The “Vertical” (generalization of) the Binary Goldbach’s conjecture as applied on “iterative” primes with (recursive) prime indexes (i-primeths)[1,2,3]

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Andrei-Lucian Drăgoi (January-February 2017) (study research interval: 2007-2017 - ?) [4,5]

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The latest variants of this article can be found at: URL1 (the conjecture only), URL2 (full version)

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Abstract

This paper proposes the generalization of the both binary (strong) and ternary (weak) Goldbach’s Conjectures (**BGC and TGC)[1,2,3][4,5,6,7]**, briefly called “the Vertical Goldbach’s Conjectures” (**VBGC and VTGC**), which are essentially meta-conjectures (as VBGC states an infinite number of conjectures stronger than BGC). VBGC was discovered in 2007**[1]** and perfected until 2017**[3]** by using the arrays (S_p and $S_{i,p}$) of Matrix of Goldbach index-partitions (**GIPs**) (simple $M_{p,n}$ and recursive $M_{i,p,n}$, with iteration order $i \geq 0$), which are a useful tool in studying BGC by focusing on prime indexes (as the function P_n that numbers the primes is a bijection). Simple M $\left(M_{p,n}\right)$ and recursive M $\left(M_{i,p,n}\right)$ are related to the ***concept of generalized “primeths”*** (a term first introduced by N. J. A. Sloane and Robert G. Wilson in their “primeth recurrence” concept [A007097]; the term “primeth” was then used from 1999 by Fernandez N. in his “The Exploring Primeness Project” **[8]**), *which is the generalization with iteration order $i \geq 0$ of the known “higher-order prime numbers” (alias “super-prime numbers”, “super-prime numbers”, ”super-primes”, ” super-primes” or “prime-indexed primes[PIPs]”) as a subset of (simple or recursive) primes with (also) prime indexes (${}^{i}P_x$ is the x-th i-primeth, with iteration order $i \geq 0$ as explained later on).*

The author of this article also brings in a **S-M-synthesis** of some Goldbach-like conjectures (**GLC**) (including those which are “stronger” than BGC) and a new class of **GLC**s “stronger” than BGC, from which VBGC (which is essentially a variant of BGC applied on a serial array of subsets of primeths with a general iteration order $i \geq 0$) distinguishes as a very important conjecture of primes (with great importance in the optimization of the BGC experimental verification and other potential useful theoretical and practical applications in mathematics [including cryptography and fractals] and physics [including crystallography and M-Theory]), and a very special self-similar property of the primes subset of $\mathbb{N}$ (noted/abbreviated as $\wp$ or $\wp^*$ as explained later on in this article).

Keywords: Prime (number), primes with prime indexes aka prime-index primes (**PIPs**), i-primeths (i-PIPs with iteration order i≥0), the Binary Goldbach Conjecture (**BGC**), the Ternary Goldbach Conjecture (**TGC**), Goldbach index-partition (**GIP**), fractal patterns of the number and distribution of Goldbach index-partitions, Goldbach-like conjectures (**GLC**), the Vertical Binary Goldbach Conjecture (**VBGC**) and Vertical Ternary Goldbach Conjecture (**VTGC**) the as applied on i-primeths

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[1] Online preprint – VBGC versions 1.0/1.1/1.2/1.3/1.4/1.5: DOI: 10.13140/RG.2.2.27963.62245 (full article); DOI: 10.13140/RG.2.2.14014.28484 (VBGC/conjecture only)
[2] Discovered in December 2007 as a special case of Goldbach’s Conjecture and registered in 2012 (in this initial variant) in “Plicul cu idei” (“The envelope of ideas”) (OSIM, Romania) with number: **300323/22.08.2012** (generic URL)
[3] ORDA (Romania) registration number: **4856/23.06.2016** (URL: orda.ro/cautare_cerere.aspx?mid=1&rid=1&cerere=4856)
[4] Romanian pediatrician specialist (with no additional academic title) undertaking independent research in the field of Prime Numbers
[5] Contact email: dr.dragoi@yahoo.com

Introduction

*Primes (which are considered natural numbers [positive integers] >1 that each has no positive divisors other than 1 and itself by the latest modern conventional definition, as number 1 is a special case***[9,10]** *which is considered neither prime nor composite, but the unit of all integers) are conjectured to have a sufficiently dense and (sufficiently) uniform distribution in* $\mathbb{N}$*, so that: (**1**) any natural even number* $\boxed{2n, with\ n>1}$ *can be splitted in at least one Goldbach partition/pair(GP)***[11]** OR (***2***) *any positive integer* $\boxed{n>1}$ *can be expressed as the arithmentic average of at least one pair of primes* (GC is specifically reformulated by the author of this article in order to emphasize the importance of studying the Primes Distribution (**PD**) **[12,13,14,15]** defined by a global and local density and uniformity with multiple interesting *fractal patterns* **[16]**: *GC is in fact an auto-recursive fractal property of PD in* $\mathbb{N}$ alias the *Goldbach Distribution of Primes (**GDP**)* (as the author will try to prove later on in this article), but also a property of $\wp$, a property which is indirectly expressed as GC, using the subset of even naturals).

Primes are the subject of many other conjectures [URL1, URL2] and other mathematical theorems and formulas [URL1, URL2, URL3].

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Part A.
The array S_p of the simple Matrix of Goldbach Index-Partitions $(M_{p,n})$

Definition of $\wp^*$ and $\wp$. We may define the prime subset of $\mathbb{N}$ as $\wp^* = \{P_1(=2), P_2(=3), P_3(=5), ..., P_x, ..., P_y, ...P_\infty\}$, with $x, y \in \mathbb{N}^*$ *and* $0 < x < y$, with $P_x(P_y)$ being the x-th (y-th) primes of $\wp^*$ and P_∞ marking the already proved fact that $\wp^*$ has an infinite number of (natural) elements (Euclid's 2nd theorem **[17]**). The numbering function of primes (P_n) is a bijection that interconnects $\wp^*$ with $\mathbb{N}^*$ so that each element of $\wp^*$ corresponds to only (just) one element of $\mathbb{N}^*$ and vice versa: $1 \leftrightarrow P_1(=2)$, $2 \leftrightarrow P_2(=3)$, ..., $x \leftrightarrow P_x$ (the x-th prime), $y \leftrightarrow P_y$ (the y-th prime), …, $\infty \leftrightarrow P_\infty$. Originally, Goldbach considered that number 1 was the first prime: although still debated until present, today the mainstream considers that number 1 is neither prime or composite, but the unity of all the other integers.[9,10]**.** However, in respect to the first "ternary" formulation of GC (**TGC**) (which was re-formulated by Euler as the BGC and also demonstrated by the same Euler to be stronger than TGC, as TGC is a consequence of BGC), the author of this article also defines $P_0 = 1$ (the unity of all integers, implicitly the unity of all primes) and $\wp = \{P_0(=1), P_1(=2), P_2(=3), P_3(=5), ..., P_x, ..., P_y, ...P_\infty\}$, with $x, y \in \mathbb{N}$ *and* $0 \le x < y$, although only $\wp^*$ shall be used in this work (as it is used in the mainstream of modern mathematics).

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The 1st formulation of BGC. For any even integer $\boxed{n > 2}$, it will always exist at least one pair of (other two) integers $x, y \in \mathbb{N}^*$ *with* $x \le y$ so that $P_x + P_y = n$, with $P_x(P_y)$ being the x-th (y-th) primes of $\wp^*$. **Important observation**: The author considers that analyzing those “homogeneous” triplets of three naturals (n, x, y) (no matter if primes or composites) is more convenient and has more “analytical” potential than analyzing (relatively) “inhomogeneous” triplets of type (n, P_x, P_y): that’s why the author proposes ***Goldbach index partitions (GIPs)*** as an alternative to the standard ***Goldbach partitions (GPs)*** proposed by Oliveira e Silva[11]. The existence of (at least) a triplet (n, x, y) for each even integer $\boxed{n > 2}$ (as BGC claims) may suggest that BGC is profoundly connected to the generic primality (of any P_x and P_y) and, more specifically, argues that *GC is in fact a property of PD in $\mathbb{N}$ (and a property of $\wp^*$ as composed of indexed/numbered elements).* The most important property of Primes and PD and is that $\boxed{P_x \to x \cdot \ln(x),\ for\ x \to \infty}$ **or** $\boxed{P_x \cong x \cdot \ln(x),\ for\ any\ progressively\ large\ x}$ (which is the alternative [linearithmic] expression of the Prime Number Theorem **[18]**, as if $\wp^*$ is a result of an apparently random quantized linearithmization of $\mathbb{N}^* - \{1\}$ so that $\boxed{P_n \to n \cdot \ln(n)}$. **In conclusion**: *For any even integer* $\boxed{n > 2}$, *at least one GIP exists* ***(BGC – 1st condensed formulation)***

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The 2nd formulation of BGC using *the Matrix of Goldbach index-partitions* (*M-GIP* or *M*).

[**1**] Let us consider an infinite string of matrices $S = \{M_1, M_2, M_3, ..., M_n, ...M_\infty\}$, with each generic M_n being composed of lines made by GIPs (x, y), such as:

$$\boxed{M_n = \begin{pmatrix} x_{n,1} & y_{n,1} \\ \vdots & \vdots \\ x_{n,j} & y_{n,j} \\ \vdots & \vdots \\ x_{n,m_n} & y_{n,m_n} \end{pmatrix}, \text{ with } P_{x_{n,j}} + P_{y_{n,j}} = n,\ \forall\ \mathrm{j} \in [1, m_n]}$$

(j is the index of any chosen line of M_n, $\mathrm{j} \geq 1$ and $\mathrm{j} \leq m_n$)

(m_n is the total maximum number of j-indexed lines of M_n)

($x_{n,i}, y_{n,i} \in \mathbb{N}^*$, $x_{n,i} < x_{n,i+1}$ for $m_n \geq 2$, $\forall\ i \in [1, m_n]$)

[**2**] Let us also consider the function that counts the lines of any M_n, such as: $\boxed{l(n) = m_n}$. This function (that numbers the lines of a GM) is classically named as $\boxed{r(n) = l(n) = m_n}$ **("r" stands for the number of "rows").[11]**

[**3**] An **empty matrix** $(M_\varnothing)$ is defined as a matrix with a 0 number of rows and/or columns.

Using S, M, $M_\varnothing$ and $r(n)$ as previously defined, BGC has 2 formulations sub-variants:

1. *$M_n \neq M_\varnothing$ (OR S doesn't contain any $M_\varnothing$) for any even integer $\boxed{n > 2}$ or shortly: $\boxed{\forall\, even\, integer\ n > 2 \Leftrightarrow M_n \neq M_\varnothing}$ **(the 2nd formulation of BGC – 1st sub-variant)**.*
2. *For any even integer $\boxed{n > 2}$, $r(n) > 0$ or shortly: $\boxed{\forall\, even\, integer\ n > 2 \Leftrightarrow r(n) > 0}$ **(the 2nd formulation of BGC – 2nd sub-variant)**.*

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The 3rd formulation of BGC using *the generalization of S (S_p) and the generalization of M $(M_{p,n})$ for GIPs matrix containing more than 2 columns (as based on GIPs with more than 2 elements).*

[1]Let us consider an infinite set OF infinite strings OF matrix:

a) $S_2 = \{M_{2,1}, M_{2,2}, M_{2,3}, ..., M_{2,n}, ...M_{2,\infty}\}$ (the generic $M_{2,n}$ of S_2 has 2 columns based on [binary] GIPs with 2 elements);

b) $S_3 = \{M_{3,1}, M_{3,2}, M_{3,3}, ..., M_{3,n}, ...M_{3,\infty}\}$ (the generic $M_{3,n}$ of S_3 has 3 columns based on [ternary] GIPs with 3 elements);

c) …;

d) $S_p = \{M_{p,1}, M_{p,2}, M_{p,3}, ..., M_{p,n}, ...M_{p,\infty}\}$ (the generic $M_{p,n}$ of S_p has p columns based on [p-nary] GIPs with p elements and natural p>3);

e) …,

f) $S_\infty = \{M_{\infty,1}, M_{\infty,2}, M_{\infty,3}, ..., M_{\infty,n}, ...M_{\infty,\infty}\}$ (the generic $M_{\infty,n}$ of S_∞ has potentially infinite (∞) number of columns based on $\infty - nary$ GIPs with a potentially infinite (∞) number of elements)

g) With **e**ach generic $M_{p,n}$ being composed of $m_{p,n}$ lines and p columns made by p-nary GIPs with p elements, such as:

$$\boxed{\begin{array}{l} M_{p,n} = \begin{pmatrix} x_{n,1} & \cdots & x_{n,k} & \cdots & x_{n,p} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{n,j} & \cdots & x_{n,k+j} & \cdots & x_{n,p+j} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{n,m_{p,n}} & \cdots & x_{n,k+m_{p,n}} & \cdots & x_{n,p+m_{p,n}} \end{pmatrix}, \text{ with } P_{x_{n,j}} + \ldots + P_{x_{n,j+k}} + \ldots + P_{x_{n,p+j}} = n, \\ \forall\ \mathrm{j} \in \left[1, m_{p,n}\right] and\ \forall\ k \in [1, p], \end{array}}$$

(j is the index of any chosen line of $M_{p,n}$, $\mathrm{j} \geq 1$ and $j \leq m_{p,n}$
and $m_{p,n}$ is the total maximum number of j-indexed lines of $M_{p,n}$)
(k is the index of any chosen column of $M_{p,n}$, $\mathrm{k} \geq 1$ and $k \leq p$
and p is the total number of k -indexed columns of $M_{p,n}$)
($x_{n,k+j} \in \mathbb{N}^*, x_{n,j} \leq x_{n,j+1}$ for $m_{p,n} \geq 2$, $\forall\ \mathrm{j} \in \left[1, m_{p,n}\right] and\ \forall\ \mathrm{k} \in [1, p]$)

[**2**] Let us also consider the function that counts the lines of any $M_{p,n}$, such as: $\boxed{r(p,n) = l(p,n) = m_{p,n}}$.

[**3**] An **empty matrix** $\left(M_{\varnothing}\right)$ is defined as a matrix with a 0 number of rows and/or columns.

Using S_p, $M_{p,n}$, $M_{\varnothing}$ and $r(p,n)$ as previously defined, BGC has 2 formulations sub-variants:

1. *$M_{2,n} \neq M_{\varnothing}$ (OR S_2 doesn't contain any $M_{\varnothing}$) for any even integer $n > 2$ or shortly:* $\boxed{\forall\, even\, integer\ n > 2 \Leftrightarrow M_{2,n} \neq M_{\varnothing}}$ ***(the 3rd formulation of BGC – 1st sub-variant).***
2. *For any even integer $n > 2$, $r(2,n) > 0$ or shortly:* $\boxed{\forall\, even\, integer\ n > 2 \Leftrightarrow r(2,n) > 0}$ ***(the 3rd formulation of BGC –2nd sub-variant).***

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Part B.
A synthesis and A/B classification of the main GLCs using the $M_{p,n}$ concept

The Goldbach-like conjectures (GLCs) category/class.

GLCs definition. A GLC may be defined as any additional special (observed/conjectured) property of S_p and its elements $M_{p,n}$ other that GC (with $n > 2$), with possibly other inferior limits $a \geq 2$, with $n > a \geq 2$).

GLCs classification. GLCs may be classified in two major classes using a double criterion such as:

1. **Type A GLCs (A-GLCs)** are *those GLCs that claim: [**1**] Not only that all $M_{p,n} \neq M_{\varnothing}$ for a chosen p>1 and for any / any odd / any even integer $n > a \geq 2$ (with a being any finite natural established by that A-GLC and $n > a$) BUT ALSO [**2**] any other non-trivial(**nt**) accessory property/properties of all $M_{p,n}(\neq M_{\varnothing})$ of S_p. A specific A-GLC is considered authentic if the other non-trivial accessory property/properties of all $M_{p,n}(\neq M_{\varnothing})$ (claimed by that A-GLC) isn't/aren't a consequence of the 1st claim (of the same A-GLC). Authentic (at least conjectured as such) A-GLCs are (have the potential to be) "stronger" than GC as they claim "more" than GC does.*

2. **Type B GLCs (B-GLCs)** *are those GLCs that claim: no matter if all $M_{p,n} \neq M_{\varnothing}$ or just some $M_{p,n} \neq M_{\varnothing}$ for a chosen p>1 and for some / some odd / some even integer $n > a \geq 2$ (with a being any finite natural established by that B-GLC and $n > a$), all those $M_{p,n}$ that are yet non-$M_{\varnothing}$ (for $n > a$) have (an)other non-trivial accessory property/properties. A specific B-GLC is considered authentic if the other non-trivial accessory property/properties of all $M_{p,n}(\neq M_{\varnothing})$ (claimed by that B-GLC for $n > a$) isn't/aren't a consequence of the fact that some $M_{p,n} \neq M_{\varnothing}$ for $n > a$. Authentic (at least conjectured as such) B-GLCs are "neutral" to GC (uncertainly "stronger" or "weaker" conjectures) as they claim "more" but also "less" than GC does (although they may be globally weaker and easier to formally prove than GC).*

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Other variants [1] of GC and GLCs include the statements that:

1. "*[...] Every [integer] number that is greater than 2 is the sum of three primes*" (Goldbach's original conjecture formulated in 1742, sometimes called the "ternary" Goldbach conjecture, written in a June 7, 1742 letter to Euler)[1] (which is equivalent to: *"every integer >2 is the sum of at least one triad of primes*", *with the specification that number 1 was also considered a prime by the majority of mathematicians contemporary to Goldbach, which is no longer the case now]"*). This (first) variant of GC can be formulated using (ternary) $M_{3,n}$ (based on GIPs with 3 elements) such as:
   a. **Type A formulation variant as applied to** $\wp$(*not just to* $\wp^*$) :
   "$\boxed{\forall\, integer\ n > 2 \Leftrightarrow M_{3,n} \neq M_{\varnothing}}$ *(with* $x_{n,j,k} \geq 0$ *and* $P_{x_{n,j,k}} \in \wp$ *)"*
   b. **Type B (neutral) formulation variant**: not supported.

2. "*Every even integer $n > 4$ is the sum of 2 odd primes.*" (Euler's binary reformulation of the original GC, which was initially expressed by Goldbach in a ternary form as previously explained) [1]. Since BGC (as originally reformulated by Euler) contains the obvious triviality that there are infinite many even positive integers of form $2p = p + p$ (with p being any prime), the

non-trivial BGC (**ntBGC**) sub-variant that shall be used in this article (alias "BGC" or "ntBGC") is that**: "*every even integer* $n > 6$ *is the sum of at least one pair of distinct odd primes*" [19,20]** (which is equivalent to: ***"every even integer*** $m > 3$ ***is the arithmetic average of at least one pair of distinct odd primes"***). Please note that ntBGC doesn't support the definition of a GLC, as $2p = p + p$ is a trivial property of some even integers implying the complementary relative triviality that: $2c \neq 2p \neq p + p$ (with c being any composite natural number and p being any prime). ntBGC can be formulated using (binary) $M_{2,n}$ (based on GIPs with 2 elements) such as:

a. **Type A formulation variant**: *"* $\boxed{\forall \text{even integer } n > 6}$, $M_{2,n}(M_n) \neq M_\varnothing$ *AND* $M_{2,n}(M_n)$ *contains at least one line with both elements (GIPs)≠1 (as* $P_1 = 2$ *is the only even prime) AND distinct to each other (as distinct GIPs means distinct primes as based on the bijection of the prime numbering function)"*

b. **Type B (neutral) formulation variant**: "$\boxed{\forall \text{even integer } n > 6}$, *all* $M_{2,n}(M_n)$ *that are non-empty (as* S_p *may also contain empty* $M_{2,n}(M_n) = M_\varnothing$ *for some specific [but still unfound]* n *values) will contain at least one line with both elements (GIPs)≠1 (as* $P_1 = 2$ *is the only even prime) AND distinct to each other (as distinct GIPs means distinct primes as based on the bijection of the prime numbering function)"*

3. "$\boxed{\forall \text{odd integer } n > 5}$, n *is the sum of 3 (possibly identical) primes.*" **[1,21]** (the [weak] Ternary Goldbach's conjecture [**TGC/TGT – Ternary Goldbach's conjecture/theorem**]; formally proved by Harald Helfgott in 2013 **[22,23,24]**, so that TGC is very probably [*but not surely however*] a proved theorem, and no longer a "conjecture") (which is equivalent to: *"*$\boxed{\forall \text{odd integer } n > 5}$, n *is the sum of at least one triad of [possibly identical] primes"*). TGC can be formulated using (ternary) $M_{3,n}$ (based on GIPs with 3 elements) such as:

a. **Type A formulation variant**: **"** $\boxed{\forall \text{odd integer } n > 5 \Leftrightarrow M_{3,n} \neq M_\varnothing}$ *"*

b. **Type B (neutral) formulation variant**: not supported.

4. "$\boxed{\forall \text{integer } n > 17}$, n *is the sum of exactly 3 distinct primes.*" **[1,19]** (cited as "Conjecture 3.2" by Pakianathan and Winfree in their article, which is equivalent to: *"*$\boxed{\forall \text{integer } n > 17}$, n *is the sum of at least one triad of distinct primes"*) (this is a conjecture stronger than TGC, but weaker than BGC as it is implied by BGC). This stronger version of TGC(**sTGC**) can also be formulated using (ternary) $M_{3,n}$ (based on GIPs with 3 elements) such as:

a. **Type A formulation variant**: **"** $\boxed{\forall \text{integer } n > 17} \Rightarrow M_{3,n} \neq M_\varnothing$ *AND* $M_{3,n}$ *contains at least one line with all 3 elements (GIPs) distinct from each other"*

b. **Type B (neutral) formulation variant**: "$\boxed{\forall \text{integer } n > 17} \Rightarrow$ *those* $M_{3,n}$ which are $\neq M_\varnothing$ *will contain at least one line with all 3 elements (GIPs) distinct from each other"*

5. "$\boxed{\forall \text{odd integer } n > 5}$, n *is the sum of a prime and a doubled prime [which is twice of any prime].*" (Lemoine's conjecture [**LC**] **[25,26]** which was erroneously attributed by MathWorld to Levy H. who pondered it in 1963 **[26,27**,**28]**). LC is stronger than TGC, but weaker than BGC. LC also has an extension formulated by Kiltinen J. and Young P. (alias the "refined Lemoine conjecture" **[29]**), which is stronger than LC, but weaker than BGC and won't be discussed in this article (as I shall mainly focus on those GLCs stronger than BGC). LC can be formulated using (ternary, not binary) $M_{3,n}$ (based on GIPs with 3 elements) such as:

a. **Type A formulation variant**: **"** $\boxed{\forall \text{odd integer } n > 5} \Rightarrow M_{3,n} \neq M_\varnothing$ *AND* $M_{3,n}$ *contains at least one line with at least 2 identical elements (GIPs)"*

b. **Type B (neutral) formulation variant**: "$\boxed{\forall \, odd\ integer\ n > 5} \Rightarrow$ *those* $M_{3,n}$ which are $\neq M_{\varnothing}$ *will contain at least one line with at least 2 identical elements (GIPs)"*

6. There are also a few original conjectures**[30]** on partitions of integers as summations of primes published by Smarandache F. that won't be discussed in this article, as these conjectures depart from VBGC (as VBGC presentation is the main purpose of this article).

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There are also a number of (relative recently discovered) GLCs stronger than BGC (and implicitly stronger than TGC), that can also be synthesized using $M_{p,n}$ concept: **these stronger GLCs (as VBGC also is) are tools that can inspire new strategies in finding a formal proof for BGC, as I shall try to demonstrate next.** Additionally, there are some arguments that Twin Prime Conjecture (**TPC**) **[31]** may be also (indirectly) related to BGC as part of a more extended and profound conjecture [6] **[16,32,33, 34]**, so that any new clue for BGC formal proof may also help in TPC (formal) demonstration. Moreover, TPC may be weaker (and possibly easier to proof) than BGC (at least regarding the efforts toward the final formal proof) as the superior limit of the primes gap was recently "pushed" to be ≤246 **[35**], but the Chen's Theorem I (that *"every sufficiently large even number can be written as the sum of either 2 primes, OR a prime and a semiprime [the product of just 2 primes]*" [**36**,**37**,**38]**) has not been improved since a long time (at least by the set of proofs that are accepted in the present by the mainstream) except Cai's new proved theorem published in 2002 ("*There exists a natural number N such that every even integer n larger than N is a sum of a prime* $\leq n^{0.95}$ *and a semi-prime*" **[39,40]**, a theorem which is a similar but a weaker statement than LC that hasn't a formal proof yet).

1. "$\boxed{\forall \, even\ integer\ n > 4}$, *there is at least one prime number* p *[so that]* $\sqrt{n} < p \leq n/2$ *and* $q = n - p$ *is also prime [with* $n = p + q$ *implicitly]*" (the Goldbach-Knjzek conjecture [**GKC**] **[41]** which is stronger than BGC) **(**GKC can also be reformulated as: "*every even integer* $n > 4$ *is the sum of at least one pair of primes with at least one element in the semi-open interval* $\left(\sqrt{n}, n/2\right]$". GKC can be formulated using (binary) $M_{2,n}$ (based on GIPs with 2 elements) such as:
   a. **Type A formulation variant**: **"**$\boxed{\forall \, even\ integer\ n > 4} \Rightarrow M_{2,n}(M_n) \neq M_{\varnothing}$ *AND* $M_{2,n}(M_n)$ *contains at least one line with at least one element in the semi-opened interval* $\left(\sqrt{n}, n/2\right]$".
   b. **Type B (neutral) formulation variant**: "$\boxed{\forall \, even\ integer\ n > 4} \Rightarrow$ those $M_{2,n}(M_n)$ which are $\neq M_{\varnothing}$ *will contain at least one line with at least one element in the semi-opened interval* $\left(\sqrt{n}, n/2\right]$"

2. "$\boxed{\forall \, even\ integer\ n > 4}$, *there is at least one prime number* p *[so that]* $\sqrt{n} < p < 4\sqrt{n}$ *and* $q = n - p$ *is also prime [with* $n = p + q$ *implicitly]*" (the Goldbach-Knjzek-Rivera conjecture [**GKRC**] **[42]** which is obviously stronger than BGC, but also stronger than GKC for $n \geq 64$) **(**GKRC can also be reformulated as: "$\boxed{\forall \, even\ integer\ n > 4}$, n *is the sum of at least one pair of primes with one element in the double-open interval* $\left(\sqrt{n}, 4\sqrt{n}\right)$". GKRC can be formulated using (binary) $M_{2,n}$ (based on GIPs with 2 elements) such as:
   a. **Type A formulation variant**: **"**$\boxed{\forall \, even\ integer\ n > 4} \Rightarrow M_{2,n}(M_n) \neq M_{\varnothing}$ *AND* $M_{2,n}(M_n)$ *contains at least one line with one element in the double-open interval* $\left(\sqrt{n}, 4\sqrt{n}\right)$"

b. **Type B (neutral) formulation variant**: "$\boxed{\forall\, even\, integer\ n>4} \Rightarrow$ those $M_{2,n}(M_n)$ which are $\neq M_{\varnothing}$ *will contain at least one line with one element in the double-open interval* $\left(\sqrt{n}, 4\sqrt{n}\right)$"

3. Any other GLC that establishes an additional inferior limit $a>0$ for $r(2,n)$ so that $r(2,n)\geq a>0$ (like Woon's GLC **[43]**) can also be considered stronger that BGC, as BGC only suggests $r(2,n)>0$ for any even integer $n>6$ (which implies a greater average number of GIPs per each n than the more selective Woon's GLC does).

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There is also a remarkable set of original conjectures (many of them stronger than BGC and/or TPC) originally proposed by Sun Zhi-Wei[URL2] **[44,45]**, a set from which I shall cite **[46]** (by rephrasing) some of those conjectures that have an important element in common with the first special case of VBGC: the recursive P_{P_x} function in which P_x is the x-th prime and P_{P_x} is the P_x-th prime (which is denoted in the next cited conjectures as P_q which is the q-th prime, with q being also a prime number)

1. **Conjecture 3.1 (Unification of GC and TPC, 29 Jan. 2014)**. For any integer $n>2$ there is at least one triad of primes $\left[(1<q<2n-1),(2n-q),\left(P_{q+2}+2\right)\right]$ (Sun's Conjecture 3.1 [**SC3.1 or U-GC-TPC**], which is obviously stronger than BGC and was tested up to $n=2\times10^8$)

2. **Conjecture 3.2 (Super TPC [SPTC], 5 Feb. 2014)**. For any integer $n>2$ there is at least one triad $\left[(0<k<n),\left(P_k+2=prime\right),\left(P_{P_{n-k}}+2=prime\right)\right]$ (Sun's Conjecture 3.2 [**SC3.2 or SPTC**], which is obviously stronger than TPC and was tested up to $n=10^9$) **[47,48]**

3. **Conjecture 3.3 (28 Jan. 2014)**. For any integer $n>2$ there is at least one pentad $\left[(0<k<n-1),(6k-1=prime),(6k+1=prime),\left(P_{n-k}=prime\right),\left(P_{n-k}+2=prime\right)\right]$ (Sun's Conjecture 3.3 [**SC3.3**], which is obviously stronger than TPC as it implies TPC; SC3.3 was tested up to $n=2\times10^7$)

4. **Conjecture 3.7-i (1 Dec. 2013)**. There are infinite many positive even integers $n>3$ which are associated with a hexad of primes $\left[(n+1),(n-1),(P_n+n),(P_n-n),(nP_n+1),(nP_n-1)\right]$ (Sun's Conjecture 3.7-1 [**SC3.7-i**], which is obviously stronger than TPC as it implies TPC; $n=22\,110$ is the first/smallest value of n predicted by SC3.7-I)

5. **Conjecture 3.12-i (5 Dec. 2013)**. All positive integers $n>7$ have at least one associated pair $\left[(k<n-1),\left(2^k+P_{n-k}=prime\right)\right]$ (Sun's Conjecture 3.12-i [**SC3.12-i**])

6. **Conjecture 3.12-ii (6 Dec. 2013)**. All positive integers $n>3$ have at least one associated pair $\left[(k<n-1),\left(k!+P_{n-k}=prime\right)\right]$ (Sun's Conjecture 3.12-ii [**SC3.12-ii**])

7. **Remark 3.19 (which is an implication of the Conjecture 3.19 not cited in this article)**. There is an infinite number of triads of primes $\left[(q>1),\left(r=P_q-q+1\right),\left(P_r-r+1\right)\right]$ (Sun's Remark on Sun's Conjecture 3.19 [**SRC3.19**])

8. **Conjecture 3.21-i (6 Mar. 2014)**. For any integer $n > 5$ there will always exist at least one triad $\left[(0 < k < n),(2k+1 = prime),\left(P_{k\cdot n} + k \cdot n = prime\right)\right]$ (Sun's Conjecture 3.21-i [**SC3.21-i**])

9. **Conjecture 3.23-i (1 Feb. 2014)**. For any integer $n > 13$ there is at least one triad of primes $\left[(1 < q < n),(q+2),\left(P_{n-q} + q + 1\right)\right]$ (the Sun's Conjecture 3.23-i [**SC3.23-i**])

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Part C.
The 'i-primeths' (${}^{i}\wp^{*}$) definition

The definition of "i-primeths", which is slightly different from Fernandez's definition[8]

I have chosen to use the term "primeth(s)" (Fernandez N. introduced it for the first time in 1999, in his "The Exploring Primeness Project" [8]) because this is the shortest and also the most suggestive of all the alternatives **[49]** used until now (as the "th" suffix includes by abbreviation the idea of "index of primes").

Primeths were originally defined by Fernandez N. as a subset of primes with (also) prime indexes [8] (the numbering of the elements of $\wp^{*}$ starts with $P_1 = 2$). As primes are in fact those positive integers with a prime index [8] (the "prime index" being non-tautological defined as a positive integer >1 that has only 2 distinct divisors: 1 and itself), all the standard primes may be considered primeths with iteration order i=0 (or shortly: 0-primeths) NOT with i=1 (as Fernandez first considered[8]) (as the i=0 marks the genesis of $\wp^{*}$ from the ordinary $\mathbb{N} \supset \wp^{*}$ and cannot be considered an iteration on $\wp^{*}$). This new definition of i-primeths (${}^{i}P$ containing ${}^{i}P_x$ elements with $i \geq 0$ and $x \in \mathbb{N}^{*}$) has three advantages:

1. the iteration order i is also the number of ("vertical") iterations for producing the i-primeths from the 0-primeths $\left({}^{0}P = \wp^{*}\right)$ (as in the Fernandez's original primeths definition, the standard primes were considered 1-primeths not 0-primeths, as if they were produced from $\mathbb{N}$ using 1 vertical iteration, but $\mathbb{N}$ doesn't contain just primes, as $\wp^{*} \neq \mathbb{N}$);
    a. these iterations numbered by order i are easy to follow when implemented in different algorithms using a programming language on a computer
2. the concept of primes can be generalized as i-primeths that also includes $\wp^{*}$ as the special case of 0-primeths $\left({}^{0}P = \wp^{*}\right)$;
3. this definition clearly separates $\wp^{*}$ from the ordinary $\mathbb{N}$ using 0 (not 1) as a starting order (i) for $\wp^{*}$ $\left({}^{0}P\right)$ and considering $\mathbb{N}$ as a ${}^{(-1)}P$ (a "bulky" ${}^{(-1)}P$ "contaminated" with composite positive integers that can be considered "(-1)-primeths" convertible to 0-primeths by different sieves of primes, which are another kind of iterations than those producing i-primeths from 0-primeths)
    a. ${}^{0}P$ inevitably "contains" $\mathbb{N}^{*}$ by its indexes , in the sense that ${}^{0}P$ contains all the generic ${}^{0}P_x$ elements with indexes $x \in \mathbb{N}^{*}$ (an index x that scrolls all $\mathbb{N}^{*}$). The same prime may be part of more than one i-primeths subset ${}^{i}P$, as x is not necessarily a prime.
    b. This slightly different definition of the i-primeths (${}^{i}P$ containing generic ${}^{i}P_x$ elements with $i \geq 0$ and $x \in \mathbb{N}^{*}$, as explained previously) is NOT a new "anomaly" and it was also practiced by Smarandache F. as cited by Murthy A.[**50]** and also by Seleacu V. and Bălăcenoiu I. [**51]**

The elements of the group ${}^{i}P$

$${}^{0}P = \left\{ {}^{0}P_1\left(=P_1=2\right), {}^{0}P_2\left(=P_2=3\right), {}^{0}P_3\left(=P_3=5\right), \ldots {}^{0}P_x\left(=P_x\right), \ldots \right\} \text{ (alias 0-primeths)}$$

$${}^{1}P = \left\{ {}^{1}P_1\left(=P_{p_1}=P_2=3\right), {}^{1}P_2\left(=P_{p_2}=P_3=5\right), \ldots {}^{1}P_x\left(=P_{p_x}\right), \ldots \right\} \text{ (alias 1-primeths } \textbf{[52]}\text{)}$$

$${}^{2}P = \left\{ {}^{2}P_1\left(P_{p_{P_1}}=P_3=5\right), {}^{2}P_2\left(P_{p_{P2}}=P_5=11\right), \ldots {}^{2}P_x\left(=P_{p_{Px}}\right), \ldots \right\}, \ \ldots$$

$${}^{i}P = \left\{ {}^{i}P_1 = P_{\underbrace{P_{\ldots P_1}}_{i \text{ iterations of } P}}, {}^{i}P_2 = P_{\underbrace{P_{\ldots P2}}_{i \text{ iterations of } P}}, \ldots, {}^{i}P_x = P_{\underbrace{P_{\ldots P_x}}_{i \text{ iterations of } P}}, \ldots \right\}, \text{ with } x \in \mathbb{N}^{*} - \{1,2\}$$

Part D.

Meta-conjecture VBGC - The extension and generalization of BGC as applied on i-primeths $\left({}^{i}P\right)$

Meta-conjecture VBGC – main statement:

1. **Defining i-primeths (i-prime-indexed-primes) as:** ${}^{0}P_{x} = P\left(\underset{\substack{0\,iterations\\ of\,P\,on\,P}}{\underline{x}}\right)$, ${}^{1}P_{x} = P\left(\underset{\substack{1\,iteration\\ of\,P\,on\,P}}{\underline{P(x)}}\right)$,

   ${}^{2}P_{x} = P\left(\underset{\substack{2\,iterations\\ of\,P\,on\,P}}{\underline{P(P(x))}}\right) \ldots\ {}^{i}P_{x} = P\left(\underset{(i\geq 0)\,iterations}{\underline{P(P(\ldots P(x)))}}\right)$, with $P(x)$ being the x-th prime in the set of standard primes (usually denoted as $P(x)$ or P_x and equivalent to ${}^{0}P_{x}$ alias "0-primeths") and the generic ${}^{i}P_{x}$ being named the generic set of i-primeths (with" i" being the "iterative"/recursive order of that i-primeth which measures the number of P-on-P iterations associated with that specific i-primeth subset).

   a. I have used the notation ${}^{0}P_{x}$ and ${}^{i}P_{x}$ instead of the standard notation $P^{1}(x) = P(x)\left[= {}^{0}P_{x}\right]$ AND $P^{i}(x) = \underset{i\,nested\,\,functions\,P}{\underline{P(P..P(x))}}\left[= {}^{(i-1)}P_{x}\right]$, so that to strictly measure the number of P-on-P recursive steps (iterations) to produce a generic set ${}^{i}P$ from ${}^{0}P$ AND ALSO to not generate the confusion between $P^{i}(x) = \underset{i\,nested\,\,functions\,P}{\underline{P(P..P(x))}}$ and the exponential product $\left[P(x)\right]^{i} = \underset{i\,times}{\underline{P(x)\cdot P(x)\cdot \ldots P(x)}}$.

   b. It is also true that producing the elements of the (prime) function $P(x)$ from the natural set $\mathbb{N}*$ is also like selecting just the naturals with prime indexes from $\mathbb{N}*$, so that ${}^{0}P$ can be theoretically identified with $\mathbb{N}*$ and the set of primes $\wp*$ can be identified with ${}^{1}P$: however, $\mathbb{N}*$ is not a set of primes and that is why I have avoided to note $\mathbb{N}*$ with ${}^{0}P$ but to ${}^{(-1)}P$ (like the result of an inverse iteration) AND ALSO decided to count the sets of i-primeths starting from 0 (so that ${}^{0}P_{x} = P(x)$) in the purpose to strictly measure the number of P-on-P iterations starting from 1, so that ${}^{1}P_{x} = P\left(\underset{1iteration}{\underline{P(x)}}\right)$.

2. **The inductive variant of (the meta-conjecture) VBGC (iVBGC) states that:**
   "Any/every even positive integer $\boxed{2m \geq 2\cdot fx(a,b)}$ **AND (also)** $\boxed{2m \geq 2\cdot fx_{2}(a,b)}$**, with**

$$\boxed{fx(a,b) = \begin{cases} 2^{(a+1)(b+1)(a+b+2)} & for\,(a=b=0) \\ 2^{[(a+1)(b+1)(a+b+3)/a]-a} & for\,(a=b)\,AND\,(a>0) \\ 2^{(a+1)(b+1)(a+b+2)-(a+b-2)} & for\,(a\neq b)\,AND\left[(a>0)\,OR\,(b>0)\right] \end{cases}}$$ **AND**

$$fx_2(a,b)=\begin{cases}2^{(a+1)(b+1)(a+b+2)} & for\,(a=b=0)\\ 2^{[(a+1)(b+1)(a+b+3)/a]-2a} & for\,(a=b)\,AND\,(a>0)\\ 2^{(a+1)(b+1)(a+b+2)-(a+b-2)} & for\,(a\neq b)\,AND\left[(a>0)OR(b>0)\right]\end{cases}$$ **, can be written as the sum of at least one pair of DISTINCT odd i-primeths** $\boxed{{}^{a}P_x > {}^{b}P_y}$ **, with the positive integers pair** $\boxed{(a,b), with\, a\geq b\geq 0}$ **defining the (recursive) orders of each of those i-primeths pair AND the pair of distinct positive integers** $\boxed{(x,y), with\, x>y>1}$ **defining the indexes of each of those i-primeths pair.".**

a. A 2nd inverse formulation of iVBGC based on the previously defined $fx(a,b)$ **and** $fx_2(a,b)$ **: "Any/every positive integer** $\boxed{m\geq fx(a,b)}$ **AND (also)** $\boxed{m\geq fx_2(a,b)}$**, can be written as the arithmetic average/mean of at least one pair of DISTINCT odd i-primeths** $\boxed{{}^{a}P_x > {}^{b}P_y}$ **"**

3. Alternative formulation for the inductive variant of (the meta-conjecture) VBGC (iVBGC), using the standard notation (1-prime) $P^1(x)=P(x)\left[={}^{0}P_x\right]$, (2-prime) $P^2(x)=P(P(x))\left[={}^{1}P_x\right]$ and (a-prime and any analogous b-prime and generic i-prime [**equivalent to a (i-1)-primeth**]) $P^a(x)=P(P(...P(x)))\left[={}^{(a-1)}P_x\right]$:

"Any even positive integer $\boxed{2m\geq 2\cdot fx_{\mathrm{var}}(a,b)}$ **AND even** $\boxed{2m\geq 2\cdot fx_{\mathrm{var}2}(a,b)}$**, with**

$$fx_{\mathrm{var}}(a,b)=\begin{cases}2^{a\cdot b\cdot(a+b)} & for\,(a=b=1)\\ 2^{[a\cdot b\cdot(a+b+1)/(a-1)]-(a-1)} & for\,(a=b)\,and\,(a>1)\\ 2^{a\cdot b\cdot(a+b)-(a+b-4)} & for\,(a\neq b)\,AND\left[(a>0)OR(b>0)\right]\end{cases}$$ **AND**

$$fx_{\mathrm{var}2}(a,b)=\begin{cases}2^{a\cdot b\cdot(a+b)} & for\,(a=b=1)\\ 2^{[a\cdot b\cdot(a+b+1)/(a-1)]-2(a-1)} & for\,(a=b)\,and\,(a>1)\\ 2^{a\cdot b\cdot(a+b)-(a+b-4)} & for\,(a\neq b)\,AND\left[(a>0)OR(b>0)\right]\end{cases}$$ **, can be written as the sum of at least one pair of DISTINCT odd i-primes** $\boxed{P^a(x)>P^b(y)}$**, with the positive integers pair** $\boxed{(a,b), with\, a\geq b\geq 1}$ **defining the (recursive) orders of each of those i-primes pair** $\boxed{\left[P^a(x),P^b(y)\right]}$ **AND the distinct positive integers pair** $\boxed{(x,y), with\, x>y>1}$ **defining the indexes of each of those i-primes pair".**

4. A secondary inductive (form of) (the meta-conjecture) VBGC (siVBGC[a,0]) states that:

"Any/every even positive integer $\boxed{2m\geq 2\cdot \mathrm{int}\left[fy(a)\right]}$**, with** $\boxed{fy(a)=e^{4a}}$ **, can be written as**

<u>the sum of at least one pair of DISTINCT odd i-primeths</u> $\boxed{{}^{a}P_x > {}^{0}P_y}$ **<u>, with the positive integers pair</u>** $\boxed{(a,0), with\, a>0}$ **<u>defining the (recursive) orders of the i-primeths pair</u>** $\left({}^{a}P_x, {}^{0}P_y\right)$ **<u>AND the distinct positive integers pair</u>** $\boxed{(x,y), with\, x>y>1}$ **<u>defining the indexes of each of those i-primeths</u>.".**

a. **<u>A 2nd inverse formulation of siVBGC[a,0] based on the previously defined</u>** $fy(a)$ **<u>:</u>** **<u>"Any/every positive integer</u>** $\boxed{m \geq \text{int}\left[fy(a,0)\right]}$ **<u>can be written as the arithmetic average/mean of at least one pair of DISTINCT odd i-primeths</u>** $\boxed{{}^{a}P_x > {}^{0}P_y}$ **"**

5. **<u>The analytical variant of (the meta-conjecture) VBGC (aVBGC) (from which the inductive VBGC can be intuitively inducted) states that</u>: "For any pair of finite positive integers** $\boxed{(a,b), with\, a \geq b \geq 0}$ defining the (recursive) orders of an a-primeth $\left({}^{a}P\right)$ and a b-primeth respectively $\left({}^{b}P\right)$, **there will always exist a single finite positive integer** $\boxed{\left(n_{a,b} = n_{b,a}\right) \geq 3}$ **so that, for any positive integer** $\boxed{m > n_{a,b}}$ **it will always exist at least one pair of finite *distinct* positive integers** $\boxed{(x,y), with\, x>y>1}$ (indexes of distinct odd i-primeths) **so that:** $\boxed{{}^{a}P_x + {}^{b}P_y = 2m}$ **AND** $\boxed{{}^{a}P_x > {}^{b}P_y}$ **AND the function** $\boxed{f(a,b) = f(b,a) = \left(n_{a,b} = n_{b,a}\right) \geq 3}$ **has a finite positive integer value for any combination of finite positive integers** $\boxed{(a,b)}$ **, without any catastrophic-like infinities for any** $\boxed{(a,b)}$ **pair of finites positive integers.**

a. **<u>Important note</u>. I have chosen the additional conditions** $\boxed{(a \geq b \geq 0) \wedge (x>y>1)} \Leftrightarrow \boxed{{}^{a}P_x > {}^{b}P_y}$ **so that to lower the nof. lines per each GM and to simplify the algorithm of searching** $\left({}^{a}P_x, {}^{b}P_y\right)$ **pairs, as the set** ${}^{a}P$ **is much less dense that the set** ${}^{b}P$ for $a>b$ **AND the sieve using** ${}^{a}P$ **(which searches an** ${}^{a}P$ **starting from** $2m\ to\ 3$**) finds a** $\left({}^{a}P_x, {}^{b}P_y\right)$ **pair much more quicker than a sieve using** ${}^{b}P$ **(if** $a>b$**).**

b. $\boxed{\boxed{\mathbf{f(0,0)} = \left(n_{0,0}\right) = \mathbf{3}}}$

c. $\boxed{\boxed{\mathbf{f(1,0)} = f(0,1) = \left(n_{1,0} = n_{0,1}\right) = \mathbf{3}}}$ **(** $f(1,0)$ **is smaller than** $f(2,0) = 2564$ **)**

d. $\boxed{\boxed{\mathbf{f(2,0)} = f(0,2) = \left(n_{2,0} = n_{0,2}\right) = \mathbf{2\ 564}}}$ **(** $f(2,0)$ **is smaller than** $f(1,1)$ **)**

e. $\boxed{\boxed{\mathbf{f(1,1)} = \left(n_{1,1}\right) = \mathbf{40\ 306}}}$ **(** $f(1,1)$ **is larger than** $f(2,0) = 2564$**, as also predicted by** $fx(1,1) > fx(2,0)$ **)**

f. $\boxed{\boxed{\mathbf{f}\left(\mathbf{3,0}\right)= f\left(0,3\right)=\left(n_{3,0}=n_{0,3}\right)=\mathbf{125\ 771}}}$ **(** $f\left(3,0\right)$ **is obviously larger than** $f\left(2,0\right)=2564$**, as also predicted by** $fx\left(3,0\right)> fx\left(2,0\right)$**;** $f\left(3,0\right)$**is smaller than** $f\left(2,1\right)=1\ 765\ 126$, **as also predicted by** $fx\left(3,0\right)< fx\left(2,1\right)$**;** $f\left(3,0\right)$ **is ALSO larger than** $f\left(1,1\right)$=40 306**, as also predicted by** $fx\left(3,0\right)> fx\left(1,1\right)$)

g. $\boxed{\boxed{\mathbf{f}\left(\mathbf{2,1}\right)= f\left(1,2\right)=\left(n_{2,1}=n_{1,2}\right)=\mathbf{1\ 765\ 126}}}$ **(** $f\left(2,1\right)$ **is larger than** $f\left(3,0\right)=125\ 771$, **as also predicted by** $fx\left(2,1\right)> fx\left(3,0\right)$**;** $f\left(2,1\right)$**is obviously smaller than** $f\left(2,2\right)=161\ 352\ 166$, **as also predicted by** $fx\left(2,1\right)< fx\left(2,2\right)$)

h. $\boxed{\boxed{\mathbf{f}\left(\mathbf{4,0}\right)= f\left(0,4\right)=\left(n_{4,0}=n_{0,4}\right)=\mathbf{6\ 204\ 163}}}$ **(** $f\left(4,0\right)$ **is obviously larger than** $f\left(3,0\right)=125\ 771$**, as also predicted by** $fx\left(4,0\right)> fx\left(3,0\right)$**;** $f\left(4,0\right)$ **is smaller than** $f\left(2,2\right)=161\ 352\ 166$**, which is also predicted by** $fx\left(4,0\right)< fx\left(2,2\right)$)

i. $\boxed{\boxed{\mathbf{f}\left(\mathbf{3,1}\right)= f\left(1,3\right)=\left(n_{3,1}=n_{1,3}\right)=\mathbf{32\ 050\ 472(?)}}}$ **(verifying in progress;** $f\left(3,1\right)$ **is obviously larger than** $f\left(2,1\right)=1\ 765\ 126$, **as also predicted by** $fx\left(3,1\right)> fx\left(2,1\right)$; $f\left(3,1\right)$ **is larger than** $f\left(4,0\right)=6\ 204\ 163$ **as also predicted by** $fx\left(3,1\right)> fx\left(4,0\right)$; $fx\left(3,1\right)> fx\left(2,2\right)$ **probably erroneously predicts that** $f\left(3,1\right)$ **is larger than** $f\left(2,2\right)=161\ 352\ 166$**, BUT this prediction is contradicted by computing until present and also by the "step 4 rule" (see next)**; **the function** fx **estimates** $f\left(3,1\right)$ **at** $fx\left(3,1\right)\cong 7.04\times 10^{13}$ **which exceeds the limit of computations of our software** $2m=10^{10}$ **: however,** $fx\left(3,1\right)\cong 7.04\times 10^{13}$ **surely overestimates** $f\left(3,1\right)$ **which has a relatively high probability to be under** $2m=10^{10}$ **, as also predicted by the "step 4 rule" [see next]**)

j. $\boxed{\boxed{\mathbf{f}\left(\mathbf{2,2}\right)=\left(n_{2,2}\right)=\mathbf{161\ 352\ 166}}}$ **(** $f\left(2,2\right)$ **is obviously larger than** $f\left(2,1\right)=1\ 765\ 126$**, as also predicted by** $fx\left(2,2\right)> fx\left(2,1\right)$, $f\left(2,2\right)$ **is also larger than** $f\left(4,0\right)=6\ 204\ 163$**, as also predicted by** $fx\left(2,2\right)> fx\left(4,0\right)$; $f\left(2,2\right)$ **is smaller than** $f\left(5,0\right)=260\ 535\ 479$**, which is also predicted by** $fx\left(2,2\right)< fx\left(5,0\right)$)

k. $\boxed{\boxed{\mathbf{f}\left(\mathbf{5,0}\right)= f\left(0,5\right)=\left(n_{5,0}=n_{0,5}\right)=\mathbf{260\ 535\ 479}}}$ **(coincidentally or not,** $f\left(5,0\right)$ **is a prime/0-primeth**; $f\left(5,0\right)$**is obviously larger than** $f\left(4,0\right)=6\ 204\ 163$**, as also predicted by** $fx\left(5,0\right)> fx\left(4,0\right)$**; however,** $fx\left(5,0\right)\cong 5.5\times 10^{11}$ **overestimates** $f\left(5,0\right)$ **over** $2m=10^{10}$ **, which may also be the case of** $fx\left(3,1\right)\cong 7.04\times 10^{13}$ **overestimating** $f\left(3,1\right)$)

l. $\boxed{\boxed{\mathbf{f}\left(\mathbf{4,1}\right)= f\left(1,4\right)=\left(n_{4,1}=n_{1,4}\right)=\mathbf{?}}}$ **(computing in progress; expected to be smaller than** $f\left(3,2\right)$ **according to the prediction** $fx\left(4,1\right)< fx\left(3,2\right)$; **obviously expected to be**

larger than $f(3,1)$ **as also according to the prediction** $fx(4,1) > fx(3,1)$; **ALSO expected to be larger than** $f(3,3)$ **as according to the prediction** $fx(4,1) > fx(3,3)$; **however,** $fx(4,1) \cong 1.5 \times 10^{20}$ **surely overestimates** $f(4,1)$**)**

m. $\boxed{\mathbf{f}(\mathbf{3},\mathbf{2}) = f(2,3) = (n_{3,2} = n_{2,3}) = \mathbf{?}}$ **(computing in progress;** $fx(3,2) \cong 2.4 \times 10^{24}$ **probably overestimates** $f(3,2)$**)**

n. $\boxed{\mathbf{f}(\mathbf{3},\mathbf{3}) = (n_{3,3}) = \mathbf{?}}$ **(computing in progress;** $fx(3,3) \cong 3.5 \times 10^{13}$ **probably overestimates** $f(3,3)$ **over** $2m = 10^{10}$ **)**

o.

p. **…[working progress on other higher indexes function values]**

q. **The 2D matrix/array of the finite values** $\boxed{f(a,b)}$ **can be organized in a both square or triangular (Pascal-like) table and was proposed to OEIS as the sequence A281929, BUT rejected in the meantime (see the review-history[URL2, URL3, URL4, URL5, URL6] for the whole discussion, ALSO in downloadable pdf format containing the email-history and my final conclusions), with the main argument that OEIS doesn't accept conjectured meta-sequences and that it wasn't in an "appropriate" form (although I have strictly respected all the given indications), although OEIS doesn't mention this (main) exclusion-criterion (applied to VBGC f[a,b] meta-sequence) explicitly in their publishing policy[URL2, URL3]; my meta-sequence proposed as A281929 is also related to other integer sequences: A000040, A006450, A038580, A049090, A049203, A049202, A057849, A057850, A057851, A057847, A058332, A093047, A002372, A002375)** (A281929 is now occupied with another sequence that was approved in the meantime by OEIS)

r. **The conjectured sequence of all even integers that cannot be expressed as the sum of two distinct 2-primeth and 0-primeth** $\boxed{{}^{2}P_x > {}^{0}P_y}$ **was also submitted to OEIS as A282251 (review completed and approved; see also review history, which is also available in downloadable/printable pdf format at this URL)**

s. **The wiki user page of Andrei-Lucian Drăgoi on OEIS can be found at: URL1, URL2 (I don't have the permission to edit those pages, as I don't belong yet to the official approved Wiki Users List). The list of all sequences submitted by Andrei-Lucian Drăgoi can be found at this URL.**

t. Interestingly, $\boxed{f(a,b)}$ **applied on** $a \in [0,5]$ **and** $b \in [0,5]$ **has its values in the set**

$$\boxed{\mathbf{F} = \left\{ \begin{matrix} (\mathbf{3}), (\mathbf{2564}), (\mathbf{40\,306}), (\mathbf{125\,771}), (\mathbf{1\,765\,126}), \\ (\mathbf{6\,204\,163}), (\mathbf{32\,050\,472}), (\mathbf{161\,352\,166}), (\mathbf{260\,535\,479}) \end{matrix} \right\}}$$

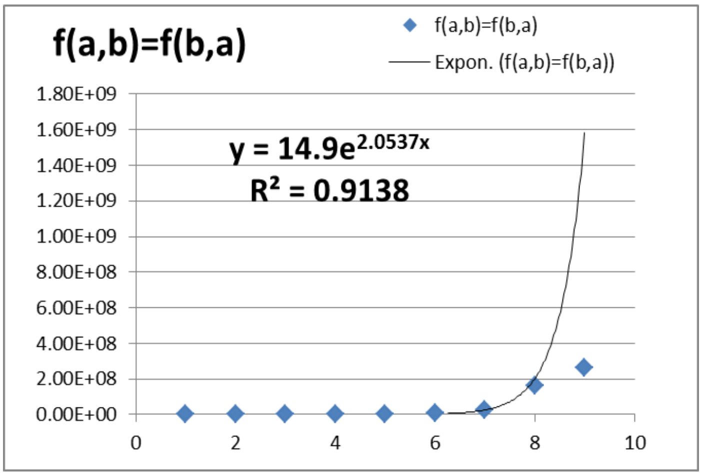
which has an exponential (relatively) compact pattern such as:

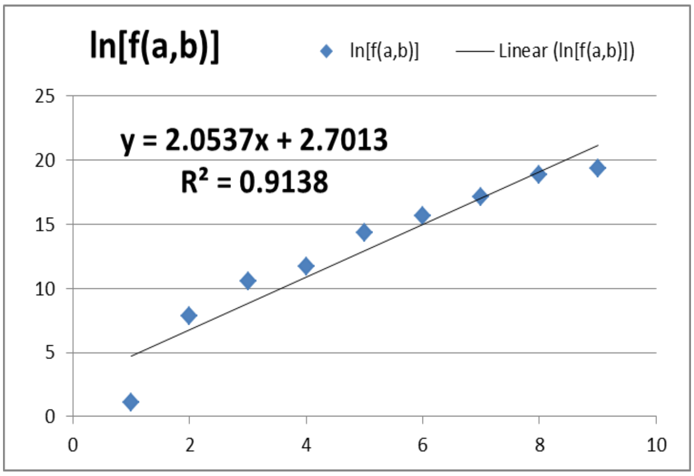
$$\boxed{\mathbf{E_F} \overset{\cong}{=} \{(\mathbf{1.1}), (\mathbf{7.8}), (\mathbf{10.6}), (\mathbf{11.7}), (\mathbf{14.4}), (\mathbf{15.6}), (\mathbf{17.3}), (\mathbf{18.9}), (\mathbf{19.4})\}}$$

, with a relatively constant geometric progression (of about 1.2 ± 0.15**) between its last 7 elements so that**

$$\boxed{\left[\begin{matrix} ({}^{\cong}19.4 / {}^{\cong}18.9) \cong ({}^{\cong}18.9 / {}^{\cong}17.3) \cong ({}^{\cong}17.3 / {}^{\cong}15.6) \cong ({}^{\cong}15.6 / {}^{\cong}14.4) \cong \\ ({}^{\cong}14.4 / {}^{\cong}11.7) \cong ({}^{\cong}11.7 / {}^{\cong}10.6) \cong ({}^{\cong}10.6 / {}^{\cong}7.8) \end{matrix} \right] \cong (\mathbf{1.2 \pm 0.15})}.$$

The single exception of this rule is the gap between the exponents $\cong 1.1$ and $\cong 7.8$. See the next figures.





u. $\mathbf{F}=\left\{\begin{matrix}(3),(2\,564),(40\,306),(125\,771),(1\,765\,126),\\(6\,204\,163),(32\,050\,472),(161\,352\,166),(260\,535\,479)\end{matrix}\right\}$ **has ALSO a correspondent matrix in which a is a column index and b is a line index**

$$M_{f(a,b)}=\begin{bmatrix}(n_{0,0}) & (n_{1,0}=n_{0,1}) & (n_{2,0}=n_{0,2}) & (n_{3,0}=n_{0,3}) & (n_{4,0}=n_{0,4}) & (n_{5,0}=n_{0,5})\\ (n_{0,1}=n_{1,0}) & (n_{1,1}) & (n_{2,1}=n_{1,2}) & (n_{3,1}=n_{1,3}) & (n_{4,1}=n_{1,4}) & (n_{5,1}=n_{1,5})\\ (n_{0,2}=n_{2,0}) & (n_{1,2}=n_{2,1}) & (n_{2,2}) & (n_{3,2}=n_{2,3}) & (n_{4,2}=n_{2,4}) & (n_{5,2}=n_{2,5})\\ (n_{0,3}=n_{3,0}) & (n_{1,3}=n_{3,1}) & (n_{2,3}=n_{3,2}) & (n_{3,3}) & (n_{4,3}=n_{3,4}) & (n_{5,3}=n_{3,5})\\ (n_{0,4}=n_{4,0}) & (n_{1,4}=n_{4,1}) & (n_{2,4}=n_{4,2}) & (n_{3,4}=n_{4,3}) & (n_{4,4}) & (n_{5,4}=n_{4,5})\\ (n_{0,5}=n_{5,0}) & (n_{1,5}=n_{5,1}) & (n_{2,5}=n_{5,2}) & (n_{3,5}=n_{5,3}) & (n_{4,5}=n_{5,4}) & (n_{5,5})\end{bmatrix}$$

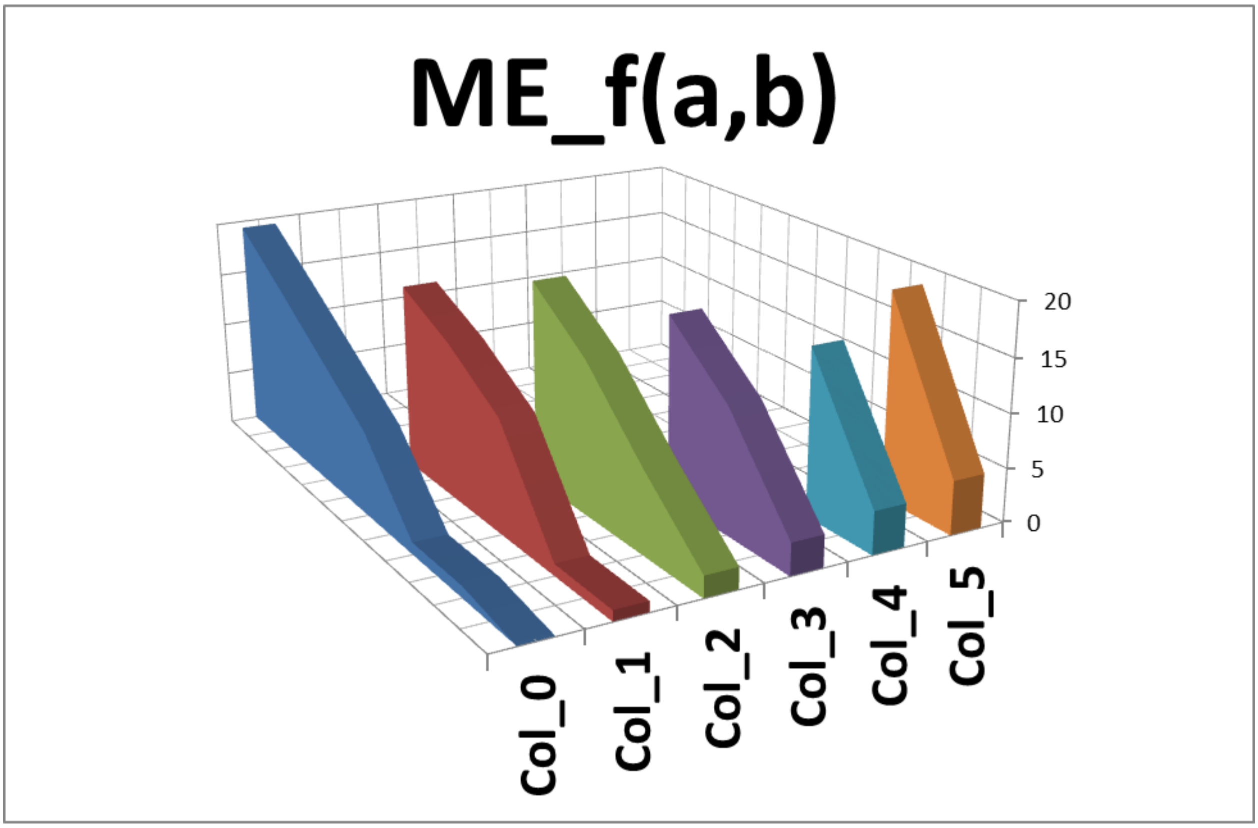
$$=\begin{bmatrix}(3) & (3) & (2564) & (125\,771) & (6\,204\,163) & (260\,535\,479)\\ (3) & (40\,306) & (1\,765\,126) & (32\,050\,472) & (?) & (?)\\ (2564) & (1\,765\,126) & (161\,352\,166) & (?) & (?) & (?)\\ (125\,771) & (32\,050\,472) & (?) & (?) & (?) & (?)\\ (6\,204\,163) & (?) & (?) & (?) & (?) & (?)\\ (260\,535\,479) & (?) & (?) & (?) & (?) & (?)\end{bmatrix}$$

and a matrix of exponents in which a is also a column index and b is also a line index

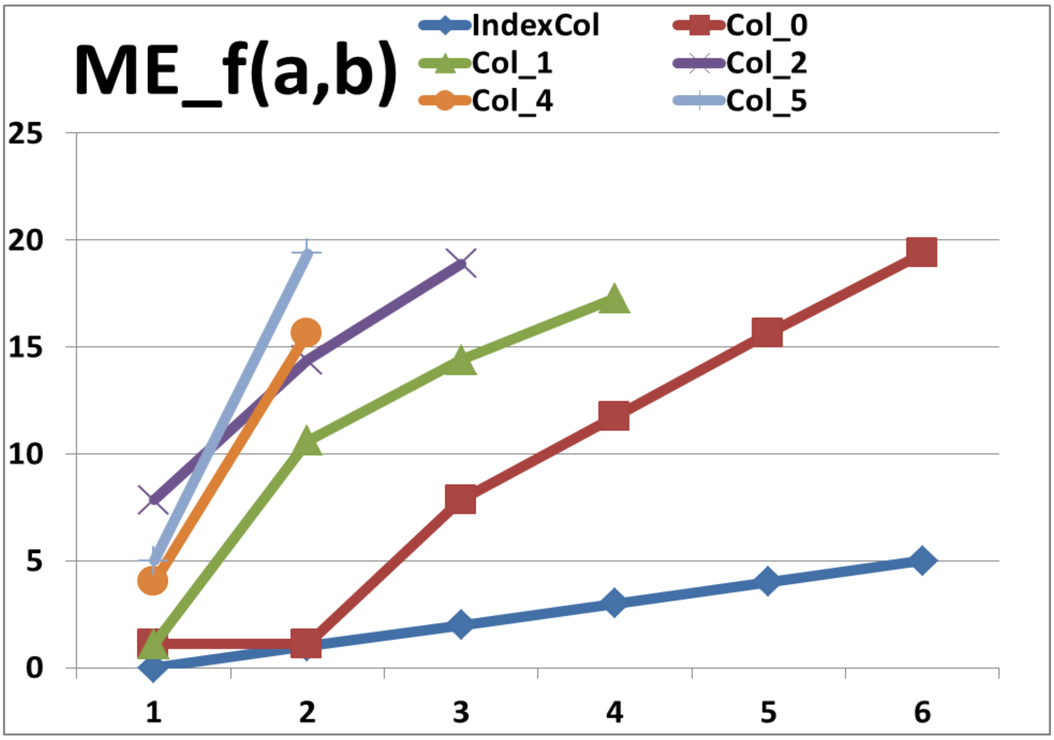
$$ME_{f(a,b)}=LN\begin{bmatrix}(n_{0,0}) & (n_{1,0}=n_{0,1}) & (n_{2,0}=n_{0,2}) & (n_{3,0}=n_{0,3}) & (n_{4,0}=n_{0,4}) & (n_{5,0}=n_{0,5})\\ (n_{0,1}=n_{1,0}) & (n_{1,1}) & (n_{2,1}=n_{1,2}) & (n_{3,1}=n_{1,3}) & (n_{4,1}=n_{1,4}) & (n_{5,1}=n_{1,5})\\ (n_{0,2}=n_{2,0}) & (n_{1,2}=n_{2,1}) & (n_{2,2}) & (n_{3,2}=n_{2,3}) & (n_{4,2}=n_{2,4}) & (n_{5,2}=n_{2,5})\\ (n_{0,3}=n_{3,0}) & (n_{1,3}=n_{3,1}) & (n_{2,3}=n_{3,2}) & (n_{3,3}) & (n_{4,3}=n_{3,4}) & (n_{5,3}=n_{3,5})\\ (n_{0,4}=n_{4,0}) & (n_{1,4}=n_{4,1}) & (n_{2,4}=n_{4,2}) & (n_{3,4}=n_{4,3}) & (n_{4,4}) & (n_{5,4}=n_{4,5})\\ (n_{0,5}=n_{5,0}) & (n_{1,5}=n_{5,1}) & (n_{2,5}=n_{5,2}) & (n_{3,5}=n_{5,3}) & (n_{4,5}=n_{5,4}) & (n_{5,5})\end{bmatrix},$$

$$ME_{f(a,b)} \cong \begin{bmatrix} (\mathbf{1.1}) & (\mathbf{1.1}) & (\mathbf{7.85}) & (\mathbf{11.74}) & (\mathbf{15.64}) & (\mathbf{19.38}) \\ (\mathbf{1.1}) & (\mathbf{10.6}) & (\mathbf{14.38}) & (\mathbf{17.28}) & (?) & (?) \\ (\mathbf{7.85}) & (\mathbf{14.38}) & (\mathbf{18.9}) & (?) & (?) & (?) \\ (\mathbf{11.74}) & (\mathbf{17.28}) & (?) & (?) & (?) & (?) \\ (\mathbf{15.64}) & (?) & (?) & (?) & (?) & (?) \\ (\mathbf{19.38}) & (?) & (?) & (?) & (?) & (?) \end{bmatrix}.$$

$ME_{f(a,b)}$ **can be graphed as a surface (see the next figure).**



v. **The previous matrices generate half-dome-like graphs, apparently with no closed “depression” regions, as all elements tend to become greater when: moving on the lines from left to right, moving on the columns from up to down, moving on the diagonals, from sides to the center. The exponents from each column of** $ME_{f(a,b)}$ **tend to grow linearly from up to down (but also on diagonals, from left to center-right and vice versa): see the next figure.**



w. The first line of $ME_{f(a,b)}$ (which is identical to its first column), has sufficiently many terms to create a function that reasonably approximates the elements on this first line/column, such as: $\boxed{\boxed{fey(a)=4a}} = [0 \quad 4 \quad 8 \quad 12 \quad 16 \quad 20]$**, with**

$\boxed{\boxed{fy(a)=e^{fey(a)}=e^{4a}}} \overset{\cong}{=} [1 \quad 54.6 \quad 2981 \quad 162754.8 \quad 8.8\times10^6 \quad 485\times10^6]$**, which is very close to the first of** $ME_{f(a,b)}$ $[(1.1) \quad (1.1) \quad (7.85) \quad (11.74) \quad (15.64) \quad (19.38)]$

and the first line of

$M_{f(a,b)}$ $[(3) \quad (3) \quad (2564) \quad (125\ 771) \quad (6\ 204\ 163) \quad (260\ 535\ 479)]$ **respectively.**

i. $fy(6)$ **predicts a value for** $\boxed{f(6,0)\cong fy(6)\cong 2.65\times10^{10}}$ **which is beyond the verification-capabilities of our current software. We have ALSO verified this hypothesis with our software AND confirmed that** $f(6,0)$ **is larger than the limit** $2m=10^{10}$. The exception of VBGC(6,0) smaller-and-closest to $2m=10^{10}$ is **9 997 202 434 = 2×4 998 601 217**

ii. $fy(7)$ **predicts a value for** $\boxed{f(7,0)\cong fy(7)\cong 1.45\times10^{12}}$ **which is far beyond the verification capabilities of our current software.**

iii. **On the 2nd line/column of** $ME_{f(a,b)}$ $[(1.1) \quad (10.6) \quad (14.38) \quad (?) \quad (?) \quad (?)]$**, the elements may also grow in a arithmetical progression with an (exponential) step** $\boxed{s \overset{<}{\cong} 4}$ **(starting from** $Ef(1,1)\cong10.6$ to $Ef(2,1)\cong14.38\cong10.6+s$**), with the exception of a first gap between** $Ef(0,1)\cong1.1$ **and** $Ef(1,1)\cong10.6$**, which is correspondent to the gap between** $Ef(1,0)\cong1.1$ and $Ef(2,0)\cong7.85$**. As**

observed, the step $\boxed{s \cong 4}$ is conserved on all lines, columns and secondary diagonals, so that the main diagonal probably has a step of $\boxed{2s \cong 8}$.

1. **The 5th unknown element $Ef\left(4,1\right) = (\mathbf{?})$ from the 2nd line may have a value of** $Ef\left(4,1\right) \overset{?}{\cong} \left[ef\left(2,1\right) + 2s = 14.38 + 8 \cong 22.38\right]$ as predicted by the same step $\boxed{s \cong 4}$. An $Ef\left(r,1\right) \overset{?}{\cong} 18.38$ corresponds to a hypothetical $f\left(4,1\right) \overset{?}{=} \left[e^{Ef(4,1)} \cong e^{22.38} \cong 5\ 242\ 162\ 809 \cong 5.2 \times 10^{9}\right]$ which is ALSO under the limit $2m = 10^{10}$ and may also be (relatively) verified with our software. However, as $f\left(4,1\right) \overset{?}{\cong} 5.2 \times 10^{9}$ is probably very close to the limit $2m = 10^{10}$, the conjecture VBGC[4,1] may not be testified by a "sufficiently" large gap)

2. Other values of $f\left(a,b\right)$ which are predicted to be under the limit $2m = 10^{10}$ are: $\left[f\left(1,4\right) = f\left(4,1\right)\right] \overset{?}{\cong} \mathbf{5.2 \times 10^{9}}$, $\left[f\left(3,2\right) = f\left(2,3\right)\right] \overset{?}{\cong} \left(\mathbf{5.2}\ \mathit{to}\ \mathbf{8.8}\right) \times \mathbf{10^{9}}$. **See the next table.**

Table D-1. The verified values of $\boxed{f(a,b), with\, a \geq b \geq 0}$ (written as exact positive integers: the shaded cells of the table) and the estimated maximum values of $f(a,b)$ using the step $\boxed{s \cong 4}$ "rule" (written in exponential format)

$f(a,b)$	0	1	2	3	4	5	6	7
0	3	3	2,564	125,771	6,204,163	260,535,479	1.4E+10	7.8E+11
1	3	40,306	1,765,126	32,050,472	5.2E+09	2.9E+11	1.6E+13	8.5E+14
2	2,564	1,765,126	161,352,166	5.2E+09	2.9E+11	1.6E+13	8.5E+14	4.7E+16
3	125,771	32,050,472	8.8E+09	2.9E+11	1.6E+13	8.5E+14	4.7E+16	2.5E+18
4	6,204,163	5.2E+09	4.8E+11	1.6E+13	8.5E+14	4.7E+16	2.5E+18	1.4E+20
5	260,535,479	2.9E+11	2.6E+13	8.5E+14	4.7E+16	2.5E+18	1.4E+20	7.6E+21
6	1.4E+10	1.6E+13	1.4E+15	4.7E+16	2.5E+18	1.4E+20	7.6E+21	4.1E+23
7	7.8E+11	8.5E+14	7.8E+16	2.5E+18	1.4E+20	7.6E+21	4.1E+23	2.3E+25

x. $\boxed{fy(a) = e^{fey(a)} = e^{4a}}$ **predicts so accurately the first line of $M_{f(a,b)}$, so that I also propose a secondary inductive (form of) VBGC (siVBGC[a,0]) which states that: "Any/every even positive integer $\boxed{2m \geq 2 \cdot \text{int}\left[fy(a)\right]}$, with $\boxed{fy(a) = e^{4a}}$, can be written as the sum of at least one pair of DISTINCT odd i-primeths $\boxed{{}^{a}P_{x} > {}^{0}P_{y}}$, with the positive integers pair $\boxed{(a,0), with\, a > 0}$ defining the (recursive) orders of the i-primeths pair $\left({}^{a}P_{x}, {}^{0}P_{y}\right)$ AND the distinct positive integers pair $\boxed{(x,y), with\, x > y > 1}$ defining the indexes of each of those i-primeths.".**

i. The set of conjectures siVBGC(a,0) can be used to verify much more rapidly ntBGC, by searching using ONLY the subsets ${}^{a}P$ from the ${}^{a}P_{x}$ which is closest to $\boxed{2m \geq 2 \cdot \text{int}\left(e^{4a}\right)}$ **down to** ${}^{a}P_{2}$ **and testing the primality of** $\left(2m - {}^{a}P_{x}\right)$

y. Interestingly, the differences between consecutive elements on any line or column of

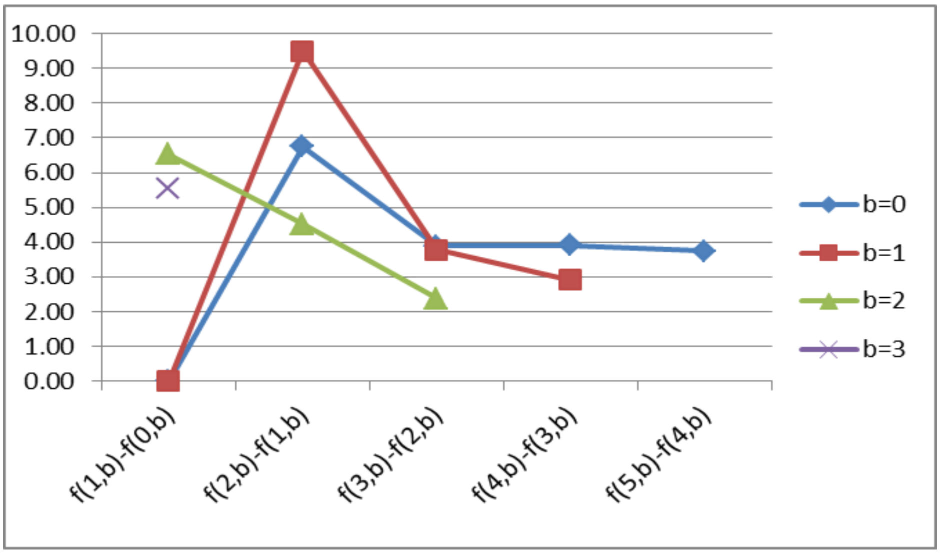
$$ME_{f(a,b)} \cong \begin{bmatrix} (1.1) & (1.1) & (7.85) & (11.74) & (15.64) & (19.38) \\ (1.1) & (10.6) & (14.38) & (17.28) & (?) & (?) \\ (7.85) & (14.38) & (18.9) & (?) & (?) & (?) \\ (11.74) & (17.28) & (?) & (?) & (?) & (?) \\ (15.64) & (?) & (?) & (?) & (?) & (?) \\ (19.38) & (?) & (?) & (?) & (?) & (?) \end{bmatrix}$$

have a 1st or a 2nd

value that is slightly above $\boxed{s \cong 4}$, with all the other values (the 2nd /3rd, the 4th etc) being smaller or approximately equal to $\boxed{s \cong 4}$: **see the next tables and graphs**.

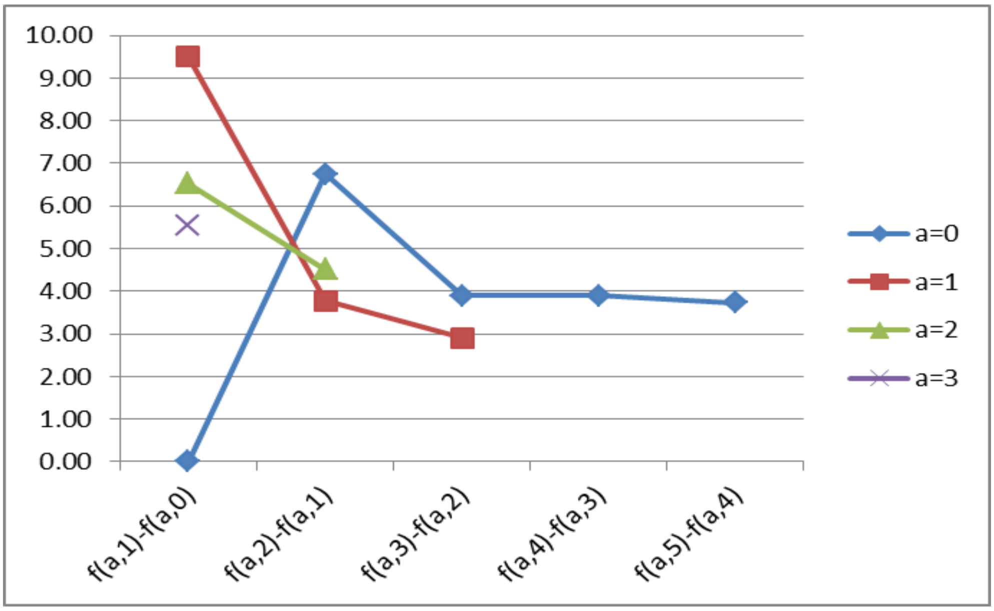
Table D-2A. The differences between consecutive (known) elements from the lines of $ME_{f(a,b)}$ (a is the index of a column of $ME_{f(a,b)}$ AND b is the index of a line of $ME_{f(a,b)}$)

$f(a+1,b)-f(a,b)$	$f(1,b)-f(0,b)$	$f(2,b)-f(1,b)$	$f(3,b)-f(2,b)$	$f(4,b)-f(3,b)$	$f(5,b)-f(4,b)$
$b=0$	**0**	**6.75**	**3.89**	**3.9**	**3.74**
$b=1$	**9.51**	**3.78**	**2.90**	**<4; <3; <2?**	**<4; <3; <2?**
$b=2$	**6.53**	**4.52**	**2.38**	**<4; <3; <2?**	**<4?; <3?; <2?; <1?**
$b=3$	**5.54**	**<5?; <4?**	**<5?; <4?**	**<4?; <3?; <2?; <1?**	
$b=4$	**<5?; <4?**	**<5?; <4?**	**<4?; <3?; <2?; <1?**		
$b=5$	**<5?; <4?**	**<4?; <3?; <2?; <1?**			



<u>Table D-2B</u>. The differences between consecutive (known) elements from the columns of $ME_{f(a,b)}$ (a is the index of a column of $ME_{f(a,b)}$ AND b is the index of a line of $ME_{f(a,b)}$)

$f(a,b+1)-f(a,b)$	$a=0$	$a=1$	$a=2$	$a=3$	$a=4$	$a=5$
$f(a,1)-f(a,0)$	**0**	**9.51**	**6.53**	**5.54**	**<6?; <5?; <4?**	**<6?; <5?; <4?**
$f(a,2)-f(a,1)$	**6.75**	**3.78**	**4.52**	**<6; <5?; <4?**	**<6?; <5?; <4?**	**<6?; <5?; <4?**
$f(a,3)-f(a,2)$	**3.89**	**2.90**	**<5; <4?; <3?**	**<6; <5?; <4?**	**<6?; <5?; <4?**	**<6?; <5?; <4?**
$f(a,4)-f(a,3)$	**3.9**	**<4; <3; <2?**	**<5; <4?; <3?**	**<6; <5?; <4?**	**<6?; <5?; <4?**	
$f(a,5)-f(a,4)$	**3.74**	**<4; <3; <2?**	**<4; <3; <2?**	**<6; <5?; <4?**		



z. Furthermore, this symmetrical function (with the property $fx(a,b) = fx(b,a)$**)**

$$\boxed{fx(a,b) = \begin{cases} 2^{(a+1)(b+1)(a+b+2)} & for\,(a=b=0) \\ 2^{[(a+1)(b+1)(a+b+3)/a]-a} & for\,(a=b)\,AND\,(a>0) \\ 2^{(a+1)(b+1)(a+b+2)-(a+b-2)} & for\,(a\neq b)\,AND\left[(a>0)\,OR\,(b>0)\right] \end{cases}}$$

generates positive integer values that are relatively close BUT strictly larger than the values of $f(a,b)$ **for** $a \in [0,5]$ **and** $b \in [0,5]$**, with also a half-dome-like graph, so that the author proposes a variant of inductive VBGC (iVBGC) stating that:**

"<u>Every even positive integer</u> $\boxed{2m \geq 2 \cdot fx(a,b)}$ **<u>, with</u>**

$$\boxed{fx(a,b) = \begin{cases} 2^{(a+1)(b+1)(a+b+2)} & for\,(a=b=0) \\ 2^{[(a+1)(b+1)(a+b+3)/a]-a} & for\,(a=b)\,AND\,(a>0) \\ 2^{(a+1)(b+1)(a+b+2)-(a+b-2)} & for\,(a\neq b)\,AND\left[(a>0)\,OR\,(b>0)\right] \end{cases}}$$

<u>, can</u>

<u>be written as the sum of at least one pair of distinct i-primeths</u> $\boxed{{}^{a}P_{x} > {}^{b}P_{y}}$ **<u>, with the positive integers pair</u>** $\boxed{(a,b),\,with\,a \geq b \geq 0}$ **<u>defining the (recursive) orders of each of those i-primeths AND the pair of distinct positive integers</u>** $\boxed{(x,y),\,with\,x > y > 1}$ **<u>defining the indexes of each of those i-primeths</u>."**

aa. The function

$$\boxed{fx(a,b) = \begin{cases} 2^{(a+1)(b+1)(a+b+2)} & for\,(a=b=0) \\ 2^{[(a+1)(b+1)(a+b+3)/a]-a} & for\,(a=b)\,AND\,(a>0) \\ 2^{(a+1)(b+1)(a+b+2)-(a+b-2)} & for\,(a\neq b)\,AND\left[(a>0)\,OR\,(b>0)\right] \end{cases}}$$

has its values in the matrix

$$\boxed{M_{fx(a,b)} \cong} \begin{bmatrix} (4) & (128) & (4\,096) & (5.2\times10^{5}) & (2.7\times10^{8}) & (5.5\times10^{11}) \\ (128) & (5.2\times10^{5}) & (5.4\times10^{8}) & (7\times10^{13}) & (...) & (...) \\ (4\,096) & (5.4\times10^{8}) & (7.6\times10^{8}) & (...) & (...) & (...) \\ (5.2\times10^{5}) & (7\times10^{13}) & (...) & (...) & (...) & (...) \\ (2.7\times10^{8}) & (...) & (...) & (...) & (...) & (...) \\ (5.5\times10^{11}) & (...) & (...) & (...) & (...) & (...) \end{bmatrix}$$

in which each element is strictly larger than its correspondent element from $M_{f(a,b)}$

$$\boxed{M_{f(a,b)} = \begin{bmatrix} (3) & (3) & (2564) & (125\,771) & (6\,204\,163) & (260\,535\,479) \\ (3) & (40\,306) & (1\,765\,126) & (32\,050\,272) & (?) & (?) \\ (2564) & (1\,765\,126) & (161\,352\,166) & (?) & (?) & (?) \\ (125\,771) & (32\,050\,272) & (?) & (?) & (?) & (?) \\ (6\,204\,163) & (?) & (?) & (?) & (?) & (?) \\ (260\,535\,479) & (?) & (?) & (?) & (?) & (?) \end{bmatrix}}$$

bb. Additionally, the function

$$\boxed{fx_2(a,b)=\begin{cases}2^{(a+1)(b+1)(a+b+2)} & for\,(a=b=0)\\ 2^{[(a+1)(b+1)(a+b+3)/a]-2a} & for\,(a=b)\,AND\,(a>0)\\ 2^{(a+1)(b+1)(a+b+2)-(a+b-2)} & for\,(a\neq b)\,AND\left[(a>0)\,OR\,(b>0)\right]\end{cases}}$$ **has**

strictly larger but even more closer values to the values of $M_{f(a,b)}$**, BUT predicts distorted inequalities between some terms (when compared to the inequalities between some elements of** $M_{f(a,b)}$**).**

$$\boxed{M_{fx_2(a,b)}\stackrel{\cong}{=}}\begin{bmatrix}(\mathbf{4}) & (\mathbf{128}) & (\mathbf{4\ 096}) & (\mathbf{5.2\times10^5}) & (\mathbf{2.7\times10^8}) & (\mathbf{5.5\times10^{11}})\\ (\mathbf{128}) & \color{red}{(\mathbf{2.6\times10^5})} & (\mathbf{5.4\times10^8}) & (\mathbf{7\times10^{13}}) & (...) & (...)\\ (\mathbf{4\ 096}) & (\mathbf{5.4\times10^8}) & \color{red}{(\mathbf{1.8\times10^8})} & (...) & (...) & (...)\\ (\mathbf{5.2\times10^5}) & (\mathbf{7\times10^{13}}) & (...) & (...) & (...) & (...)\\ (\mathbf{2.7\times10^8}) & (...) & (...) & (...) & (...) & (...)\\ (\mathbf{5.5\times10^{11}}) & (...) & (...) & (...) & (...) & (...)\end{bmatrix}$$

<u>VBGC – secondary statements (also part of VBGC)</u>:

<u>1.</u> for $\boxed{(a,b)=(1,0)}$ **AND** $\boxed{m>28}$**, it will always exist at least one pair of finite distinct positive integers** $\boxed{(x,y), with\, x>y>1}$ **AND** $\boxed{{}^1P_x+{}^0P_y=2m}$ **AND x (or y) in the double-open interval** $\boxed{(\ln(2m),2m/\ln(2m))}$**.**

- **<u>a.</u> Important note:** VBGC is much “stronger” and general than BGC and proposes a much more rapid and efficient (at-least-one-GIP)-sieve than the GKRC. The GM of GIPs generated by VBGC has a smaller nof. lines than the GM of GIPs generated by GKRC. VBGC is a useful optimized sieve to push forward the limit $4\cdot10^{18}$ to which BGC was verified to hold **[53]**. When verifying BGC for a very large number N, one can use the VBGC(a,b) with a minimal positive value for the difference $[N-f(a,b)]$.

<u>2.</u> Important note: VBGC essentially (and alternatively) states that there is an infinite number of conjectures indexable as VBGC(a,b), all stronger than BGC, EACH of if associated with a pair $\boxed{(a,b), with\, a\geq b>0}$ AND a finite positive integer $\boxed{n_{a,b}=f(a,b)}$.

- **<u>a.</u>** VBGC(0,0) is in fact ntBGC.

<u>3.</u> The different special cases of VBGC can be named after the pair (a,b) [VBGC(a,b)] AND:

- **<u>a.</u> VBGC(0,0)** is in fact ntBGC (defined in the Part B of this article)
- **<u>b.</u> VBGC(1,0)[1]** is a GLC stronger and more elegant than ntBGC, as it acts on a limit $\boxed{2f(1,0)=6}$ identical to ntBGC inferior limit (which is $\boxed{2f(0,0)=6}$) BUT the associated $\boxed{G_{1,0}(m)}$ (which counts the number of pairs of possible GIPs for any even integer $m>3$) has significantly smaller values than the function $\boxed{G_{0,0}(m)}$ of ntBGC [which is VBGC(0,0)]

c. **VBGC(2,0)** is obviously a stronger GLC than VBGC(1,0) is AND ALSO $\boxed{G_{2,0}(m)}$ has smaller non-0 values than $\boxed{G_{1,1}(m)}$ for $m \in \left(f(2,0),\infty\right)$

d. **VBGC(1,1)** (anticipated by my discovery of **VBGC(1,0) from 2007 and officially registered in 2012 at OSIM** [1]) is an obviously stronger GLC than VBGC(1,0) and is equivalent to Bayless-Klyve-Oliveira e Silva Goldbach-like Conjecture (BKOS-GLC) published in Oct. 2013 [**54]** alias "Conjecture 9.1" (rephrased) (tested by these authors up to $2m = 10^9$): all even integers $2m > \left[2 \cdot 40306 \left(= 2f(1,1)\right)\right]$ can be expressed as the sum of at least one pair of prime-indexed primes [PIPs] (1-primeths ${}^1P_x$ and ${}^1P_y$). This article of Bayless. Klyve and Oliveira (2012, 2013) was based on a previous article by Barnett and Broughan (published in 2009) **[55]**, but BKOS-GLC was an additional result to this 2009 article. Mr. George Anescu (a friend and collaborator) have also helped me to retest VBGC(1,1) up to $2m = 10^{10}$, but also helped me verifying all VBGC(a,b) for all pairs $\boxed{(a,b) \in \{(1,0),(1,1),(2,0),(2,1),(2,2)\}}$ [6].

4. When $a \to \infty, b \to \infty \, and \, m \to \infty$, $\boxed{G_{a,b}\left(f(a,b)+1\right) \underset{\geq}{\to} 1}$ and the "comets" of VBGC(a,b) tend to narrow progressively for each pair of positive integers $(a_2, b_2), with \, a_2 > a_1 \, and \, b_2 > b_1$.

5. All VBGC(a>0,b≥0) can be used to produce more rapid algorithms for the experimental verification of ntBGC for very large positive integers

a. For VBGC(1,0), the average number of attempts (ANA) to find the first pair (x,y) for each integer m, in the interval [3,2m] tends asymptotically to $\boxed{\ln\left(\sqrt{n}\right) = \ln(n)/2}$ when searching just the 1-primeths subset in descending manner, starting from the largest 1-primeth ≤2m-1 and verifying if $\left(2m - {}^1P_x\right)$ is a 0- primeth)

Conclusions on VBGC:

1. VBGC(a,b) is essentially an extension and generalization of BGC as applied on (the extended and generalized concept of) all subsets of i-primeths.

2. VBGC can be considered a "meta-conjecture", as it states an infinite number of BGC-like conjectures (stronger than BGC) which are generically named as VBGC(a,b), with $a,b \in \mathbb{N}$ and with VBGC(0,0) being equivalent to the non-trivial variant of BGC (**ntBGC).**

a. VBGC has an inductive variant and an analytical variant, which both apply to any super-prime family (of any iteration order i, generically named "i-primeths" by the author of VBGC)

3. VBGC (especially siVBGC) can be used to optimize (by speeding up) the algorithms used to verify BGC on very large numbers. BGC was tested until present up to $2m = 4 \times 10^{18}$ [**56]**. A first experiment would be to re-test BGC up to that limit $2m = 4 \times 10^{18}$ alternatively using siVBGC and to compare the global times of computing.

4. VBGC distinguishes as a very important (unified) conjecture of primes and a very special self-similar property of the primes as the rarefied ${}^iP$ is self-similar to the more dense ${}^{(i-1)}P$ in respect to the ntBGC. In other words, each of the i-primeths sets behaves as a "summary of" the 0-primeths set in

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[6] The code-source (written by Mr. George Anescu in Microsoft Studio 2015 - Visual C++ language/environment using parallel processing) that was used to test BKOS-GLC up to n=10^{10} (using a laptop PC with an Intel^R Core^TM processor i7-3630 QM CPU at 2.4 GHz with 4 processors (8 hyper-threads), can be found at this URL (the old variant can be found at this URL-old) respect to the ntBGC: this is a (quasi)fractal-like BGC-related behavior of the infinite number of the i-primeths sets. **Essentially, VBGC conjectures that ntBGC is a common property of all the i-primeths sets (for any positive integer order i), differing just by the inferior limit of each VBGC(a,b) defined by the function $f(a,b)$). I have called VBGC as "vertical" motivated by the fact that VBGC is a "vertical" (recursive) generalization of the ntBGC on the infinite super-set of i-primeths sets.**

a. The set of values of $f(a,b)$ is a set of critical density thresholds/points of each i-primeths set in respect to the set VBGC(a,b) conjectures.

b. Batchko R.G. has also reported other quasi-fractal/quasi-self-similar structure in the distribution of the prime-indexed primes **[57]:** Batchko also used a similar general definition for primes with (recursive) prime indexes (PIPs), briefly named in my article as "i-primeths".

c. Carlo Cattani and Armando Ciancio also reported a quasi-fractal distribution of primes (including i-primeths) similar to a Cantor set (Cantor dust) by mapping primes and i-primeths into a binary image which visualizes the distribution of i-primeths **[58]**. VBGC may be an intrinsic property of all sets of i-primeths that can also explain OR be explained by this Cantor dust-like distribution of these i-primeths.

5. All sets ${}^{(i>0)}P$ are subsets of ${}^{0}P = \wp^*$ and come in an infinite number: this family of subsets is governed/defined by the Prime number theorem. There is a potential infinite number of rules/criterions/theorems to extract an infinite number of subsets from ${}^{0}P$ (grouped in a family of subsets defined by that specific rule/criterion/theorem), like the Dirichlet's theorem on arithmetic progressions for example[URL2] OR other prime formulas[URL2, URL3] that generate infinite subsets of primes. It would be an interesting research subfield of BGC to test what are those families (of subsets of primes) that respect ntBGC and generate functions with finite values similar to $f(a,b) = n_{a,b} = n_{b,a}$. This potential future research subfield may also help in optimizing the algorithms used in the present for ntBGC verification on large numbers. However, one special property of the family ${}^{(i>0)}P$ is that each subset of this family is a commutative monoid[[URL2]].

6. It is an interesting fact per se that all ${}^{(i>0)}P$ subsets have very low densities (when compared to ${}^{0}P$ and $\mathbb{N}^*$) BUT NOT sufficiently low densities to NOT generate a function $f(a,b)$ with finite values for any pair of finites (a,b).

Future challenges for VBGC (to be also approached in the next versions of this article):

1. To calculate the values of the function $f(a,b) = n_{a,b} = n_{b,a}$ and test/verify VBGC(a,b) for large positive integers pairs $(a > 2, b > 2)$ (a,b), but also for the pairs (a,b) with large $(a-b)$ differences.

Potential applications of VBGC (to also be created in the next versions of this article):

1. **VBGC can offer a potential infinite set of Goldbach Comets[URL2, URL3a, URL3b], one for each conjecture VBGC(a,b) applied on each order of i-primeths. The number of possible decompositions of any even integer 2m>2 in two primes/0-primeths is the string A045917[URL2] in OEIS.**

2. **VBGC can be used to optimize the algorithms of finding/verifying very large primes (i-primeths)/potential primes (i-primeths)**

3. **As TGC/TGT is considered a consequence of BGC, VBGC can be used as a model to also formulate a Vertical (generalization) of the Ternary Goldbach Conjecture/Theorem (VTGC) as an analogous consequence of VBGC (with a corresponding meta-sequence f_VTGC[a,b])**

4. **VBGC can be theoretically used to optimize the algorithms of prime/integer factorization[URL2,URL3] (the main tool of cryptography)**

5. **VBGC can offer a rule of decomposition of Euclidean[URL2,URL3,URL4]/non-Euclidean[URL2] spaces/volumes with a finite 2N (positive) integer number of dimensions into pair of spaces, both with a (positive) i-primeth number of dimensions. According to VBGC, an Euclidian/non-Euclidean (hyper)space/(hyper)volume with 2N dimensions V_{2N} (with N>2) can always be decomposed such as:**

$$\boxed{V_{2N} = k \cdot \left(V_{\left({}^{a}P_x\right)} \times V_{\left({}^{b}P_y\right)} \right) = k \cdot \left(r^{\left({}^{a}P_x\right)} \times r^{\left({}^{b}P_y\right)} \right), with\ k = volume - specific\ constant}$$

6. **VBGC can be used in M-Theory to simulate decompositions of 2N-branes (with a finite 2N [positive] integer number of dimensions) into pair of branes both with a (positive) i-primeth number of dimensions: VBGC can be also used to predict possible symmetries/asymmetries in crystallography, as based on i-primeths.**

7. **This type of vertical generalization (generating a meta-conjecture) may be the start of a new research sub-field in which other conjectures may be hypothesized to also have vertical generalizations applied on i-primeths. For example, a hypothetical vertical Polignac's conjecture (a "minus" version of BGC) may speed up the searching algorithms to find very large primes (larger than a given limit m).**

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Competing interests

Author has declared that no competing interests exist.

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Addendum

Method for verifying VBGC. We have used Microsoft Visual C++. First, we have created (and stored on hard-disk) a set of ".bin" files containing all these primes in the double-open interval $(1,10^{10})$:

(1) standard primes (alias 0-primeths) (a non-archived "p1_10000000000.bin" bin file of ~3.55GigaBytes)

(2) super-primes (alias 1-primeths) (a non-archived "p2_10000000000.bin" file of ~188MegaBytes, containing **24,106,415 1-primeths)**

(3) 2-super-primes (2-primeths) (a non-archived "p3_10000000000.bin" file of ~12MegaBytes, containing **1,513,371 2-primeths)**

(4) 3-super-primes (3-primeths) (a non-archived "p4_10000000000.bin" file of ~900KiloBytes, containing **115,127 3-primeths)**

(5) 4-super-primes (4-primeths) (a non-archived "p5_10000000000.bin" file of ~86KiloBytes, containing **10,883 4-primeths)**

(7) 5-super-primes (5-primeths) (a non-archived "p6_10000000000.bin" file of ~11KiloBytes, containing **1,323 5-primeths)**

(8) 6-super-primes (6-primeths) (a non-archived "p7_10000000000.bin" file of ~2KiloBytes, containing **216 6-primeths)**

(9) 7-super-primes (7-primeths) (a non-archived "p8_10000000000.bin" file of ~1KiloBytes, containing **47 7-primeths)**

For every (a,b) pair with $a \geq b$, we have verified each ${}^{a}P_{x}\left(> {}^{b}P_{x}\right)$ from the (less) dense subset of ${}^{a}P$ superposing the double-open interval $(2, 2m \geq 6)$ (starting from that ${}^{a}P_{x}$ which was the closest to $2m-1$ in descending order): we have then verified if the difference $(2m - {}^{a}P_{x})$ is an element in the (more) dense set ${}^{b}P$ by using binary section method.

We have then computed each value of $f(a,b)$ (with the additional condition ${}^{a}P_{x} \neq {}^{b}P_{y} \Leftrightarrow {}^{a}P_{x} > {}^{b}P_{y}$ in at least one Goldbach partition for any $m > f(a,b)$, with ${}^{a}P_{x} + {}^{b}P_{y} = 2m$). The computing

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[7] The CV of Professor Albu T. is also available online (URL)

[8] The CV of Professor Strătilă Ş-V. is also available online (URL) time for determining and verifying $\boxed{f(2,1) = f(1,2) = (n_{2,1} = n_{1,2}) = 1\ 765\ 126}$ and $\boxed{f(2,2) = (n_{2,2}) = 161\ 352\ 166}$ was about 30 hours. The computing time for determining and verifying $\boxed{f(3,0) = f(0,3) = (n_{3,0} = n_{0,3}) = \mathbf{125\ 771}}$, $\boxed{f(4,0) = f(0,4) = (n_{4,0} = n_{0,4}) = \mathbf{6\ \ 204\ 163}}$ and $\boxed{f(5,0) = f(0,5) = (n_{5,0} = n_{0,5}) = \mathbf{260\ 535\ 479}}$ was also about 30 hours.

$\boxed{\boxed{\mathbf{f}(\mathbf{3,1}) = f(1,3) = (n_{3,1} = n_{1,3}) = \mathbf{32\ 050\ 472(?)}}}$ **is still in a verification process, which started in the second week of February 2017. No exceptions found until present between** $2m = 2\cdot\mathbf{f}(\mathbf{3,1})$ and $2m = 2\cdot(49.1\times10^6)$ so that $\mathbf{f}(\mathbf{3,1})$ may be a veritable last exception of VBGC[3,1] or just the start of a large gap until the next possible exception (which may be found in the future).

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