A COLLECTION OF DOUBLE INTEGRALS INVOLVING PI , BAILEY'S ISSUE

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1. Introducción

En la referencia (1) D.H.Bailey (2010) , pag.6 , fórmula (58) aparece la integral:

$$I=-\frac{\pi}{6}+\ln\left(2+\sqrt{3}\right)=\int_0^1\int_0^1\frac{1}{\sqrt{1+x^2+y^2}}\,dx\,dy$$

En esta nota mostramos una colección de integrales

dobles equivalentes a I.

2. Integrales

$$I=\int_0^1\int_1^\infty\frac{1}{x\sqrt{1+x^2+x^2y^2}}\,dx\,dy$$

$$I=\int_1^\infty\int_1^\infty\frac{1}{x\,y\sqrt{x^2+y^2+x^2y^2}}\,dx\,dy$$

$$I=\frac{1}{2}\int_0^1\int_0^1\frac{1}{\sqrt{x}\sqrt{1+x+y^2}}\,dx\,dy$$

$$I=\frac{1}{4}\int_0^1\int_0^1\frac{1}{\sqrt{x\,y}\sqrt{1+x+y}}\,dx\,dy$$

$$I=\frac{1}{4\sqrt{2}}\int_1^3\int_1^3\frac{1}{\sqrt{x+y}\sqrt{(x-1)(y-1)}}\,dx\,dy$$

$$I=\frac{1}{4\sqrt{2}}\int_{-1}^1\int_{-1}^1\frac{1}{\sqrt{(1+x)(1+y)}\sqrt{4+x+y}}\,dx\,dy$$

ABSTRACT:

In this note presents a collection of double integrals involving constant pi.

RESUMEN:

En esta nota mostramos una colección de integrales dobles que involucran a la constante pi.

$$I=\int_0^1\int_0^\infty \frac{e^{-x}}{\sqrt{1+e^{-2x}+y^2}}\,dx\,dy$$

$$I=\int_0^\infty\int_0^\infty \frac{e^{-x-y}}{\sqrt{1+e^{-2x}+e^{-2y}}}\,dx\,dy$$

$$I=\int_{-\infty}^\infty\int_{-\infty}^\infty \frac{e^{x+y-e^x-e^y}}{\sqrt{1+e^{-2e^x}+e^{-2e^y}}}\,dx\,dy$$

$$I=\frac{1}{4}\int_{e^{-1}}^1\int_{e^{-1}}^1 \frac{1}{x\,y\,\sqrt{\ln x\ln y}\,\sqrt{1-\ln x\,y}}\,dx\,dy$$

$$I=\int_0^1\int_0^{\ln(1+\sqrt{2})} \frac{\cosh x}{\sqrt{(\cosh x)^2+y^2}}\,dx\,dy$$

$$I=\int_0^{\ln(1+\sqrt{2})}\int_0^{\ln(1+\sqrt{2})} \frac{\cosh x\ \cosh y}{\sqrt{(\cosh x)^2+(\sinh y)^2}}\,dx\,dy$$

$$I=\frac{1}{2}\int_0^1\int_{-1}^1 \frac{1}{\sqrt{1+x^2+y^2}}\,dx\,dy$$

$$I=\frac{1}{4}\int_{-1}^1\int_{-1}^1 \frac{1}{\sqrt{1+x^2+y^2}}\,dx\,dy$$

$$I=n\,k\int_0^1\int_0^1 \frac{x^{n-1}y^{k-1}}{\sqrt{1+x^{2n}+y^{2k}}}\,dx\,dy\ \ ,n,k>0$$

$$I=\int_1^2\int_1^2 \frac{1}{\sqrt{3-2x-2y+x^2+y^2}}\,dx\,dy$$

$$I=\int_{-1}^0\int_{-1}^0 \frac{1}{\sqrt{3+2x+2y+x^2+y^2}}\,dx\,dy$$

$$I=\ln\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)+\int_0^1\int_0^1 \frac{x^2}{(1+x^2+y^2)^{3/2}}\,dx\,dy$$

$$I = \ln\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right) + \frac{1}{6\sqrt{3}} + \int_0^1 \int_0^1 \frac{x^4}{(1+x^2+y^2)^{5/2}}\,dx\,dy$$

$$I = 2\int_0^1 \int_0^y \frac{1}{\sqrt{1+x^2+y^2}}\,dx\,dy$$

$$I = \frac{1}{4}\int_0^1 \int_0^y \frac{1}{\sqrt{x\,y}\,\sqrt{x+y+y^2}}\,dx\,dy$$

$$I = \int_1^\infty \int_y^\infty \frac{1}{x\,\sqrt{x^2+x^2y^2+y^4}}\,dx\,dy$$

$$I = \int_0^{1/\sqrt{2}} \int_0^{1/\sqrt{2}} \frac{1}{\sqrt{1+x^2+y^2}}\,dx\,dy + 2\int_{1/\sqrt{2}}^{1} \int_0^{y} \frac{1}{\sqrt{1+x^2+y^2}}\,dx\,dy$$

$$I = \int_0^1 \int_y^{\sqrt{1+y^2}} \frac{x}{\sqrt{(x^2-y^2)(1+x^2)}}\,dx\,dy$$

$$I = \int_0^1 \int_1^{\sqrt{2}} \frac{x}{\sqrt{(x^2+y^2)(x^2-1)}}\,dx\,dy$$

$$I = \frac{1}{4}\int_0^\infty \int_0^\infty \sqrt{\frac{e^{-x-y}}{1+e^{-x}+e^{-y}}}\,dx\,dy$$

$$I = 4\int_0^{1/2} \int_y^{1-y} \frac{1}{\sqrt{1+2x^2+2y^2}}\,dx\,dy$$

$$I = \frac{1}{\sqrt{1+a}}\int_0^1 \int_0^1 F\left(\frac{1}{2},1;1;\frac{a-x^2-y^2}{1+a}\right)dx\,dy \ \ ,a > 1/2$$

donde F es la función hipergeométrica de Gauss.

$$I = \frac{\pi\left(\sqrt{2}-1\right)}{2} + \int_0^1 \int_{\sqrt{1-y^2}}^{1} \frac{1}{\sqrt{1+x^2+y^2}}\,dx\,dy$$

$$I=\int_{\pi}^{3\pi/2}\int_{0}^{1}\frac{r}{\sqrt{3+r^2+2r(\sin\theta+\cos\theta)}}dr\,d\theta+\int_{0}^{1}\int_{0}^{1-\sqrt{2y-y^2}}\frac{1}{\sqrt{1+x^2+y^2}}dx\,dy$$

$$I=\int_{\pi/2}^{\pi}\int_{0}^{1}\frac{r}{\sqrt{2+r^2+2r\cos\theta}}dr\,d\theta+\int_{0}^{1}\int_{0}^{1-\sqrt{1-y^2}}\frac{1}{\sqrt{1+x^2+y^2}}dx\,dy$$

$$I=\int_{3\pi/2}^{2\pi}\int_{0}^{1}\frac{r}{\sqrt{2+r^2+2r\sin\theta}}dr\,d\theta+\int_{0}^{1}\int_{\sqrt{2y-y^2}}^{1}\frac{1}{\sqrt{1+x^2+y^2}}dx\,dy$$

3. Serie para I

$$I=\frac{1}{\sqrt{2}}\sum_{n=0}^{\infty}\binom{2n}{n}2^{-3n}f(n)$$

donde

$$f(n)=\sum_{k=0}^{n}\binom{n}{k}(-1)^k\sum_{m=0}^{k}\binom{k}{m}\frac{1}{(2k-2m+1)(2m+1)}$$

$$f(n)=\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{2k+1}F(\left\{\frac{1}{2},-k,-k-\frac{1}{2}\right\},\left\{\frac{3}{2},-k+\frac{1}{2}\right\},-1)$$

donde F es la función hipergeométrica.

4. Integrales simples relacionadas

$$I=\int_{0}^{1}\sinh^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)dx$$

$$I=\int_{1/\sqrt{2}}^{1}\frac{\sinh^{-1}x}{x^2\sqrt{1-x^2}}dx$$

$$I=\int_{0}^{\pi/4}\frac{\sinh^{-1}(\cos x)}{(\cos x)^2}dx$$

$$I = \int_{\pi/4}^{\pi/2} \frac{\sinh^{-1}(\sin x)}{(\sin x)^2} dx$$

$$I = \int_a^b \frac{x \cosh x}{(\sinh x)^2 \sqrt{1-(\sinh x)^2}} dx$$

donde $a = \ln(1+\sqrt{2})$, $b = \ln\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)$.

$$\int \sinh^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right) dx = x \sinh^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right) + \sinh^{-1}\left(\frac{x}{\sqrt{2}}\right) - \tan^{-1}\left(\frac{x}{\sqrt{2+x^2}}\right)$$

5. Una integral doble

$$\int_0^1 \int_0^1 \frac{1}{\sqrt{1+x^2+(\sqrt{2}-1)y^2}} dx\, dy$$

$$= \sqrt{\sqrt{2}+1}\ \ln\left(\sqrt{\sqrt{2}+1}\right) + \ln\left(\frac{\sqrt[4]{8}+\sqrt{4+2\sqrt{2}}}{2}\right) - \frac{\pi}{8}\sqrt{\sqrt{2}+1}$$

REFERENCIAS

1.D.H. Bailey,"A collection of mathematical formulas involving π ",November 4 ,2010,Lawrence Berkeley National Laboratory, Berkeley,CA 94720,dhbailey@lbl.gov.

2.J.M. Borwein, D.H. Bailey and Roland Girgensohn, Experimentation in Mathematics: Computational Paths to Discovery, AK Peters,Natick, MA 2004.