xZ_e} \tag{7}$$

The first and second derivatives of this function will be:

$$\frac{d}{dx}{Z_e}^{x} = \frac{d}{dx}e^{xZ_e} = Z_e e^{xZ_e} = Z_e {Z_e}^{x} = Z_e^{(x+1)} \tag{8}$$

$$\frac{d^2}{dx^2}{Z_e}^{x} = \frac{d}{dx}{Z_e}^{(x+1)} = Z_e e^{(x+1)Z_e} = Z_e Z_e^{(x+1)} = Z_e^{(x+2)} \tag{9}$$

And in general:

$$\frac{d^n}{dx^n}{Z_e}^{x} = Z_e^{(x+n)} \tag{10}$$

Therefore the derivative will shift the function by the value *n* (where *n* can be any complex number [1]). Relationships to the natural logarithm are as follows:

$$\ln(a{Z_e}^{x}) = \ln(ae^{xZ_e}) = \ln(a) + xZ_e \tag{11}$$

$$\log_{Z_e}(x) = \log_{Z_e}\left(e^{\frac{\ln(x)}{Z_e}Z_e}\right) = \log_{Z_e}\left({Z_e}^{\frac{\ln(x)}{Z_e}}\right) = \frac{1}{Z_e}\ln(x) \tag{12}$$

Finally, the fact that $\ln(Z_e) = Z_e$ tells us that Z_e can be expressed in terms of the Lambert-W function (W) as follows:

$$Z_e = -W_{-1}(-1) \tag{13}$$

References

[1] Lavoie, J. L., Osler, T. J., Tremblay, R.: "Fractional Derivatives and Special Functions", SIAM Review, Vol. 18, Issue 2, 240-268 (1976)