Exact Differentials in General Relativity

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Abstract

The article seeks to define and analyze work in the context of General Relativity. The definition of work in General Relativity as considered with this article is an extrapolation of what we have in Special Relativity.This definition as brought out in this paper takes into account the involvement of the curvature effects into the definition of work. The paper also considers the weak field limit of work in relation to Schwarzschild's Geometry. In the classical limit of weak space time curvature our definition produces the classical energy conservation formula: the sum of potential and kinetic energy as defined classically is conserved when Schwarzschild geometry is treated in the weak field limit with our definition of work.

Keywords: Exact Differentials, Special Relativity, General Relativity, Schwarzschild's Geometry, Work, Metric

Introduction

Work as defined in Special relativity may be extrapolated into the realm of General Relativity taking into account the effect of the metric coefficients characterizing the General Relativity metric. The basic tool used is the invariance of the four dot product. In the classical limit of weak space time curvature our definition produces the classical energy conservation formula: the sum of potential and kinetic energy as defined classically is conserved when Schwarzschild geometry is treated in the weak field limit woth our definition of work.

Rudimentary Notions

The definition of four velocity is identical in Special and in General Relativity. But the definition of four acceleration is different. Curvature effects are involved in acceleration and hence in the concept of work in General relativity.

We first consider the differential relation

$$d\bar{x}^{\mu} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} dx^{\alpha}$$

The above relation is a general mathematical statement for the transformation of rank one tensors.

Dividing both sides by invariant proper time interval we have, $\frac{d\bar{x}^{\mu}}{d\tau} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} \frac{dx^{\alpha}}{d\tau} \Rightarrow \bar{v}^{\mu} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} v^{\alpha}$

The same definition for four velocity is used for flat space time and curved space time since the above relation represents a tensor transformation in any situation

Differentiating $\frac{d\bar{x}^\mu}{d\tau}=\frac{\partial\bar{x}^\mu}{\partial x^\alpha}\frac{dx^\alpha}{d\tau}$ with respect to proper time τ we have:$\frac{d^2\bar{x}^\mu}{d\tau^2}=\frac{\partial\bar{x}^\mu}{\partial x^\alpha}\frac{d^2\bar{x}^\alpha}{d\tau^2}+\frac{d}{d\tau}\left(\frac{\partial\bar{x}^\mu}{\partial x^\alpha}\right)\frac{dx^\alpha}{d\tau}$

$\frac{d^2\bar{x}^\mu}{d\tau^2}$does not behave as a rank one tensor unless $\frac{\partial\bar{x}^\mu}{\partial x^\alpha}$ is a constant in which case $\frac{d}{d\tau}\left(\frac{\partial\bar{x}^\mu}{\partial x^\alpha}\right)\frac{dx^\alpha}{d\tau}=0$ [Example: Lorentz Transformations: $\frac{\partial\bar{x}^\mu}{\partial x^\alpha}$ are constants] If $\frac{\partial\bar{x}^\mu}{\partial x^\alpha}$ =constant,$\frac{d}{d\tau}\left(\frac{\partial\bar{x}^\mu}{\partial x^\alpha}\right)$being zero we do have a tensor transformation given by the equations :$\frac{d^2\bar{x}^\mu}{d\tau^2}=\frac{\partial\bar{x}^\mu}{\partial x^\alpha}\frac{d^2\bar{x}^\mu}{d\tau^2}$

The quantity $\frac{d^2x^\alpha}{d\tau^2}+\Gamma^\alpha{}_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\alpha}{d\tau}=\frac{D^2x^\alpha}{D\tau^2}$ is defined as four acceleration which behaves as a tensor .

Thus Four acceleration [1]is defined by:

(1) $\frac{d^2x^\alpha}{d\tau^2}+\Gamma^\alpha{}_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\alpha}{d\tau}=\frac{D^2x^\alpha}{D\tau^2}$

It is important to take note of the fact that four acceleration=0 does not mean that four velocity,$\frac{dx^\alpha}{d\tau}$, is constant as we have in Special Relativity. Rather four acceleration=0 implies

Or,

$$\frac{d^2x^\alpha}{d\tau^2}=-\Gamma^\alpha{}_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\alpha}{d\tau}$$

Proper velocity components $\frac{dx^\alpha}{d\tau}\neq c$onstant when four acceleration is zero.

When we see an apple falling from a tree its radial component of four acceleration is *exactly* zero----quite different from what we observe:9.8m/s^2.The quantity $\frac{d^2x^\alpha}{d\tau^2}\left(=\frac{d^2r}{d\tau^2}\right)$ in the geodesic equation $\frac{d^2x^\alpha}{d\tau^2}+\Gamma^\alpha{}_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\alpha}{d\tau}=0$,relates to 9.8m/s^2 in the weak space time context when physical and coordinate values are identical [approximately identical as for weak space time]: $\frac{d^2x^\alpha}{d\tau^2}\left(=\frac{d^2r}{d\tau^2}\right)$ is the quantity we measure for acceleration when we observe an apple falling from a tree. The important point to appreciate is that gravity[space time curvature effects] causes acceleration that we measure[Example: $\frac{d^2x^\alpha}{d\tau^2}\left(=\frac{d^2r}{d\tau^2}\right)$ for a falling apple]. For physical separations inj the strict sense of strong curvature, we have to use $dx^\alpha{}_{physical}=g_{\alpha\alpha}dx^\alpha$ [no summation on α: relations (5.1) to (5.4) coming up subsequently have to be considered]

If the earth were a million times denser the radial component of a freely falling object would be exactly zero, the value of 'g' the acceleration due to becoming enormously large.

Work In General Relativity

Considering curvature effects we define Work , as

$\Delta W\ (per\ unit\ rest\ mass)=g_{ii}\left(\frac{d^2x^i}{d\tau^2}+\Gamma^i{}_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\alpha}{d\tau}\right)dx^i$,$i$ running over *spatial* components.

We have used ΔW instead of dW since we do not know whether the right side is a perfect differential.

For gravity acting alone,$\frac{d^2x^i}{d\tau^2} + \Gamma^i{}_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\alpha}{d\tau} = 0$.Four acceleration in presence of gravity alone is zero though it causes measurable amount of three acceleration as observed in the free fall of a body. Then $\frac{d^2x^i}{d\tau^2} + \Gamma^i{}_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\alpha}{d\tau} \neq 0$ implies the involvement of non gravity agents .For non geodesic paths we may write:

Then $\frac{d^2x^i}{d\tau^2} + \Gamma^i{}_{\alpha\beta}\frac{dx^\beta}{d\tau}\frac{dx^\alpha}{d\tau} = f \neq 0$, or $\frac{d^2x^i}{d\tau^2} = -\Gamma^i{}_{\alpha\beta}\frac{dx^\beta}{d\tau}\frac{dx^\alpha}{d\tau} + f$, interpreting $-\Gamma^i{}_{\alpha\beta}\frac{dx^\beta}{d\tau}\frac{dx^\alpha}{d\tau}$ as acceleration due to gravity [accelerating due to space time Geometry effects]and f as acceleration due to non gravity agents. If gravity is turned off the path [world line on the manifold]itself changes resulting in a geodesic. In presenceof gravity alone $\frac{d^2x^i}{d\tau^2}$ and $\Gamma^i{}_{\alpha\beta}\frac{dx^\beta}{d\tau}\frac{dx^\alpha}{d\tau}$ cancel out. Hence net work done is zero. but classical work should be in consideration of either $\frac{d^2x^i}{d\tau^2}$ or $\Gamma^i{}_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\alpha}{d\tau}$. It is not expected to be zero. Rather it should be in conformity with the work energy theorem

Since $\frac{d^2x^i}{d\tau^2} + \Gamma^i{}_{\alpha\beta}\frac{dx^\beta}{d\tau}\frac{dx^\alpha}{d\tau} \neq 0$ in presence of non gravitational agents we may define

(3) $\Delta W = g_{ii}\left(\frac{d^2x^i}{d\tau^2} + \Gamma^i{}_{\alpha\beta}\frac{dx^\beta}{d\tau}\frac{dx^\alpha}{d\tau}\right)dx^i[i \text{ running over } spatial \text{ components}]$.If four acceleration is not zero,). In presence of gravity alone ΔW will be zero In presence of non gravity agents W_{non_Grav} will be represented by a non zero value.

Special Relativity Perspectives

The work energy theorem is an accepted idea in Special Relativity. It is used to deduce the kinetic energy[2] formula in Special relativity:

KE=$m_0c^2\left(\frac{1}{\sqrt{1-v^2/c^2}} - 1\right) = mc^2 - m_0c^2$

Change in KE= $m_0c^2(\gamma_{final} - \gamma_{finitial})$

To apply the same in General Relativity we have to modify the General Relativity metric by suitable transformations to obtain the flat space *form* of metric. This will enable us apply Special relativity formulas in the general relativity context, using the transformed metric

GR metric:

(4) $c^2d\tau^2 = g_{tt}d(ct)^2 - g_{xx}dx^2 - g_{yy}dy^2 - g_{zz}dz^2$

(4.1)$c^2 = g_{tt}\left(\frac{d(ct)}{d\tau}\right)^2 - g_{xx}\left(\frac{dx}{d\tau}\right)^2 - g_{yy}\left(\frac{dy}{d\tau}\right)^2 - g_{zz}\left(\frac{dz}{d\tau}\right)^2$

The quantity ct has the dimension of length and τ has the dimension of time . Consequently $\frac{d(ct)}{d\tau}$ has the dimension of speed like $\frac{dx^i}{d\tau}$; i running over the spatial components. Thus in relations (1), (2) etc we have to use ct for t keeping τ undisturbed.

We use the following Transformations:

(5.1)$dT = \sqrt{g_{tt}}dt$

(5.2) $dX = \sqrt{g_{xx}}dx$

(5.3) $dY = \sqrt{g_{yy}}dy$

(5.4) $dZ = \sqrt{g_{zz}}dz$

The metric GR metric now has the form of that of Minkowski space

(6) $c^2d\tau^2 = d(cT)^2 - dX^2 - dY^2 - dZ^2$

Now we may apply the work energy theorem to derive the formula for Kinetic energy in the curved space time context[using the same method as applied for Special relativity] since the metric represented by (6) has the form of the flat space time metric

(7.1) $\Delta W_{grav+non\ grav} = \frac{dP^i}{d\tau}dX^i$[Minkowski form of metric is used]

Three momentum component: $p^i = m\frac{dx^i}{dt} = m_0\gamma\frac{dx^i}{dt} = m_0\frac{dt}{d\tau}\frac{dx^i}{dt} = m_0\frac{dx^i}{d\tau}$

m:relativistic mass;m_0: rest mass

Three force in :$F^i = \frac{dP^i}{dt} = m_0\frac{d}{dt}\left(\gamma\frac{dx^i}{dt}\right) = m_0\frac{d}{dt}\left(\frac{dt}{d\tau}\frac{dx^i}{dt}\right) = m_0\frac{d}{dt}\left(\frac{dx^i}{d\tau}\right) = m_0\frac{d}{d\tau}\left(\frac{dx^i}{d\tau}\right)\frac{d\tau}{dt} = \frac{m_0}{\gamma}\frac{d^2x^i}{d\tau^2}$;i=1,2,3[spatial components]

[Remembering , $\gamma = \frac{dt}{d\tau}$, using Special relativity concepts; m:relativistic mass;m_0:rest mass]

Four momentum component: $p^i = m_0\frac{dx^i}{d\tau} = m_0\frac{dx^i}{dt}\frac{dt}{d\tau} = m_0\gamma\frac{dx^i}{dt} = m\frac{dx^i}{dt}$

Four Force: $\frac{dP^i}{d\tau} = m_0\frac{d}{d\tau}\left(\frac{dx^i}{d\tau}\right) = m_0\frac{d^2x^i}{d\tau^2}$;$[i = 0,1,2,3;]$

The Metric and Energy Considerations

We start with the metric:

$$c^2d\tau^2 = d(ct)^2 - dx^2 - dy^2 - dz^2$$

$$c^2 = \left(\frac{d(ct)}{d\tau}\right)^2 - \left(\frac{dx}{d\tau}\right)^2 - \left(\frac{dy}{d\tau}\right)^2 - \left(\frac{dy}{d\tau}\right)^2$$

Differentiating the above with respect to proper time τ , we obtain:

$$0 = c^2 \frac{dt}{d\tau}\frac{d^2t}{d\tau^2} - \frac{dx}{d\tau}\frac{d^2x}{d\tau^2} - \frac{dy}{d\tau}\frac{d^2y}{d\tau^2} - \frac{dz}{d\tau}\frac{d^2z}{d\tau^2} \Rightarrow c^2 \frac{dt}{d\tau}\frac{d^2t}{d\tau^2} = \frac{dt}{d\tau}\frac{d^2x}{d\tau^2} + \frac{dt}{d\tau}\frac{d^2y}{d\tau^2} + \frac{dt}{d\tau}\frac{d^2z}{d\tau^2}$$

$$(7.2) c^2 \frac{d^2t}{d\tau^2} = \frac{d^2x}{d\tau^2} + \frac{d^2y}{d\tau^2} + \frac{d^2z}{d\tau^2}$$

We have,

$$\frac{dE}{dt} = \frac{d}{dt}(m_0\gamma c^2) = m_0c^2\frac{d}{dt}\left(\frac{dt}{d\tau}\right) = m_0c^2\frac{d}{d\tau}\left(\frac{dt}{d\tau}\right)\frac{d\tau}{dt} = m_0c^2\frac{1}{\gamma}\frac{d^2t}{d\tau^2}$$

$\frac{dE}{dt} = m_0c^2\frac{1}{\gamma}\frac{d^2t}{d\tau^2}$ or, $\gamma\frac{dE}{dt} = m_0c^2\frac{d^2t}{d\tau^2}$ or, $\frac{dE}{d\tau} = m_0c^2\frac{d^2t}{d\tau^2}$

$$(8) \frac{dE}{d\tau} = m_0c^2\frac{d^2t}{d\tau^2}$$

Defining Work for General Relativity

Let us consider the four dot product[invariant]

(9) $g_{ii}\left(\frac{d^2x^i}{d\tau^2} + \Gamma^i{}_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau}\right)dx^i = INV$, i running over **spatial** and ***time*** components

Four dot product is independent of the choice of reference frames and transformations between them.
Four acceleration[] $\frac{d^2x^i}{d\tau^2} + \Gamma^i{}_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau}$ transforms like a tensor irrespective of the nature of transformation so long as the determinant of the Jacobian is not zero. Same is true of the tensor $(dt.dx, dy, dz)$ is independent of the choice of transformation that takes us to a different frame of reference frame so long as the non singularity of the transformation matrix is maintained.

For any instant during the motion of particle we transform to a frame where the spatial components of four velocity are momentarily zero;[for different instants we may choose different transformations]

According to our choice

(10) $\frac{dx}{d\tau} = \frac{dy}{d\tau} = \frac{dz}{d\tau} = 0$

[*But* $\frac{d^2x^i}{d\tau^2}$ *may not be zero*]

We have from equation (9),

$$\left[g_{ii}\left(\frac{d^2x^i}{d\tau^2}+\Gamma^i{}_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau}\right)dx^i\right]_{4-dot\ product:arbitrary\ frame}$$
$$=\left[g_{tt}\left(\frac{d^2(ct)}{d\tau^2}+\Gamma^t{}_{tt}\left(\frac{d(ct)}{d\tau}\right)^2\right)d(ct)-g_{xx}\frac{d^2x}{d\tau^2}dx-g_{yy}\frac{d^2y}{d\tau^2}dy-g_{xx}\frac{d^2z}{d\tau^2}dz\right]_{rest\ fraqme\ of\ particle}$$

[i on the left side running over space and time coordinates]

We set out to calculate the right side dropping the long suffix

$$(10.1)g_{tt}\left(\frac{d^2(ct)}{d\tau^2}+\Gamma^t{}_{tt}\left(\frac{d(ct)}{d\tau}\right)^2\right)d(ct)-g_{xx}\frac{d^2x}{d\tau^2}dx-g_{yy}\frac{d^2y}{d\tau^2}dy-g_{xx}\frac{d^2z}{d\tau^2}dz=4dot\ product$$

$$g_{tt}\left(\frac{d^2(ct)}{d\tau^2}+\Gamma^t{}_{tt}\left(\frac{d(ct)}{d\tau}\right)^2\right)\frac{d(ct)}{dt}dt-\left[g_{xx}\frac{d^2x}{d\tau^2}\frac{dx}{dt}+g_{yy}\frac{d^2y}{d\tau^2}\frac{dy}{dt}+g_{yy}\frac{d^2z}{d\tau^2}\frac{dz}{dt}\right]dt$$
$$=g_{tt}\left(\frac{d^2(ct)}{d\tau^2}+\Gamma^t{}_{tt}\left(\frac{d(ct)}{d\tau}\right)^2\right)\frac{d(ct)}{dt}dt=g_{tt}\left(\frac{d^2(ct)}{d\tau^2}+\Gamma^t{}_{tt}\left(\frac{d(ct)}{d\tau}\right)^2\right)d(ct)$$
$$=INV[4dot\ product]$$

The above invariant quantity calculated in any frame of reference will apply to all other frames. Our strategy would be to calculate the four dot product in a frame where the particle is momentarily at rest . The result will hold for all other frames.

For a particle momentarily at rest[) $\frac{dx}{d\tau}=\frac{dy}{d\tau}=\frac{dz}{d\tau}=0$] the stated invariant quantity[four dot product] is equal to

$$g_{tt}\left(\frac{d^2(ct)}{d\tau^2}+\Gamma^t{}_{tt}\left(\frac{d(ct)}{d\tau}\right)^2\right)\frac{d(ct)}{dt}dt=g_{tt}\left(\frac{d^2(ct)}{d\tau^2}+\Gamma^t{}_{tt}\left(\frac{d(ct)}{d\tau}\right)^2\right)d(ct)$$

$$g_{tt}\left(\frac{d^2(ct)}{d\tau^2}+\Gamma^t{}_{tt}\left(\frac{d(ct)}{d\tau}\right)^2\right)\frac{d(ct)}{d\tau}d\tau=g_{tt}\left(\frac{d^2(ct)}{d\tau^2}\frac{d(ct)}{d\tau}+\Gamma^t{}_{tt}\left(\frac{d(ct)}{d\tau}\right)^3\right)d\tau=INV$$

$$(11)g_{tt}\left(\frac{1}{2}\frac{d\left(\frac{d(ct)}{d\tau}\right)^2}{d\tau}+\Gamma^t{}_{tt}\left(\frac{d(ct)}{d\tau}\right)^3\right)d\tau=INV[four\ dot\ product]$$

as observed from (10) and (10.1)

Again from the metric we have:

$$c^2=g_{tt}\left(\frac{dct}{d\tau}\right)^2-g_{xx}\left(\frac{dx}{d\tau}\right)^2-g_{yy}\left(\frac{dy}{d\tau}\right)^2-g_{zz}\left(\frac{dz}{d\tau}\right)^2$$

For a particle momentarily at rest: $\frac{dx}{d\tau}=\frac{dy}{d\tau}=\frac{dz}{d\tau}=0$. The above metric gives us.

(12.1)$c^2 = g_{tt}\left(\frac{d(ct)}{d\tau}\right)^2$

Applying (12.1) on (11)

$$g_{tt}\left(\frac{1}{2}\frac{d}{d\tau}\left(\frac{c^2}{g_{tt}}\right)+\Gamma^t{}_{tt}\left(\frac{d(ct)}{d\tau}\right)^3\right)d\tau=\left(\frac{1}{2}g_{tt}\frac{d}{d\tau}\left(\frac{c^2}{g_{tt}}\right)+g_{tt}\Gamma^t{}_{tt}\left(\frac{d(ct)}{d\tau}\right)^3\right)d\tau$$
$$=\left(\frac{1}{2}g_{tt}\frac{d}{d\tau}\left(\frac{c^2}{g_{tt}}\right)+c^2\Gamma^t{}_{tt}\frac{d(ct)}{d\tau}\right)d\tau=INV$$

Now in orthogonal coordinates we have :

$$\Gamma^t{}_{tt}=\frac{1}{2g_{tt}}\frac{\partial g_{tt}}{\partial(ct)}$$

$\frac{1}{2g_{tt}}\frac{\partial g_{tt}}{\partial(ct)} \neq 0$*for time dependent metrics*

Therefore

$$g_{tt}\left(\frac{1}{2}\frac{d}{d\tau}\left(\frac{c^2}{g_{tt}}\right)+\Gamma^t{}_{tt}\left(\frac{d(ct)}{d\tau}\right)^3\right)d\tau=\left(\frac{1}{2}g_{tt}\frac{d}{d\tau}\left(\frac{c^2}{g_{tt}}\right)+g_{tt}\Gamma^t{}_{tt}\left(\frac{d(ct)}{d\tau}\right)^3\right)d\tau$$
$$=\left(\frac{1}{2}g_{tt}\frac{d}{d\tau}\left(\frac{c^2}{g_{tt}}\right)+c^2\Gamma^t{}_{tt}\frac{d(ct)}{d\tau}\right)d\tau=INV$$

Therefore,

Four Dot product: $\left(-\frac{c^2}{2g_{tt}}\frac{dg_{tt}}{d\tau}+\frac{1}{2g_{tt}}\frac{\partial g_{tt}}{\partial ct}c^2\frac{d(ct)}{d\tau}\right)d\tau=\left(-\frac{c^2}{2g_{tt}}\frac{dg_{tt}}{d\tau}+\frac{c^2}{2g_{tt}}\frac{\partial g_{tt}}{\partial ct}\frac{d(ct)}{d\tau}\right)d\tau$

Now,

$$\frac{dg_{tt}}{d\tau}=\frac{\partial g_{tt}}{\partial(ct)}\frac{d(ct)}{d\tau}+\frac{\partial g_{tt}}{\partial x}\frac{dx}{d\tau}+\frac{\partial g_{tt}}{\partial y}\frac{dy}{d\tau}+\frac{\partial g_{tt}}{\partial z}\frac{dz}{d\tau}$$

But $\frac{dx}{d\tau}=\frac{dy}{d\tau}=\frac{dz}{d\tau}=0$ according to our scheme[metric may or may not be time dependent]

We have

$$\frac{dg_{tt}}{d\tau}=\frac{\partial g_{tt}}{\partial(ct)}\frac{d(ct)}{d\tau}$$

and consequently

$$\left(-\frac{c^2}{2g_{tt}}\frac{dg_{tt}}{d\tau}+\frac{c^2}{2g_{tt}}\frac{\partial g_{tt}}{\partial ct}\frac{d(ct)}{d\tau}\right)d\tau=0$$

The concerned four dot product

$$g_{ii}\left(\frac{d^2x^i}{d\tau^2}+\Gamma^i{}_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau}\right)dx^i=0$$

[$i = 0,1,2,3$ indexing time and spatial components]

The result is true in presence of gravity alone or in presence of gravity along with other agents[non gravity agents] irrespective of time independence or time dependence of the metric

$(12.2)\Delta W = g_{ii}\left(\frac{d^2x^i}{d\tau^2} + \Gamma^i{}_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau}\right)dx^i = g_{tt}\left(\frac{d^2(ct)}{d\tau^2} + \Gamma^t{}_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau}\right)d(ct)$; on the left side $i = 1,2,3$; here i runs only over spatial components[General relation valid for time dependent or time independent cases].Absolute values of metric coefficients have been considered in the above.

lsThe above equation is true for an arbitrary reference frame as well as for a metric which may be a time dependent or a time independent one . Proper speed components in an arbitrary frame of reference , generally speaking, are not zero when non gravity agents are in action.

In presence of gravity alone $\frac{d^2x^i}{d\tau^2} + \Gamma^i{}_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} = 0$ for each $'i'$. Therefore, $\Delta W = 0$ when gravity is acting alone. We must keep in mind that four force is different from three force in that four acceleration is quite different from what we understand by three acceleration in the usual sense: an apple dropping radially has four acceleration component exactly zero but three acceleration=9.8m/s^2

when it is close to the earth's surface.

Calculating the Exact differentials $g_{tt}\left(\frac{d^2(ct)}{d\tau^2} + \Gamma^t{}_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau}\right) = g_{tt}\left(\frac{d^2ct}{d\tau^2} + 2\Gamma^t{}_{tx}\frac{dct}{d\tau}\frac{dx}{d\tau} + 2\Gamma^t{}_{ty}\frac{dct}{d\tau}\frac{dy}{d\tau} + 2\Gamma^t{}_{tz}\frac{dct}{d\tau}\frac{dz}{d\tau} + \Gamma^t{}_{tt}\left(\frac{dct}{d\tau}\right)^2 + \Gamma^t{}_{xx}\left(\frac{dx}{d\tau}\right)^2 + \Gamma^t{}_{yy}\left(\frac{dy}{d\tau}\right)^2 + \Gamma^t{}_{zz}\left(\frac{dz}{d\tau}\right)^2 + 2\Gamma^t{}_{xy}\frac{dx}{d\tau}\frac{dy}{d\tau} + 2\Gamma^t{}_{yz}\frac{dy}{d\tau}\frac{dz}{d\tau} + 2\Gamma^t{}_{zx}\frac{dz}{d\tau}\frac{dx}{d\tau}\right)$

What you have on the right side of the equation above is for frames where the proper speed components [spatial ones] are not zero

Now some results [3] for orthogonal systems,

$\Gamma^a{}_{bc} = 0$ if$a \neq b \neq c$; $\Gamma^a{}_{aa} = \frac{1}{2g_{aa}}\frac{\partial g_{aa}}{\partial x^a}$; $\Gamma^a{}_{ab} = \frac{1}{2g_{aa}}\frac{\partial g_{aa}}{\partial x^b}$;for$a \neq b$;$\Gamma^a{}_{bb} = -\frac{1}{2g_{aa}}\frac{\partial g_{bb}}{\partial x^a}$;for $a \neq b$;and$\Gamma^t{}_{bb} = 0$ for time independent metrics when $t \neq b$.$\Gamma^a{}_{aa} = \frac{1}{2g_{aa}}\frac{\partial g_{aa}}{\partial x^a}$and for For a=t, $\Gamma^t{}_{tt} = \frac{1}{2g_{tt}}\frac{\partial g_{tt}}{\partial t}$

In general involving, independent metrics

$$\Delta W = g_{tt}\left(\frac{d^2(ct)}{d\tau^2} + \Gamma^t{}_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau}\right)d(ct)$$
$$= g_{tt}\left(\frac{d^2(ct)}{d\tau^2} + 2\Gamma^t{}_{tt}\left(\frac{dct}{d\tau}\right)^2 2\Gamma^t{}_{tx}\frac{d(ct)}{d\tau}\frac{dx}{d\tau} + 2\Gamma^t{}_{ty}\frac{d(ct)}{d\tau}\frac{dy}{d\tau} + 2\Gamma^t{}_{tz}\frac{d(ct)}{d\tau}\frac{dz}{d\tau}\right)d(ct)$$

(13) $g_{tt}\left(\frac{d^2(ct)}{d\tau^2}+\Gamma^t{}_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau}\right)\frac{d(ct)}{d\tau}d\tau = g_{tt}\left(\frac{d^2(ct)}{d\tau^2}+\frac{1}{g_{tt}}\frac{\partial g_{tt}}{\partial ct}\left(\frac{dct}{d\tau}\right)^2+\frac{1}{g_{tt}}\frac{\partial g_{tt}}{\partial x}\frac{d(ct)}{d\tau}\frac{dx}{d\tau}+\frac{1}{g_{tt}}\frac{\partial g_{tt}}{\partial y}\frac{d(ct)}{d\tau}\frac{dy}{d\tau}+\frac{1}{g_{tt}}\frac{\partial g_{tt}}{\partial z}\frac{d(ct)}{d\tau}\frac{dz}{d\tau}\right)\frac{d(ct)}{d\tau}d\tau$

$$= g_{tt}\left(\frac{d^2(ct)}{d\tau^2}\frac{d\tau}{d(ct)}+\frac{1}{g_{tt}}\frac{\partial g_{tt}}{\partial t}\left(\frac{dct}{d\tau}\right)+\frac{1}{g_{tt}}\frac{\partial g_{tt}}{\partial x}\frac{dx}{d\tau}+\frac{1}{g_{tt}}\frac{\partial g_{tt}}{\partial y}\frac{dy}{d\tau}+\frac{1}{g_{tt}}\frac{\partial g_{tt}}{\partial z}\frac{dz}{d\tau}\right)\left(\frac{d(ct)}{d\tau}\right)^2 d\tau$$

$$= g_{tt}\left(\frac{1}{2}\frac{d\left(\frac{d(ct)}{d\tau}\right)^2}{d\tau}\left(\frac{d\tau}{d(ct)}\right)^2+\frac{1}{g_{tt}}\frac{\partial g_{tt}}{\partial t}\left(\frac{dct}{d\tau}\right)+\frac{\partial\ln(g_{tt})}{\partial x}\frac{dx}{d\tau}+\frac{\partial\ln(g_{tt})}{\partial y}\frac{dy}{d\tau}+\frac{\partial\ln(g_{tt})}{\partial z}\frac{dz}{d\tau}\right)\left(\frac{d(ct)}{d\tau}\right)^2 d\tau$$

To note that $\frac{dln(g_{tt})}{d\tau}=\frac{\partial\ln(g_{tt})}{\partial t}\frac{dt}{d\tau}+\frac{\partial\ln(g_{tt})}{\partial x}\frac{dx}{d\tau}+\frac{\partial\ln(g_{tt})}{\partial y}\frac{dy}{d\tau}+\frac{\partial\ln(g_{tt})}{\partial z}\frac{dz}{d\tau}$

$$(14)\quad \Delta W = g_{tt}\left(\frac{1}{2}\frac{d\left(\frac{d(ct)}{d\tau}\right)^2}{d\tau}\left(\frac{d\tau}{d(ct)}\right)^2+\frac{d\ln(g_{tt})}{d\tau}\right)\left(\frac{d(ct)}{d\tau}\right)^2 d\tau$$

The above is valid in a general manner irrespective of time dependence or time independence of the metric

(14.1) $\Delta W = g_{tt}\left(\frac{d^2(ct)}{d\tau^2}+\Gamma^t{}_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau}\right)\frac{dt}{d\tau}d\tau = g_{tt}\left(\frac{1}{2}\frac{d\left(\frac{d(ct)}{d\tau}\right)^2}{d\tau}\left(\frac{d\tau}{d(ct)}\right)^2+\frac{d\ln(g_{tt})}{d\tau}\right)\left(\frac{d(ct)}{d\tau}\right)^2 d\tau$

Let us go back to the metric for a moment

$$c^2 = g_{tt}\left(\frac{dct}{d\tau}\right)^2 - g_{xx}\left(\frac{dx}{d\tau}\right)^2 - g_{yy}\left(\frac{dy}{d\tau}\right)^2 - g_{zz}\left(\frac{dz}{d\tau}\right)^2$$

$$g_{tt}\left(\frac{dct}{d\tau}\right)^2 = c^2 + g_{xx}\left(\frac{dx}{d\tau}\right)^2 + g_{yy}\left(\frac{dy}{d\tau}\right)^2 + g_{zz}\left(\frac{dz}{d\tau}\right)^2$$

$$c^2 g_{tt}\left(\frac{dct}{d\tau}\right)^2 = c^4 + g_{xx}c^2\left(\frac{dx}{d\tau}\right)^2 + c^2 g_{yy}\left(\frac{dy}{d\tau}\right)^2 + g_{zz}\left(\frac{dz}{d\tau}\right)^2$$

Using (5.1) to (5.4)

$$m_0{}^2 c^2 g_{tt}\left(\frac{dct}{d\tau}\right)^2 = m_0{}^2 c^4 + c^2\vec{p}^2$$

Where $\vec{p}^2 = g_{xx}\left(\frac{dx}{d\tau}\right)^2 + g_{yy}\left(\frac{dy}{d\tau}\right)^2 + g_{zz}\left(\frac{dz}{d\tau}\right)^2$

[above is the dot product between $\vec{p}$ and $\vec{p}$: norm squared in the curved space time context; $\vec{v} = \left(\frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau}\right)$

But ${m_0}^2c^4 + c^2\vec{p}^2 = E^2$:Relaivistic expression for Energy

$$E^2 = {m_0}^2c^2 g_{tt}\left(\frac{dct}{d\tau}\right)^2 \Longrightarrow E^2 = {m_0}^2c^2\left(\frac{dcT}{d\tau}\right)^2$$

$$E = m_0 c\sqrt{g_{tt}}\frac{dct}{d\tau} = m_0 c\frac{dcT}{d\tau} \Longrightarrow E = m_0 c\frac{d^2cT}{d\tau^2}$$

(15) $\left(\frac{dcT}{d\tau}\right)^2 = \frac{E^2}{{m_0}^2c^2}$

[for unit rest mass $m_0 = 1unit$]

We re write (14.1) [Time independent metrics]

$$\Delta W = g_{tt}\left(\frac{d^2ct}{d\tau^2} + \Gamma^t{}_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau}\right)\frac{dct}{d\tau}d\tau = \left(\frac{1}{2}\frac{d\left(\frac{d(ct)}{d\tau}\right)^2}{d\tau}\left(\frac{d\tau}{d(ct)}\right)^2 + \frac{d\ln(g_{tt})}{d\tau}\right)\left(\frac{d(cT)}{d\tau}\right)^2 d\tau$$

Since $g_{tt}\left(\frac{dct}{d\tau}\right)^2 = \left(\frac{dcT}{d\tau}\right)^2$[using (5.1)

$$\Delta W = g_{tt}\left(\frac{d^2ct}{d\tau^2} + \Gamma^t{}_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau}\right)\frac{dt}{d\tau}d\tau = \left(\frac{1}{2}\frac{d\left[\frac{1}{g_{tt}}\left(\frac{dcT}{d\tau}\right)^2\right]}{d\tau}\left(\frac{d\tau}{dct}\right)^2 + \frac{d\ln(g_{tt})}{d\tau}\right)\left(\frac{dcT}{d\tau}\right)^2 d\tau \quad \text{[since } dT = \sqrt{g_{tt}}dt\text{]}$$

Important to note:

$\frac{dg_{tt}}{d\tau} = \frac{\partial g_{tt}}{\partial(ct)}\frac{d(ct)}{d\tau}$*for the special form where proper speed components[spatial] are zero*

$$\Delta W = \left(\frac{1}{2}\frac{1}{g_{tt}}\frac{d\left(\frac{dcT}{d\tau}\right)^2}{d\tau}\left(\frac{d\tau}{dct}\right)^2 - \frac{1}{2{g_{tt}}^2}\frac{dg_{tt}}{d\tau}\left(\frac{dcT}{d\tau}\right)^2\left(\frac{d\tau}{dct}\right)^2 + \frac{d\ln(g_{tt})}{d\tau}\right)\left(\frac{dcT}{d\tau}\right)^2 d\tau$$

Work done per unit rest mass

$$\Delta W = \left(\frac{1}{2}\frac{1}{g_{tt}}\frac{d\left(\frac{dcT}{d\tau}\right)^2}{d\tau}\left(\frac{d\tau}{dcT}\right)^2 g_{tt} - \frac{1}{2{g_{tt}}^2}\frac{dg_{tt}}{d\tau}\left(\frac{dcT}{d\tau}\right)^2\left(\frac{d\tau}{dcT}\right)^2 g_{tt} + \frac{d\ln(g_{tt})}{d\tau}\right)\left(\frac{dcT}{d\tau}\right)^2 d\tau$$

[From the second last step to the last one we have replaced dt by$\frac{1}{\sqrt{g_{tt}}}dT$in $\frac{d\tau}{dct}$]

$$\Delta W=\left(\frac{1}{2}\frac{d\left(\frac{dcT}{d\tau}\right)^2}{d\tau}\left(\frac{d\tau}{dcT}\right)^2-\frac{1}{2g_{tt}}\frac{dg_{tt}}{d\tau}+\frac{d\ln(g_{tt})}{d\tau}\right)\left(\frac{dcT}{d\tau}\right)^2 d\tau$$

$$\Delta W=\left(\frac{1}{2}\frac{d\left(\frac{dcT}{d\tau}\right)^2}{d\tau}-\frac{1}{2g_{tt}}\frac{dg_{tt}}{d\tau}\left(\frac{dcT}{d\tau}\right)^2+\frac{d\ln(g_{tt})}{d\tau}\left(\frac{dcT}{d\tau}\right)^2\right)d\tau$$

Applying (15) on the above,

$$\Delta W=\left(\frac{1}{2}\frac{1}{(c^2{m_0}^2)}\frac{d(E^2)}{d\tau}-\frac{1}{2g_{tt}}\frac{dg_{tt}}{d\tau}\frac{E^2}{{m_0}^2c^2}+\frac{d\ln(g_{tt})}{d\tau}\frac{E^2}{{m_0}^2c^2}\right)d\tau$$

$$\Delta W=\left(\frac{1}{2}\frac{1}{(c^2{m_0}^2)}\frac{d(E^2)}{d\tau}-\frac{1}{2}\frac{d\ln(g_{tt})}{d\tau}\frac{E^2}{{m_0}^2c^2}+\frac{d\ln(g_{tt})}{d\tau}\frac{E^2}{{m_0}^2c^2}\right)d\tau=$$

$$\Delta W=\left(\frac{1}{2}\frac{1}{(c^2{m_0}^2)}\frac{d(E^2)}{d\tau}+\frac{1}{2}\frac{d\ln(g_{tt})}{d\tau}\frac{E^2}{c^2{m_0}^2}\right)d\tau$$

Therefore,

$$c^2{m_0}^2\frac{dW}{E^2}=\frac{1}{2}\frac{d(E^2)}{E^2}+\frac{1}{2}dlng_{tt}\Rightarrow c^2{m_0}^2\frac{dW}{E^2}=\frac{dE}{E}+\frac{1}{2}dlng_{tt}$$

$$\Longrightarrow c^2{m_0}^2\frac{\Delta W}{E^2}=dlnE+\frac{1}{2}dlng_{tt}\Rightarrow c^2{m_0}^2\frac{dW}{E^2}=dln(m_0\gamma c^2)+\frac{1}{2}dlng_{tt}-\frac{\partial\ln(g_{tt})}{\partial t}dt$$

$$\Rightarrow c^2{m_0}^2\frac{\Delta W}{E^2}=d(lnm_0c^2)+dln\gamma|+\frac{1}{2}dlng_{tt}$$

$$\Rightarrow c^2{m_0}^2\frac{\Delta W}{E^2}=dln\gamma+\frac{1}{2}dlng_{tt}$$

(16)$c^2\frac{g_{tt}\Delta W}{E^2}=d\left[ln\gamma+\frac{1}{2}lng_{tt}\right]$;Unit rest mass being considered.

Relation (16) is valid irrespective of the time dependence or time independence of the metric.

For geodesics $\Delta W=0$ *since the four acceleration components involved in our definition of* ΔW *are zero.*

$$(17)\ c^2\frac{g_{tt}\Delta W}{E^2}=d\left[ln\gamma+\frac{1}{2}lng_{tt}\right]=0\Rightarrow ln\gamma+\frac{1}{2}lng_{tt}=Constant\Rightarrow ln\gamma=Constant-\frac{1}{2}lng_{tt}$$

In presence of gravity alone:

$$ln\gamma=Constant-\frac{1}{2}lng_{tt}$$

The same relation holds for the time independent case.

The physical velocity depends only on g_{tt}

It is getting uniquely determined by g_{tt}Components may change preserving the spatial magnitude [so long as gravity being the only agent in action]. If you push up a particle it will come down **to** the same point with the same speed. A satellite always has the same speed at the same point of the orbit. Actually the constant will be different for different motions[different initial conditions]. If I throw up a particle by exerting a greater force, it will move upwards with a greater speed and come down to the same spatial point[lng_{tt} being independent of time since we are using time independent metrics] with the same speed at which it passed up. A particle may have different speeds at the same point due to different initial conditions brought about by nom gravity agents. Different initial conditions will change the value of the constant in (17)

$$ln\gamma = Constant(initial\ conditions) - \frac{1}{2} lng_{tt}$$

(18) $\int_A^B c^2 \frac{g_{tt} dW}{E^2} = (ln\gamma_B - ln\gamma_A) + \frac{1}{2}(lng_{tt:B} - lng_{tt:A})$

The right side is a function of only the initial point (A) and the final point (B). It may be used to develop a rigorous potential function in general relativity

Verifying with Classical limit [as a limiting case of Schwarzschild Geometry]:

$$\int_A^B c^2 \frac{g_{tt} dW}{E^2} = (ln\gamma_B - ln\gamma_A) + \frac{1}{2}(lng_{tt:B} - lng_{tt:A})$$

For geodesic motion $dW = 0$

$$\Rightarrow (ln\gamma_B - ln\gamma_A) + \frac{1}{2}(lng_{tt:B} - lng_{tt:A}) = 0$$

1)$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \approx 1 + \frac{v^2}{2c^2}$ [First order approximation: binomial series]

where $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$

Therefore,

$$ln\gamma = ln\left[1 + \frac{v^2}{2c^2}\right] = \frac{1}{2}\left(\frac{v^2}{c^2}\right): log\ series\ approximation, first\ order$$

Therefore,

$$(ln\gamma_B - ln\gamma_A) = \frac{1}{2c^2}({v_B}^2 - {v_A}^2)$$

2) $lng_{tt:B} - lng_{tt:A} = \ln \frac{g_{tt:B}}{g_{tt:A}} = ln \frac{1-\frac{2GM}{c^2 r_B}}{1-\frac{2GM}{c^2 r_A}} = ln\left(1 - \frac{2GM}{c^2 r_B}\right)\left(1 + \frac{2GM}{c^2 r_A}\right) = ln\left(1 + \frac{2GM}{c^2 r_A} - \frac{2GM}{c^2 r_B} + higherorderterms\right) \approx \frac{2GM}{c^2 r_A} - \frac{2GM}{c^2 r_B}$

Therefore,

$lng_{tt:B} - lng_{tt:A} = \ln\frac{g_{tt:B}}{g_{tt:A}} = \frac{2GM}{c^2 r_A} - \frac{2GM}{c^2 r_B} = -\frac{2GM}{c^2}\left(\frac{1}{r_B} - \frac{1}{r_A}\right)$; [First order approximation: log series expansion]

Therefore,

$$\frac{1}{2c^2}({v_B}^2 - {v_A}^2) - \frac{GM}{c^2}\left(\frac{1}{r_B} - \frac{1}{r_A}\right) = 0$$

$\frac{1}{2}{v_A}^2 + \frac{-GM}{r_A} = \frac{1}{2}{v_B}^2 + \frac{-GM}{r_B}$*[valid result in the classical limit]*

Supplementary Material

From the invariance of the four dot product it follows that

$$\frac{dP^i}{d\tau}dX^i = g_{ii}\left(\frac{d^2x^i}{d\tau^2} + \Gamma^i{}_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\alpha}{d\tau}\right)dx^i \quad (1)$$

The same does not hold when i runs only over spatial indices

$$\frac{dP^i}{d\tau}dX^i \neq g_{ii}\left(\frac{d^2x^i}{d\tau^2} + \Gamma^i{}_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\alpha}{d\tau}\right)dx^i \ (2)$$

Metric

$$c^2 d\tau^2 = g_{tt}d(ct)^2 - g_{xx}dx^2 - g_{yy}dy^2 - g_{zz}dz^2 \ (3)$$

Using the transformations(5.1) to (5.4) in the original paper we obtain an equivalent metric Lorentzian in form

Transformations:

$$dT = \sqrt{g_{tt}}dt$$

(5.2) $dX = \sqrt{g_{xx}}dx$

(5.3) $dY = \sqrt{g_{yy}}dy$

(5.4) $dZ = \sqrt{g_{zz}}dz$

Equivalent metric:

$$c^2 d\tau^2 = d(cT)^2 - dX^2 - dY^2 - dZ^2 \ (4)$$

$$c^2 = \left(\frac{d(cT)}{d\tau}\right)^2 - \left(\frac{dX}{d\tau}\right)^2 - \left(\frac{dY}{d\tau}\right)^2 - \left(\frac{dZ}{d\tau}\right)^2 \ (4.1)$$

Differentiating with respect to proper time

$$0 = 2\frac{d(cT)}{d\tau}\frac{d^2(cT)}{d\tau^2} - 2\frac{dX}{d\tau}\frac{d^2X}{d\tau^2} - 2\frac{dY}{d\tau}\frac{d^2Y}{d\tau^2} - 2\frac{dZ}{d\tau}\frac{d^2Z}{d\tau^2}$$

$$\frac{dX}{d\tau}\frac{d^2X}{d\tau^2} + \frac{dY}{d\tau}\frac{d^2Y}{d\tau^2} + \frac{dZ}{d\tau}\frac{d^2Z}{d\tau^2} = \frac{d^2(cT)}{d\tau^2}\frac{d(cT)}{d\tau}$$

Four dot product $\frac{dP^i}{d\tau}dX^i = 0 \Longrightarrow g_{ii}\left(\frac{d^2x^i}{d\tau^2} + \Gamma^i{}_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\alpha}{d\tau}\right)dx^i = 0$

Nevertheless an alternative mechanism has been provided in the original paper to show

$$g_{ii}\left(\frac{d^2x^i}{d\tau^2} + \Gamma^i{}_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\alpha}{d\tau}\right)dx^i = 0$$

to corroborate the overall effectiveness of the transformed metric.

Work for unit rest mass:

$$dW = \frac{dP^i}{d\tau}dX^i = \frac{d^2X}{d\tau^2}dX + \frac{d^2Y}{d\tau^2}dY + \frac{d^2Z}{d\tau^2}dZ = \frac{d^2(cT)}{d\tau^2}d(cT) = \frac{dP^T}{d\tau}dT$$

$$dW = \frac{dP^i}{d\tau}dX^i = \frac{dP^T}{d\tau}dT \quad (5)$$

[In the above $'i'$ is the spatial index. T refers to time with the transformed metric: no summation on T]

$$\frac{dP^T}{d\tau}dT = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{dT}{d\tau}\right)dT = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\sqrt{g_{tt}}dt}{d\tau}\right)\sqrt{g_{tt}}dT = \left[\sqrt{g_{tt}}\frac{d^2t}{d\tau^2} + \frac{1}{2\sqrt{g_{tt}}}\frac{dg_{tt}}{d\tau}\frac{dt}{d\tau}\right]\sqrt{g_{tt}}dt$$

$$dW = \frac{dP^T}{d\tau}dT = g_{tt}\left[\frac{d^2t}{d\tau^2} + \frac{1}{2g_{tt}}\frac{dg_{tt}}{d\tau}\frac{dt}{d\tau}\right]dt \quad (6)$$

We digress for calculating the Lorentz factor in General Relativity

$$c^2d\tau^2 = c^2g_{tt}dt^2 - g_{xx}dx^2 - g_{yy}dy^2 - g_{zz}dz^2$$

$$\left(\frac{d\tau}{dt}\right)^2 = g_{tt} - \frac{1}{c^2}\frac{g_{xx}dx^2 + dy^2 + g_{zz}dz^2}{dt^2}$$

$$\frac{1}{g_{tt}}\left(\frac{d\tau}{dt}\right)^2 = 1 - \frac{1}{c^2}\frac{g_{xx}dx^2 + dy^2 + g_{zz}dz^2}{g_{tt}dt^2}$$

$$\left(\frac{d\tau}{\sqrt{g_{tt}}dt}\right)^2 = 1 - \frac{{v_{ph}}^2}{c^2}$$

$$v_{ph} = \frac{g_{xx}dx^2 + dy^2 + g_{zz}dz^2}{(\sqrt{g_{tt}}dt)^2}$$

$$\left(\frac{d\tau}{dT}\right)^2 = 1 - \frac{{v_{ph}}^2}{c^2}$$

$$\gamma_{ph} = \frac{dT}{d\tau} = \frac{1}{\sqrt{1 - \frac{{v_{ph}}^2}{c^2}}} \quad (7)$$

Again

$$\left(\frac{d\tau}{dt}\right)^2 = g_{tt} - \frac{1}{c^2} g_{tt} \frac{g_{xx}dx^2 + dy^2 + g_{zz}dz^2}{g_{tt}dt^2}$$

$$\left(\frac{d\tau}{dt}\right)^2 = g_{tt}\left[1 - \frac{1}{c^2}\frac{g_{xx}dx^2 + dy^2 + g_{zz}dz^2}{g_{tt}dt^2}\right]$$

$$\left(\frac{d\tau}{dt}\right)^2 = g_{tt}\left[1 - \frac{{v_{ph}}^2}{c^2}\right]$$

Again

$$\gamma = \frac{dt}{d\tau} = \frac{1}{\sqrt{g_{tt}}} \frac{1}{\sqrt{1 - \frac{{v_{ph}}^2}{c^2}}} = \frac{1}{g_{tt}} \times \gamma_{ph} \quad (7.1)$$

Using work energy theorem we obtain by usual process the formula for kinetic energy. Since the converted/transformed metric is of Minkowski form we simply repeat the procedure for Special Relativity

KE=$mc^2 - m_0c^2 = m_0\gamma_{ph}c^2 - m_0c^2$

Work done =$m_0\gamma'_{ph}c^2 - m_0\gamma_{ph}c^2$

Differential amount of work for unit mass

$$dW = \frac{dP^T}{d\tau} dT = c^2 d\gamma_{ph}(t, x, y, z)$$

$$dW = {\gamma_{ph}}^3 v_{ph} dv_{ph}$$

Again

$$\frac{dP^T}{d\tau} dT = g_{tt}\left[-\Gamma^t{}_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\alpha}{d\tau} + \frac{1}{2}\frac{dln g_{tt}}{d\tau}\frac{dt}{d\tau}\right] dt \quad (8)$$

Using from the original paper,

$$d\left[ln\gamma_{ph} + \frac{1}{2} ln g_{tt}\right] = 0 \quad (9)$$

for geodesic motion, we obtain

$$\frac{dP^T}{d\tau}dT = g_{tt}\left[-\Gamma^t{}_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\alpha}{d\tau} - \frac{dln\gamma_{ph}}{d\tau}\frac{dt}{d\tau}\right]dt$$

$$\frac{dP^T}{d\tau}dT = g_{tt}\left[-\Gamma^t{}_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\alpha}{d\tau} - {\gamma_{ph}}^2\frac{1}{c^2}\frac{d(v_{ph})}{d\tau}\gamma\right]dt$$

$$\frac{dP^T}{d\tau}dT = g_{tt}\left[-\Gamma^t{}_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\alpha}{d\tau} - \frac{1}{\gamma_{ph}c^2}\frac{dW}{d\tau}\gamma\right]dt$$

$$\frac{dP^T}{d\tau}dT + g_{tt}\frac{1}{\gamma_{ph}c^2}\frac{dW}{d\tau}\gamma dt = -g_{tt}\Gamma^t{}_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\alpha}{d\tau}dt$$

$$dW + \frac{1}{c^2}\frac{dW}{d\tau}dt = -g_{tt}\Gamma^t{}_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\alpha}{d\tau}dt$$

$$dW\left[1 + \frac{1}{c^2}\frac{dt}{d\tau}\right] = -g_{tt}\Gamma^t{}_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\alpha}{d\tau}dt$$

$$dW = \frac{1}{1 + \frac{1}{c^2}\gamma\frac{dt}{d\tau}} \times \left[-g_{tt}\Gamma^t{}_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\alpha}{d\tau}dt\right] \neq 0 \quad (10)$$

$\left[-g_{tt}\Gamma^t{}_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\alpha}{d\tau}dt\right]$:Work done by Gravity: Christoffel Symbols are directly involved

$$\frac{1}{1 + \frac{1}{c^2}\gamma\frac{dt}{d\tau}} = \frac{1}{1 + \frac{1}{c^2}\gamma^2} : Scale\ factor : \gamma = \frac{dt}{d\tau}$$

The scale factor is approximately unity for n on relativistic motion.

$dW \neq 0$ for geodesic motion when the definition of work is taken as $dW = \frac{dP^i}{d\tau}dX^i$, I being the spatial index.

Classically and in Special Relativity work done is either change in Kinetic energy or change of potential energy when conservative forces are in action

$$\Delta\mathrm{W} = \Delta(\text{Kinetic energy})$$

For conservative forces: KE+PE=constant

$$\Delta\mathrm{KE} + \Delta\mathrm{PE} = 0$$

In the General Relativity scenario: $\Delta\mathrm{W} = 0$in presence of gravity alone [geodesics]when

$$\Delta\mathrm{W} = g_{ii}\left(\frac{d^2x^i}{d\tau^2} + \Gamma^i{}_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\alpha}{d\tau}\right); i\ spatial\ index$$

But

$$\Delta W \neq 0$$

when ΔW is defined by

$$\Delta W = \frac{dP^i}{d\tau} dX^i; i\ spatial\ index$$

<u>On Physical Speed</u>:

Referring to the metric:

$$c^2 d\tau^2 = g_{tt} d(ct)^2 - g_{xx} dx^2 - g_{yy} dy^2 - g_{zz} dz^2$$

and the subsequent metric of flay space time orm as obtained by transformations 95.1) to (5.4) we have

$$c^2 d\tau^2 = d(cT)^2 - dX^2 - dY^2 - dZ^2$$

$$c^2 d\tau^2 = d(cT)^2 - dL^2$$

$$dL^2 = dX^2 - dY^2 - dZ^2$$

dL Spatial separation

For null paths

$$d\tau = 0$$

We have,

$$0 = c^2 dT^2 - dL^2$$

For null geodesics we have

$$\frac{dL}{dT} = c$$

This formulation does not upset the speed barrier in that the metric is of Minkowski form.

Conclusion

Our definition of work is hinged on the invariance of dot product. In the weak field limit It reduces to the classical result of energy conservation. Schwarzschild Geometry in the weak field limit has been used for the verification. Our definition of work also coincides with that in Special Relativity in the limit of zero curvature with the disappearance of curvature in the flat space limit. The definition is an extrapolation of what we understand by work in Special Relativity.

References

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