R.B. Easter, E.Hitzer, Double Conformal Space-Time Algebra, S. Sivasundaram (ed.), International Conference in Nonlinear Problems in Aviation and Aerospace ICNPAA 2016, AIP Conf. Proc., Vol. 1798, 020066 (2017); 10 pages, doi: 10.1063/1.4972658.

Double Conformal Space-Time Algebra

Robert Benjamin Easter[1,b)] and Eckhard Hitzer[2,a)]

[1] *Bangkok, Thailand*
[2] *International Christian University, Osawa 3-10-2, 181-8585 Mitaka, Tokyo , Japan*

[a)] Corresponding author: hitzer@icu.ac.jp
[b)] dirname@yahoo.com

Abstract. The Double Conformal Space-Time Algebra (DCSTA) is a high-dimensional 12D Geometric Algebra $\mathcal{G}_{4,8}$ that extends the concepts introduced with the Double Conformal / Darboux Cyclide Geometric Algebra (DCGA) $\mathcal{G}_{8,2}$ with entities for Darboux cyclides (incl. parabolic and Dupin cyclides, general quadrics, and ring torus) in spacetime with a new boost operator. The base algebra in which spacetime geometry is modeled is the Space-Time Algebra (STA) $\mathcal{G}_{1,3}$. Two Conformal Space-Time subalgebras (CSTA) $\mathcal{G}_{2,4}$ provide spacetime entities for points, flats (incl. worldlines), and hyperbolics, and a complete set of versors for their spacetime transformations that includes rotation, translation, isotropic dilation, hyperbolic rotation (boost), planar reflection, and (pseudo)spherical inversion in rounds or hyperbolics. The DCSTA $\mathcal{G}_{4,8}$ is a doubling product of two $\mathcal{G}_{2,4}$ CSTA subalgebras that inherits doubled CSTA entities and versors from CSTA and adds new bivector entities for (pseudo)quadrics and Darboux (pseudo)cyclides in spacetime that are also transformed by the doubled versors. The "pseudo" surface entities are spacetime hyperbolics or other surface entities using the time axis as a pseudospatial dimension. The (pseudo)cyclides are the inversions of (pseudo)quadrics in rounds or hyperbolics. An operation for the directed non-uniform scaling (anisotropic dilation) of the bivector general quadric entities is defined using the boost operator and a spatial projection. DCSTA allows general quadric surfaces to be transformed in spacetime by the same complete set of doubled CSTA versor (i.e., DCSTA versor) operations that are also valid on the doubled CSTA point entity (i.e., DCSTA point) and the other doubled CSTA entities. The new DCSTA bivector entities are formed by extracting values from the DCSTA point entity using specifically defined inner product extraction operators. Quadric surface entities can be boosted into moving surfaces with constant velocities that display the length contraction effect of special relativity. DCSTA is an algebra for computing with quadrics and their cyclide inversions in spacetime. For applications or testing, DCSTA $\mathcal{G}_{4,8}$ can be computed using various software packages, such as *Gaalop*, the *Clifford Multivector Toolbox (for MATLAB)*, or the symbolic computer algebra system *SymPy* with the *$\mathcal{G}$Algebra* module.

Introduction

The Double Conformal Space-Time Algebra (DCSTA)[1] $\mathcal{G}_{4,8}$ [1] is a high-dimensional Geometric Algebra [2][3][4][5] over the twelve-dimensional (12D) vector space[2] $\mathbb{R}^{4,8}$, that extends the concepts introduced with the Double Conformal / Darboux Cyclide Geometric Algebra (DCGA) $\mathcal{G}_{8,2}$ [6][7][8][9][10] with entities for

[1] We use the expression *geometric algebra* for Clifford algebra, and the notation $\mathcal{G}_{p,q}$ for $Cl(p,q)$.

[2] Only in one instance do we use complex numbers, when we use a boost operation (37) to create a Lorentz dilation $d > 1$ as, e.g., in Fig. 2. The resulting dilation (15) is still real, but at an intermediate stage of the computation a complex value for the natural speed β (11) is introduced. This is clearly not physical, but (as described in the paper) a useful method for computer graphics. Only for this application, an implementation of the complex field $\mathbb{C}$ will be needed.

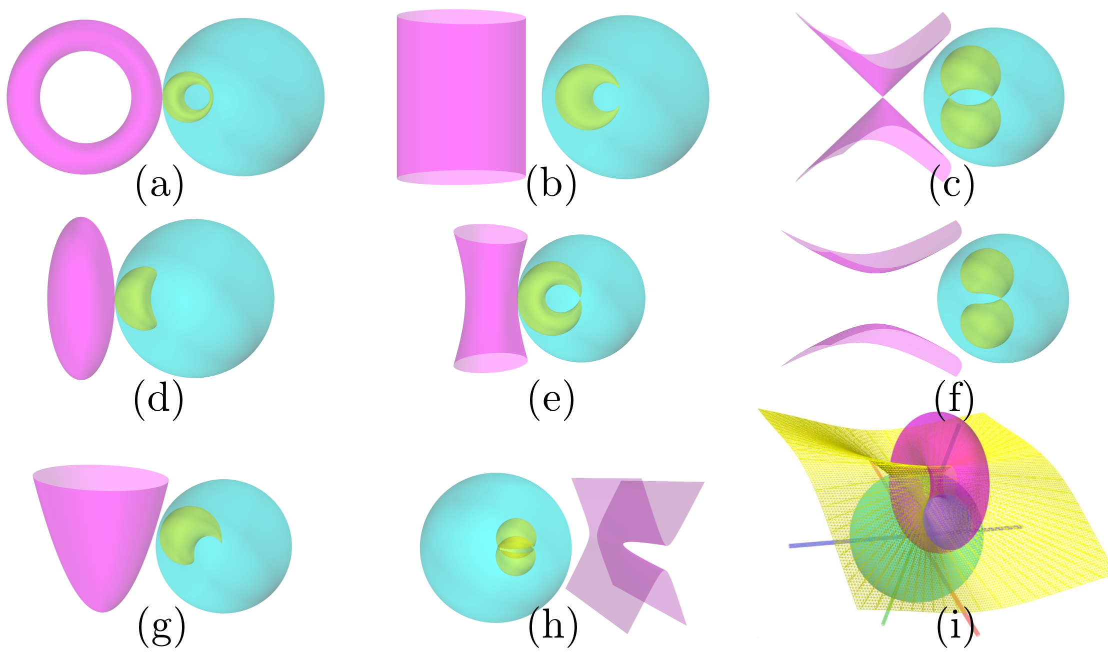

Figure 1. Quadric and ring torus surfaces $\boldsymbol{Q}$ and their inversions in sphere $\boldsymbol{S}$ as cyclides $\boldsymbol{\Omega} = \boldsymbol{SQS}^{\sim}$.

Darboux cyclides (including parabolic and Dupin cyclides, general quadrics, and ring tori) in space-time $\mathbb{R}^{1,3}$ with a new boost (rotor) operator.

The base algebra in which space-time geometry is modeled is the *Space-Time Algebra* (STA) $\mathcal{G}_{1,3}$ [11][12]. Two isomorphic *Conformal* Space-Time subalgebras (CSTA) $\mathcal{G}_{2,4} = \mathcal{G}_{1+1,3+1}$ [13] provide space-time entities for points, flats (including flat points, lines, worldlines, planes, . . . , hyperplanes), and rounds (including point pairs, circles, spheres, . . . , hyperspheres) and hyperbolics, and a complete set of versors for their space-time transformations with rotation, translation, isotropic dilation, *hyperbolic rotation* (boost), planar reflection, and (pseudo)spherical inversion in rounds or *hyperbolics*.

The double CSTA (DCSTA) $\mathcal{G}_{4,8}$ is a doubling product of two CSTA subalgebras $\mathcal{G}_{2,4}$, that inherits doubled CSTA entities and versors from CSTA and adds new bivector entities for (pseudo)quadrics and Darboux (pseudo)cyclides in space-time that are also transformed by the doubled versors. The “pseudo” surface entities are space-time hyperbolics or other surface entities using the *time axis* as a pseudospatial dimension. The (pseudo)cyclides are the inversions of (pseudo)quadrics in rounds or hyperbolics.

DCSTA allows general quadric surfaces to be transformed in space-time by the same complete set of doubled CSTA versor (i.e., DCSTA versor) operations that are also valid on the doubled CSTA point entity (i.e., DCSTA point) and the other doubled CSTA entities. Quadric surface entities can be boosted into moving surfaces with constant velocities that display the length contraction effect of special relativity. DCSTA also defines an operation for the directed non-uniform spatial scaling (anisotropic dilation) of the bivector general quadric entities using the boost operator followed by a spatial projection.

The new DCSTA bivector entities for quadrics and Darboux cyclides are formed by extracting values from the DCSTA point entity using specifically defined (inner product) extraction operators.

The DCSTA $\mathcal{G}_{4,8}$ $\mathcal{D}$ has a basis of twelve orthonormal vector elements $\mathbf{e}_i$, $1 \le i \le 12$, with metric (squares or signatures) $m_{\mathcal{D}}$:

$$\begin{aligned} m = m_{\mathcal{D}} &= \mathrm{diag}(1,-1,-1,-1,1,-1,1,-1,-1,-1,1,-1) = [m_{ij}] && (1)\\ &= \mathrm{diag}(m_{C^1}, m_{C^2}) = \mathrm{diag}(1, m_{\mathcal{CS}^1}, 1, m_{\mathcal{CS}^2}) && (2)\\ &= \mathrm{diag}(m_{\mathcal{M}^1}, 1, -1, m_{\mathcal{M}^2}, 1, -1) = \mathrm{diag}(1, m_{\mathcal{S}^1}, 1, -1, 1, m_{\mathcal{S}^2}, 1, -1), && (3)\\ m_{\mathcal{DS}} &= \mathrm{diag}(m_{\mathcal{CS}^1}, m_{\mathcal{CS}^2}), \qquad m_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j. && (4) \end{aligned}$$

The above metric also includes the metrics of the following subalgebras:

- $\mathcal{G}_{2,4}$ CSTA1 C^1: m_{C^1}
- $\mathcal{G}_{1,4}$ Conformal SA1 (CSA1) $\mathcal{CS}^1$: $m_{\mathcal{CS}^1}$
- $\mathcal{G}_{1,3}$ STA1 $\mathcal{M}^1$: $m_{\mathcal{M}^1}$
- $\mathcal{G}_{0,3}$ Space Algebra 1 (SA1) $\mathcal{S}^1$: $m_{\mathcal{S}^1}$
- $\mathcal{G}_{2,8}$ Double Conformal SA (DCSA) $\mathcal{DS}$: $m_{\mathcal{DS}}$
- $\mathcal{G}_{2,4}$ CSTA2 C^2: m_{C^2}
- $\mathcal{G}_{1,4}$ CSA2 $\mathcal{CS}^2$: $m_{\mathcal{CS}^2}$
- $\mathcal{G}_{1,3}$ STA2 $\mathcal{M}^2$: $m_{\mathcal{M}^2}$
- $\mathcal{G}_{0,3}$ SA2 $\mathcal{S}^2$: $m_{\mathcal{S}^2}$

Notation of Space-Time Algebra

The basis of the space-time algebra $\mathcal{G}_{1,3}$ STA $\mathcal{M} \cong \mathcal{G}_{1,3}$ STA1 $\mathcal{M}^1$ is $\{\boldsymbol{\gamma}_0, \boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\gamma}_3\} \cong \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$, and for the second copy of the space-time algebra $\mathcal{G}_{1,3}$ STA2 $\mathcal{M}^2$ we have the basis $\{\boldsymbol{\gamma}_0, \boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\gamma}_3\} \cong \{\mathbf{e}_7, \mathbf{e}_8, \mathbf{e}_9, \mathbf{e}_{10}\}$. The space algebra $\mathcal{G}_{0,3}$ SA $\mathcal{S}$ basis, included in the space-time algebra, is $\{\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\gamma}_3\}$. The STA unit four-dimensional pseudoscalar is $\mathbf{I}_{\mathcal{M}} = \boldsymbol{\gamma}_0\boldsymbol{\gamma}_1\boldsymbol{\gamma}_2\boldsymbol{\gamma}_3$, and for SA the unit three-dimensional pseudoscalar is $\mathbf{I}_{\mathcal{S}} = \boldsymbol{\gamma}_1\boldsymbol{\gamma}_2\boldsymbol{\gamma}_3$. Moreover, STA defines a space-time position with coordinates (w, x, y, z) by the four-dimensional vector

$$\boldsymbol{p} = \boldsymbol{p}_{\mathcal{M}} = (w = ct)\boldsymbol{\gamma}_0 + x\boldsymbol{\gamma}_1 + y\boldsymbol{\gamma}_2 + z\boldsymbol{\gamma}_3 = w\boldsymbol{\gamma}_0 + \mathbf{p}_{\mathcal{S}}, \tag{5}$$

and four-dimensional space-time velocity

$$\boldsymbol{v} = \boldsymbol{v}_{\mathcal{M}} = c\boldsymbol{\gamma}_0 + v_x\boldsymbol{\gamma}_1 + v_y\boldsymbol{\gamma}_2 + v_z\boldsymbol{\gamma}_3 = c\boldsymbol{\gamma}_0 + \mathbf{v}_{\mathcal{S}}, \tag{6}$$

with 4D STA vectors in ***bold italic***, and 3D SA spatial $v_x\boldsymbol{\gamma}_1 + v_y\boldsymbol{\gamma}_2 + v_z\boldsymbol{\gamma}_3$ vectors $\mathbf{v} = \mathbf{v}_{\mathcal{S}}$ in **bold**.

The untranslated (at origin) *observer* worldline, in the rest frame of the observer[3], is

$$\boldsymbol{o}t = ct\boldsymbol{\gamma}_0 \tag{7}$$

with *proper time* (coordinate time) t. See also, the CSTA line entity $\boldsymbol{L}_C$ of (34).

STA versors include *rotor* ($(\hat{\mathbf{n}}^*_{\mathcal{S}})^2 = -1$, see (16))

$$R = \exp\left(\theta\hat{\mathbf{n}}^*_{\mathcal{S}}/2\right) = e^{\frac{1}{2}\theta\hat{\mathbf{n}}^*_{\mathcal{S}}} = \cos(\theta/2) + \sin(\theta/2)\,\hat{\mathbf{n}}^*_{\mathcal{S}}, \tag{8}$$

and *hyperbolic rotor (boost)* ($(\hat{\mathbf{v}}\boldsymbol{\gamma}_0)^2 = +1$)

$$B = (\gamma\boldsymbol{v}/\boldsymbol{o})^{\frac{1}{2}} = \exp(\varphi\hat{\mathbf{v}}\boldsymbol{\gamma}_0/2) = \cosh(\varphi/2) + \sinh(\varphi/2)\,\hat{\mathbf{v}}\boldsymbol{\gamma}_0, \tag{9}$$

where three-dimensional *spatial speed* v in physics is

$$v = \beta c = \|\mathbf{v}\| = \sqrt{v_x^2 + v_y^2 + v_z^2}, \tag{10}$$

[3]Note that in special relativity an observer rest frame means, the observer continues to rest at the origin of the frame.

light speed is c, *natural speed* β is[4]

$$0 \leq \beta = v/c \quad \leq \quad 1, \tag{11}$$

space-time velocity is by (6)

$$\boldsymbol{v} \quad = \quad c\boldsymbol{\gamma}_0 + \beta c\hat{\mathbf{v}}, \tag{12}$$

and *rapidity* (hyperbolic angle in (9)) is

$$\varphi \quad = \quad \operatorname{atanh}(\beta). \tag{13}$$

The *Lorentz factor*

$$\gamma = \mathrm{d}t/\mathrm{d}\tau \quad = \quad 1/\sqrt{1-\beta^2} = 1/d, \tag{14}$$

is related to special relativity *length contraction* (from L_0 to L) as[5]

$$L \quad = \quad \sqrt{1-\beta^2}L_0 = L_0/\gamma = dL_0, \tag{15}$$

and τ is *proper time* of the observable with space-time velocity $\boldsymbol{v}$.

The SA *spatial dualization*

$$\hat{\mathbf{n}}^*_{\mathcal{S}} \quad = \quad -\hat{\mathbf{n}}_{\mathcal{S}}\mathbf{I}_{\mathcal{S}}^{-1} \tag{16}$$

of an SA spatial *unit vector* rotation axis $\hat{\mathbf{n}}$ is the rotation plane bivector $\hat{\mathbf{n}}^*$ that is isomorphic to a pure unit quaternion, where $(\hat{\mathbf{n}}^*)^2 = -1$.

STA *space-time dualization* is

$$\boldsymbol{v}^*_{\mathcal{M}} \quad = \quad \boldsymbol{v}_{\mathcal{M}}\mathbf{I}_{\mathcal{M}}^{-1}. \tag{17}$$

A versor operates on a vector using the *versor "sandwich" operation*

$$\boldsymbol{v}' \quad = \quad B\boldsymbol{v}\widetilde{B}, \tag{18}$$

where the *reverse* is

$$\widetilde{B} = e^{\frac{1}{2}\varphi\gamma_0\hat{\mathbf{v}}} = e^{-\frac{1}{2}\varphi\hat{\mathbf{v}}\gamma_0} = B^{-1}, \tag{19}$$

which reverses the product of all vectors in any multivector (e.g., $\widetilde{\mathbf{I}_{\mathcal{M}}} = \boldsymbol{\gamma}_3\boldsymbol{\gamma}_2\boldsymbol{\gamma}_1\boldsymbol{\gamma}_0$).

The *conjugate* $A^\dagger$ [14] of any STA multivector is (a composition of reversion with sandwiching between $\boldsymbol{\gamma}_0$ factors)

$$A^\dagger \quad = \quad \boldsymbol{\gamma}_0\widetilde{A}\boldsymbol{\gamma}_0, \tag{20}$$

and for a vector $\boldsymbol{v} = c\boldsymbol{\gamma}_0 + \beta c\hat{\mathbf{v}}$, its conjugate is (changing the sign of the spatial component)

$$\boldsymbol{v}^\dagger \quad = \quad \boldsymbol{\gamma}_0\boldsymbol{v}\boldsymbol{\gamma}_0 = c\boldsymbol{\gamma}_0 - \beta c\hat{\mathbf{v}}. \tag{21}$$

[4] A possible negative sign of β is taken care of by the direction of $\hat{\mathbf{v}}$, the unit vector in the direction of $\mathbf{v}_{\mathcal{S}}$.

[5] Note that in computer graphics applications we purposely go beyond physical limitations, and permit a complex value for β, in order to achieve length contractions with $d > 1$.

Notation of Conformal STA

The basis of CSTA $\mathcal{G}_{2,4}\ \mathcal{C} \cong$ CSTA1 $\mathcal{G}_{2,4}\ \mathcal{C}^1$, index $\gamma = 1$, is

$$\{\boldsymbol{\gamma}_0, \boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\gamma}_3, \mathbf{e}_+, \mathbf{e}_-\} \quad \cong \quad \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5, \mathbf{e}_6\}, \tag{22}$$

and for the second copy CSTA2 $\mathcal{G}_{2,4}\ \mathcal{C}^2$, index $\gamma = 2$,

$$\{\boldsymbol{\gamma}_0, \boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\gamma}_3, \mathbf{e}_+, \mathbf{e}_-\} \quad \cong \quad \{\mathbf{e}_7, \mathbf{e}_8, \mathbf{e}_9, \mathbf{e}_{10}, \mathbf{e}_{11}, \mathbf{e}_{12}\}. \tag{23}$$

The six-dimensional CSTA unit pseudoscalar is

$$\mathbf{I}_{\mathcal{C}} \quad = \quad \boldsymbol{\gamma}_0\boldsymbol{\gamma}_1\boldsymbol{\gamma}_2\boldsymbol{\gamma}_3\mathbf{e}_+\mathbf{e}_-. \tag{24}$$

CSTA defines three *geometric inner product null space* (GIPNS) [14] 1-blade *entities*.

The CSTA GIPNS 1-blade *null hypercone* entity $\boldsymbol{K}_{\mathcal{C}}$ (growing sphere in time from a point), equal to the null *point* embedding $\boldsymbol{P}_{\mathcal{C}}$, is

$$\boldsymbol{K}_{\mathcal{C}} = \boldsymbol{P}_{\mathcal{C}} = \mathcal{C}(\boldsymbol{p}_{\mathcal{M}}) \quad = \quad \boldsymbol{p}_{\mathcal{M}} + (1/2)\boldsymbol{p}_{\mathcal{M}}^2\mathbf{e}_{\infty\gamma} + \mathbf{e}_{o\gamma}, \qquad \boldsymbol{P}_{\mathcal{C}}^2 = 0, \tag{25}$$

centered at $\boldsymbol{p}_{\mathcal{M}}$ with null *infinity point*

$$\mathbf{e}_{\infty\gamma} \quad = \quad \mathbf{e}_+ + \mathbf{e}_-, \qquad \mathbf{e}_{\infty\gamma}^2 = 0, \tag{26}$$

and null *origin point*

$$\mathbf{e}_{o\gamma} \quad = \quad (\mathbf{e}_- - \mathbf{e}_+)/2, \qquad \mathbf{e}_{o\gamma}^2 = 0, \qquad \mathbf{e}_{o\gamma} \cdot \mathbf{e}_{\infty\gamma} = -1, \qquad \mathbf{e}_+\mathbf{e}_- = \mathbf{e}_{o\gamma} \wedge \mathbf{e}_{\infty\gamma}. \tag{27}$$

The CSTA GIPNS 1-blade *hyperplane* (3D subspace) entity $\boldsymbol{E}_{\mathcal{C}}$ is

$$\boldsymbol{E}_{\mathcal{C}} \quad = \quad \boldsymbol{n}_{\mathcal{M}} + (\boldsymbol{p}_{\mathcal{M}} \cdot \boldsymbol{n}_{\mathcal{M}})\mathbf{e}_{\infty\gamma}, \tag{28}$$

normal to $\boldsymbol{n}_{\mathcal{M}}$, passing through space-time position $\boldsymbol{p}_{\mathcal{M}}$.

The CSTA GIPNS 1-blade *hyperpseudosphere* entity $\boldsymbol{\Sigma}_{\mathcal{C}}$ (growing pseudosphere) is

$$\boldsymbol{\Sigma}_{\mathcal{C}} \quad = \quad \boldsymbol{P}_{\mathcal{C}} + (1/2)r_0^2\mathbf{e}_{\infty\gamma}, \tag{29}$$

centered at $\boldsymbol{P}_{\mathcal{C}}$ with initial radius r_0, which can be real or imaginary (n.b., for $r_0 = 0$, $\boldsymbol{\Sigma}_{\mathcal{C}} = \boldsymbol{P}_{\mathcal{C}}$). The outer product of two to six of the above CSTA GIPNS 1-blade entities (null-hypercones $\boldsymbol{K}_{\mathcal{C}}$, hyperplanes $\boldsymbol{E}_{\mathcal{C}}$, hyperpseudospheres $\boldsymbol{\Sigma}_{\mathcal{C}}$) forms, by intersection, more CSTA GIPNS entities of higher grades.

CSTA *dualization* of a CSTA GIPNS k-vector entity $\boldsymbol{X}_{\mathcal{C}}$ gives its dual CSTA *geometric outer product null space* (GOPNS) [14] $(6-k)$-vector entity

$$\boldsymbol{X}_{\mathcal{C}}^* \quad = \quad \boldsymbol{X}_{\mathcal{C}}\mathbf{I}_{\mathcal{C}}^{-1}. \tag{30}$$

A CSTA point $\boldsymbol{P}_{\mathcal{C}}$ is on CSTA GIPNS entity $\boldsymbol{X}_{\mathcal{C}}$ iff

$$\boldsymbol{P}_{\mathcal{C}} \cdot \boldsymbol{X}_{\mathcal{C}} \quad = \quad 0. \tag{31}$$

A CSTA point $\boldsymbol{P}_{\mathcal{C}}$ is on the corresponding dual CSTA GOPNS entity $\boldsymbol{X}_{\mathcal{C}}^*$ iff

$$\boldsymbol{P}_{\mathcal{C}} \wedge \boldsymbol{X}_{\mathcal{C}}^* \quad = \quad 0. \tag{32}$$

The outer product of up to six well-chosen CSTA points produces various CSTA GOPNS $(1\ldots 6)$-blade space-time surface entities $\boldsymbol{X}_C^* = \bigwedge \boldsymbol{P}_{C_i}$ that the points span as surface points. The CSTA GOPNS null 1-blade *point* (embedding) $\boldsymbol{P}_C$ equals the CSTA GIPNS null 1-blade *hypercone* $\boldsymbol{P}_C = \boldsymbol{K}_C$.

The CSTA GIPNS 2-blade *plane* entity

$$\boldsymbol{\Pi}_C = \boldsymbol{D}_{\mathcal{M}}^* - (\boldsymbol{p}_{\mathcal{M}} \cdot \boldsymbol{D}_{\mathcal{M}}^*)\mathbf{e}_{\infty\gamma}, \tag{33}$$

in direction of unit bivector $\boldsymbol{D}_{\mathcal{M}}$ through $\boldsymbol{p}_{\mathcal{M}}$, is the intersection (wedge) of two space-time hyperplanes (28).

The CSTA GIPNS 3-blade *line* entity

$$\boldsymbol{L}_C = \boldsymbol{d}_{\mathcal{M}}^* + (\boldsymbol{p}_{\mathcal{M}} \cdot \boldsymbol{d}_{\mathcal{M}}^*) \wedge \mathbf{e}_{\infty\gamma}, \tag{34}$$

in the direction $\boldsymbol{d}_{\mathcal{M}}$ through $\boldsymbol{p}_{\mathcal{M}} = p_w\boldsymbol{\gamma}_0 + \mathbf{p}_{\mathcal{S}}$, is the intersection of three hyperplanes (28) and represents the worldline of an observable with STA velocity $\boldsymbol{v} = \boldsymbol{d} = c\boldsymbol{\gamma}_0 + \beta c\hat{\mathbf{v}}$ and initial spatial CSTA GOPNS 2-blade *flat point*[6] [4] position

$$C(\mathbf{p}_0) \wedge \mathbf{e}_{\infty\gamma} \simeq (\boldsymbol{\gamma}_0 \wedge \boldsymbol{L}_C)\mathbf{I}_C^{-1} \tag{35}$$

at $t = 0$ ($\boldsymbol{\gamma}_0$ is $t = 0$ hyperplane $\boldsymbol{E}_C$). The boost, and the other CSTA versors, can operate on the line $\boldsymbol{L}_C$ to implement space-time transformations. A normed[7] unit plane $\hat{\boldsymbol{\Pi}}_C$ and unit line $\hat{\boldsymbol{L}}_C$ have unit direction $\hat{\boldsymbol{D}}_{\mathcal{M}}^*$ and $\hat{\boldsymbol{d}}_{\mathcal{M}}^*$, respectively.

CSTA inherits the STA 2-versor *rotor*

$$R_C = R = \exp\left(\theta\hat{\mathbf{n}}_{\mathcal{S}}^*/2\right) = \cos(\theta/2) + \sin(\theta/2)\,\hat{\mathbf{n}}_{\mathcal{S}}^*, \tag{36}$$

and STA 2-versor *hyperbolic rotor (boost)*

$$B_C = B = \exp(\varphi\hat{\mathbf{v}}\boldsymbol{\gamma}_0/2) = \cosh(\varphi/2) + \sinh(\varphi/2)\,\hat{\mathbf{v}}\boldsymbol{\gamma}_0. \tag{37}$$

CSTA introduces the CSTA 2-versor *translator*

$$T_C = \exp\left(\mathbf{e}_{\infty\gamma}\boldsymbol{d}_{\mathcal{M}}/2\right) = 1 + (1/2)\mathbf{e}_{\infty\gamma}\boldsymbol{d}_{\mathcal{M}}, \tag{38}$$

which translates by $\boldsymbol{d}_{\mathcal{M}}$. As versor compositions, CSTA also introduces the following three translated 2-versors. The CSTA 2-versor *translated-rotor* is

$$L_C = T_C R_C T_C^{-1} = \exp\left(-\theta\boldsymbol{\gamma}_0 \cdot \hat{\boldsymbol{L}}_C/2\right) = \cos(\theta/2) + \sin(\theta/2)\,\hat{\mathbf{L}}_{C\mathcal{S}}, \tag{39}$$

which rotates by angle θ anticlockwise (by right-hand rule) around the spatial CSA line $\hat{\mathbf{L}}_{C\mathcal{S}}$ through point $\boldsymbol{d}_{\mathcal{M}} = \mathbf{d}_{\mathcal{S}}$ in the rotor axis direction $\hat{\mathbf{n}}_{\mathcal{S}}$. The CSTA 2-versor *translated-boost* is

$$B_C^{\boldsymbol{d}} = \exp\left(\varphi(\hat{\mathbf{v}}\boldsymbol{\gamma}_0 - (\boldsymbol{d}_{\mathcal{M}} \cdot (\hat{\mathbf{v}}\boldsymbol{\gamma}_0))\mathbf{e}_{\infty\gamma})/2\right) = \exp\left(\varphi\hat{\boldsymbol{\Pi}}_C/2\right) = \cosh(\varphi/2) + \sinh(\varphi/2)\,\hat{\boldsymbol{\Pi}}_C, \tag{40}$$

centered on $\boldsymbol{d} = \boldsymbol{d}_{\mathcal{M}}$ and with direction $\boldsymbol{D}_{\mathcal{M}} = (\hat{\mathbf{v}}\boldsymbol{\gamma}_0)\mathbf{I}_{\mathcal{M}}$. The CSTA 2-versor *translated-isotropic dilator* is

$$D_C = \exp\left(\ln(d)\hat{\boldsymbol{P}}_C \wedge \mathbf{e}_{\infty\gamma}/2\right) = \cosh(\ln(d)/2) + \sinh(\ln(d)/2)\,\hat{\boldsymbol{P}}_C \wedge \mathbf{e}_{\infty\gamma} \tag{41}$$

for isotropic dilation by factor d relative to normalized center point $\hat{\boldsymbol{P}}_C$, i.e. $\hat{\boldsymbol{P}}_C \cdot \mathbf{e}_{\infty\gamma} = -1$. By *versor outermorphism* [14], all CSTA versors are valid on all CSTA GIPNS and dual CSTA GOPNS entities.

The *projection* (inverse of embedding) of a point $\boldsymbol{P}_C = C(\boldsymbol{p}_{\mathcal{M}})$ to its embedded STA vector is

$$\boldsymbol{p}_{\mathcal{M}} = C^{-1}(\boldsymbol{P}_C) = (\hat{\boldsymbol{P}}_C \cdot \mathbf{I}_{\mathcal{M}})\mathbf{I}_{\mathcal{M}}^{-1} = (\hat{\boldsymbol{P}}_C \wedge \mathbf{e}_+ \wedge \mathbf{e}_-)(\mathbf{e}_+ \wedge \mathbf{e}_-), \tag{42}$$

which is geometrically *projection* onto $\mathbf{I}_{\mathcal{M}}$ or *rejection* from $\mathbf{e}_+\mathbf{e}_- = \mathbf{e}_+ \wedge \mathbf{e}_- = \mathbf{e}_{\infty\gamma} \wedge \mathbf{e}_{o\gamma}$.

[6] *Flat point* $\boldsymbol{P}_C \wedge \mathbf{e}_{\infty\gamma}$ in [4] is called *homogeneous point* $\boldsymbol{p}_{\mathcal{M}} \wedge \mathbf{e}_{\infty\gamma} + \mathbf{e}_{o\gamma} \wedge \mathbf{e}_{\infty\gamma}$ in [14].

[7] This means, that the homongeneous factors of $\boldsymbol{\Pi}_C$ and $\boldsymbol{L}_C$ are chosen, such that the Euclidean carrier blades $\boldsymbol{D}_{\mathcal{M}}^*$ and $\boldsymbol{d}_{\mathcal{M}}^*$ have magnitude one.

Construction of Double CSTA

In double conformal space-time algebra (DCSTA), CSTA1 and CSTA2 are orthogonal subalgebras and their geometric or outer product forms DCSTA as a doubling extension. Any CSTA1 entity or versor A_{C^1} and its double A_{C^2} in CSTA2 (with the same scalar basis blade coefficients) can be multiplied to form the corresponding DCSTA entity or versor $A_{\mathcal{D}} = A_{C^1}A_{C^2} = A_{C^1} \wedge A_{C^2}$. By versor outermorphism, the DCSTA doubled versors operate correctly on all DCSTA entities. The DCSTA point $\boldsymbol{P}_{\mathcal{D}} = \mathcal{D}(\boldsymbol{p}_{\mathcal{M}}) = \boldsymbol{P}_{C^1}\boldsymbol{P}_{C^2}$ is a quadratic (squared, double) form of the CSTA embedding, and, as we will show, from it we can extract values that construct polynomials which in turn represent Darboux cyclides as DCSTA entities.

Table 1. DCSTA bivector *extraction elements* T_s.

$T_x = (\mathbf{e}_{\infty 2} \wedge \mathbf{e}_2 + \mathbf{e}_8 \wedge \mathbf{e}_{\infty 1})/2$	$T_y = (\mathbf{e}_{\infty 2} \wedge \mathbf{e}_3 + \mathbf{e}_9 \wedge \mathbf{e}_{\infty 1})/2$	$T_z = (\mathbf{e}_{\infty 2} \wedge \mathbf{e}_4 + \mathbf{e}_{10} \wedge \mathbf{e}_{\infty 1})/2$
$T_{x^2} = \mathbf{e}_8 \wedge \mathbf{e}_2$	$T_{y^2} = \mathbf{e}_9 \wedge \mathbf{e}_3$	$T_{z^2} = \mathbf{e}_{10} \wedge \mathbf{e}_4$
$T_{xy} = (\mathbf{e}_9 \wedge \mathbf{e}_2 + \mathbf{e}_8 \wedge \mathbf{e}_3)/2$	$T_{yz} = (\mathbf{e}_{10} \wedge \mathbf{e}_3 + \mathbf{e}_9 \wedge \mathbf{e}_4)/2$	$T_{zx} = (\mathbf{e}_8 \wedge \mathbf{e}_4 + \mathbf{e}_{10} \wedge \mathbf{e}_2)/2$
$T_{x\boldsymbol{t}^2_{\mathcal{M}}} = \mathbf{e}_{o2} \wedge \mathbf{e}_2 + \mathbf{e}_8 \wedge \mathbf{e}_{o1}$	$T_{y\boldsymbol{t}^2_{\mathcal{M}}} = \mathbf{e}_{o2} \wedge \mathbf{e}_3 + \mathbf{e}_9 \wedge \mathbf{e}_{o1}$	$T_{z\boldsymbol{t}^2_{\mathcal{M}}} = \mathbf{e}_{o2} \wedge \mathbf{e}_4 + \mathbf{e}_{10} \wedge \mathbf{e}_{o1}$
$T_1 = -\mathbf{e}_\infty = -\mathbf{e}_{\infty 1} \wedge \mathbf{e}_{\infty 2}$	$T_{\boldsymbol{t}^2_{\mathcal{M}}} = \mathbf{e}_{o2} \wedge \mathbf{e}_{\infty 1} + \mathbf{e}_{\infty 2} \wedge \mathbf{e}_{o1}$	$T_{\boldsymbol{t}^4_{\mathcal{M}}} = -4\mathbf{e}_o = -4\mathbf{e}_{o1} \wedge \mathbf{e}_{o2}$
$T_w = (\mathbf{e}_1 \wedge \mathbf{e}_{\infty 2} + \mathbf{e}_{\infty 1} \wedge \mathbf{e}_7)/2$	$T_{w^2} = \mathbf{e}_7 \wedge \mathbf{e}_1$	$T_{w\boldsymbol{t}^2_{\mathcal{M}}} = \mathbf{e}_1 \wedge \mathbf{e}_{o2} + \mathbf{e}_{o1} \wedge \mathbf{e}_7$
$T_{wx} = (\mathbf{e}_1 \wedge \mathbf{e}_8 + \mathbf{e}_2 \wedge \mathbf{e}_7)/2$	$T_{wy} = (\mathbf{e}_1 \wedge \mathbf{e}_9 + \mathbf{e}_3 \wedge \mathbf{e}_7)/2$	$T_{wz} = (\mathbf{e}_1 \wedge \mathbf{e}_{10} + \mathbf{e}_4 \wedge \mathbf{e}_7)/2$
$T_t = T_w/c$	$T_{t^2} = T_{w^2}/c^2$	$T_{t\boldsymbol{t}^2_{\mathcal{M}}} = T_{w\boldsymbol{t}^2_{\mathcal{M}}}/c$
$T_{tx} = T_{wx}/c$	$T_{ty} = T_{wy}/c$	$T_{tz} = T_{wz}/c$

The 27 inner product bivector *extraction operators* T_s for extraction of scalar value s (indicated by the indices $x, \ldots, tz$) from DCSTA test point $\boldsymbol{T}_{\mathcal{D}} = \mathcal{D}(\boldsymbol{t}_{\mathcal{M}})$ in the form of

$$s = T_s \cdot \boldsymbol{T}_{\mathcal{D}} \tag{43}$$

are all given in Table 1 (n.b., $t \neq \boldsymbol{t}$: t is the time coordinate, and $\boldsymbol{t} = \boldsymbol{t}_{\mathcal{M}}$ is the STA test point). The DCSTA GIPNS bivector entities for quadrics and cyclides can be directly written as linear combinations of extraction operators T_s. For example, an ellipsoid (centered at the origin, aligned along the SA axes $\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\gamma}_3$) is

$$\mathbf{E}_{\mathcal{D}} = T_{x^2}/a^2 + T_{y^2}/b^2 + T_{z^2}/c^2 - T_1, \tag{44}$$

and a general point $\boldsymbol{P}_{\mathcal{D}}$ is on it iff $\boldsymbol{P}_{\mathcal{D}} \cdot \mathbf{E}_{\mathcal{D}} = 0$. The DCSTA dualization of the bivector $\mathbf{E}_{\mathcal{D}}$,

$$\mathbf{E}^*_{\mathcal{D}} = \mathbf{E}_{\mathcal{D}}\mathbf{I}^{-1}_{\mathcal{D}} = \mathbf{E}_{\mathcal{D}}(\mathbf{I}_{C^1}\mathbf{I}_{C^2})^{-1}, \tag{45}$$

is a valid GOPNS 10-vector entity where $\boldsymbol{P}_{\mathcal{D}}$ is on it iff $\boldsymbol{P}_{\mathcal{D}} \wedge \mathbf{E}^*_{\mathcal{D}} = 0$. If time is always fixed as $t = 0$, then the DCSTA GIPNS bivector entities formed from the T_s correspond to entities of $\mathcal{G}_{8,2}$ DCGA [6], up to some sign differences in some scalar expressions, due to the different choice of signature. In DCSTA, the quadrics gain the ability to be boosted into a velocity using the DCSTA 4-versor boost operator $B_{\mathcal{D}} = B_{C^1}B_{C^2}$. Furthermore, inversion of quadrics in pseudospheres (circular hyperboloids) is also possible. The boost natural speed for a length contraction factor d is by (14) $\beta = \sqrt{1-d^2}$. Boost of a quadric by an imaginary β dilates by $d > 1$ and then the result can be projected to the spatial subalgebra $\mathcal{G}_{2,8}$ DCSA to discard time components and achieve directed scaling in the direction of the boost.

Figure 2 visualizes [15] a DCSTA GIPNS bivector *spherical ellipsoid* $\mathbf{E}$ dilated in situ by factor $d = 3$ in the direction $\hat{\mathbf{v}}$ as $\mathbf{E}'$ using a *translated-boost* operator $B^{\boldsymbol{d}}_{\mathcal{D}}$ centered on the position $\boldsymbol{p} = \boldsymbol{d}$ of $\mathbf{E}$ and $\mathbf{E}'$. The $\mathcal{G}_{2,8}$ DCSA projection is

$$\mathcal{P}(A) = (A \cdot \mathbf{I}_{\mathcal{DS}})\mathbf{I}^{-1}_{\mathcal{DS}}, \tag{46}$$

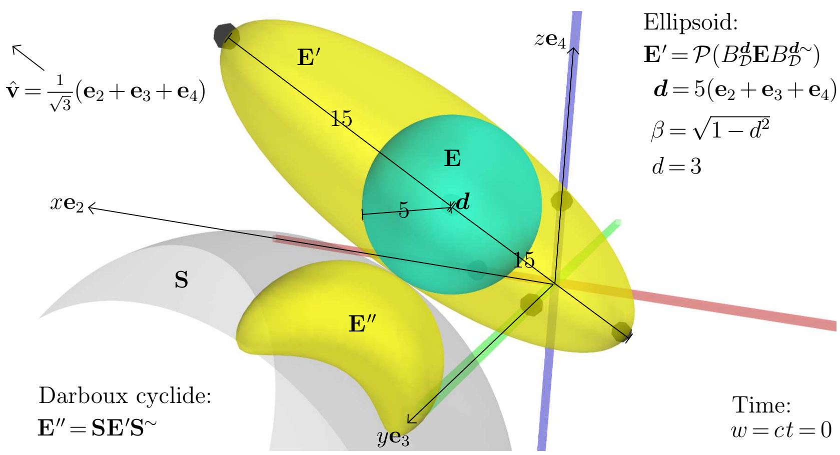

Figure 2. Spherical ellipsoid $\mathbf{E}(r = 5)$ dilated by factor $d = 3$ in direction $\hat{\mathbf{v}}$ as ellipsoid $\mathbf{E}'$ and then reflected in sphere $\mathbf{S}$ as Darboux cyclide $\mathbf{E}''$.

where the DCSA unit pseudoscalar is

$$\mathbf{I}_{\mathcal{DS}} \quad = \quad \mathbf{I}_{\mathcal{S}^1}\mathbf{e}_5\mathbf{e}_6\mathbf{I}_{\mathcal{S}^2}\mathbf{e}_{11}\mathbf{e}_{12}. \tag{47}$$

$\mathbf{E}'$ is reflected in a DCSTA GIPNS 2-blade *(hyperpseudo)sphere* $\mathbf{S} = \mathbf{\Sigma}(t = 0, r_0 = 15) = \mathbf{\Sigma}_{C^1}\mathbf{\Sigma}_{C^2}$ as $\mathbf{E}''$, which is a Darboux cyclide. All are at time $t = 0$.

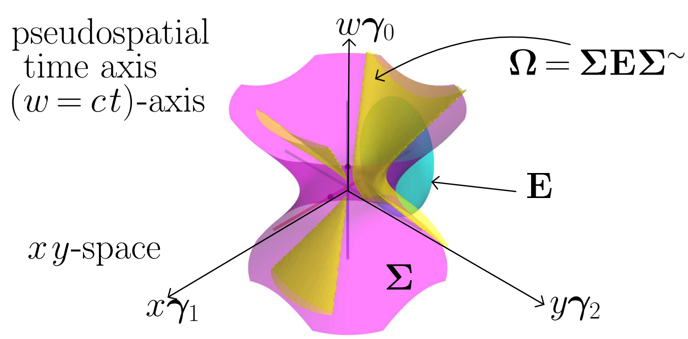

Figure 3. Ellipsoid $\mathbf{E}$ reflected in circular hyperboloid $\mathbf{\Sigma}$, $\mathbf{\Omega} = \mathbf{\Sigma}\mathbf{E}\mathbf{\Sigma}^{\sim}$.

Figure 3 shows a space-time "pseudocyclide" $\mathbf{\Omega}$, which is the reflection of the quadric $\mathbf{E}$ in a space-time hyperboloid $\mathbf{\Sigma}(z = 0)$. For graphing, the time w-axis was mapped to a vertical z-axis and the $\boldsymbol{\gamma}_3$-axis was suppressed. Figure 4 shows another space-time "pseudocyclide" $\mathbf{\Omega}$ that is the reflection of the pseudoquadric ellipsoid

$$\mathbf{E}_{\mathcal{D}}^{+} \quad = \quad T_{x^2}/a^2 + T_{y^2}/b^2 + T_{w^2}/c^2 - T_1 \tag{48}$$

in a space-time hyperboloid.

The DCSTA *differential elements* are

$$D_w = 2T_wT_{w^2}^{-1}, \qquad D_x = 2T_xT_{x^2}^{-1}, \qquad D_y = 2T_yT_{y^2}^{-1}, \qquad D_z = 2T_zT_{z^2}^{-1}, \qquad D_t = 2T_tT_{t^2}^{-1}, \tag{49}$$

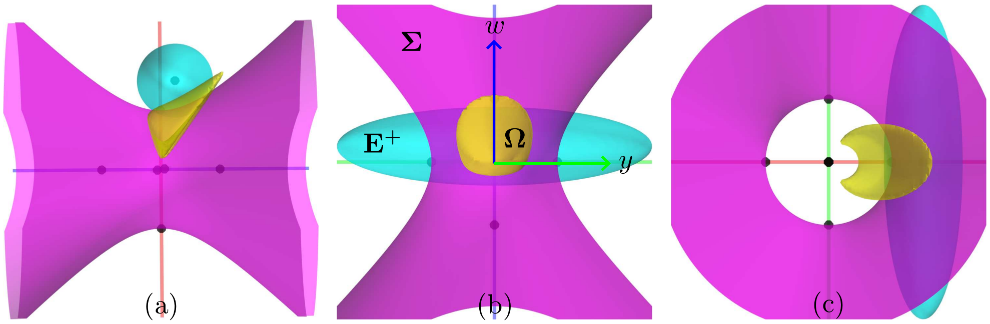

Figure 4. Pseudoquadric ellipsoid $\mathbf{E}^{+}$ reflected in circular hyperboloid $\mathbf{\Sigma}$ as $\mathbf{\Omega} = \mathbf{\Sigma}\mathbf{E}^{+}\mathbf{\Sigma}^{\sim}$.

and the commutator product $\times$ of multivectors A and B is

$$A \times B \quad = \quad (AB - BA)/2 = -B \times A. \tag{50}$$

Using the commutator product, the DCSTA differential elements can be used as *differential operators* on any bivector surface entity $\boldsymbol{E}$ that is formed as a linear combination of the DCSTA extraction elements T_s. For example, the time t derivative of $\boldsymbol{E}$ is

$$\dot{\boldsymbol{E}} \quad = \quad \partial_t \boldsymbol{E} = \frac{\partial \boldsymbol{E}}{\partial t} = D_t \times \boldsymbol{E}. \tag{51}$$

For direction $\boldsymbol{n}$ with unit magnitude $\boldsymbol{n} \cdot \boldsymbol{n}^{\dagger} = 1$, the $\boldsymbol{n}$-directional derivative operator is

$$\partial_{\boldsymbol{n}} = \frac{\partial}{\partial \boldsymbol{n}} = D_{\boldsymbol{n}} \times \quad = \quad (n_w D_w + n_x D_x + n_y D_y + n_z D_z)\times \tag{52}$$

and the $\boldsymbol{n}$-directional derivative of any bivector entity $\boldsymbol{E}$ is

$$\partial_{\boldsymbol{n}} \boldsymbol{E} \quad = \quad D_{\boldsymbol{n}} \times \boldsymbol{E}. \tag{53}$$

The entity $\boldsymbol{E}$ represents an implicit surface function $F(w, x, y, z)$, and its $\boldsymbol{n}$-directional derivative $\partial_{\boldsymbol{n}} \boldsymbol{E}$ represents the derivative implicit surface function $\partial_{\boldsymbol{n}} F$. Mixed partial derivatives are obtained by taking successive derivatives in any order.

Conclusion

The DCSTA $\mathcal{G}_{4,8}$ extends Double Conformal Space Algebra (DCSA) $\mathcal{G}_{2,8}$, which is different in space signature from the DCGA $\mathcal{G}_{8,2}$ of [10], into a high-dimensional 12D embedding of Space-Time Algebra $\mathcal{G}_{1,3}$ that has quadric surface entities with a complete set of space-time transformation operations as versors and projections.

DCSTA is an algebra for computing with quadrics and their cyclide inversions in space-time. For applications or testing, DCSTA $\mathcal{G}_{4,8}$ can be computed using various software packages, such as *Gaalop* [16] for optimization, the *Clifford Multivector Toolbox (for MATLAB)* [17], the Clifford package for MAPLE [18], or the free symbolic computer algebra system *SymPy* [19] with the *$\mathcal{G}$Algebra* [20] module.

The scheme can be generalized to any *non-Euclidean geometry*. It is possible to form higher order *Triple, Quadruple, etc. CGAs*, for handling higher order algebraic curves and surfaces. There is the potential of applications in *discretized geometry*, *topological computations*, and to *quadric and cyclidic surface patches* bounded by bi-CGA entities (via intersection), e.g. piece of cylinder bounded by intersection with pair of planes, etc.

ACKNOWLEDGMENTS

E.H. wishes to thank God: *Soli Deo Gloria*, to thank his family and H. Suzuki for their kind support, R.E. for excellent collaboration, and the ICNPAA session organizers Isabel Cacao and Joao Morais. In applications of this research, please respect the Creative Peace License [21].

References

[1] R. B. Easter, "Double Conformal Space-Time Algebra," (2016), preprint: vixra.org/abs/1602.0114.
[2] D. Hestenes and G. Sobczyk, *Clifford Algebra to Geometric Calculus, A Unified Language for Mathematics and Physics*, Fundamental Theories of Physics, Vol. 5 (Dordrecht-Boston-Lancaster: D. Reidel Publishing Company, a Member of the Kluwer Academic Publishers Group, 1984).
[3] P. Lounesto, *Clifford Algebras and Spinors*, 2nd ed. (Cambridge: Cambridge University Press, 2001), pp. ix + 338.
[4] L. Dorst, D. Fontijne, and S. Mann, *Geometric Algebra for Computer Science (Revised Edition): An Object-Oriented Approach to Geometry*, The Morgan Kaufmann Series in Computer Graphics (Elsevier Science, 2009).
[5] E. Hitzer, SICE Journal of Control, Measurement, and System Integration **51**, 338–350 (2012), Preprint: arxiv.org/abs/1306.1660v1.
[6] R. B. Easter, "G8,2 Geometric Algebra, DCGA," (2015), preprint: vixra.org/abs/1508.0086.
[7] R. B. Easter, "Conic and Cyclidic Sections in the G8,2 Geometric Algebra, DCGA," (2015), preprint: vixra.org/abs/1511.0182.
[8] R. B. Easter, "Differential Operators in the G8,2 Geometric Algebra, DCGA," (2015), preprint: vixra.org/abs/1512.0303.
[9] E. Hitzer, K. Tachibana, S. Buchholz, and I. Yu, Advances in Applied Clifford Algebras **19**, 339–364 (2009).
[10] R. B. Easter and E. Hitzer, "Double Conformal Geometric Algebra for Quadrics and Darboux Cyclides," in *Proceedings of the 33rd Computer Graphics International Conference, Heraklion, Greece*, CGI '16 (ACM, New York, 2016), pp. 93–96.
[11] D. Hestenes, *Space-Time Algebra*, Second ed. (Springer, 2015).
[12] E. Hitzer, "Relativistic Physics as Application of Geometric Algebra," in *Proceedings of the International Conference on Relativity 2005 (ICR2005), University of Amravati, India, January 2005*, edited by K. Adhav (Department of Mathematics, Amravati University, Amravati 444602, India, 2005), pp. 71–90, Preprint: arxiv.org/abs/1306.0121.
[13] C. Doran and A. Lasenby, *Geometric Algebra for Physicists*, paperback reprint of the 2003 original ed. (Cambridge: Cambridge University Press, 2007), pp. xiv + 578.
[14] C. Perwass, *Geometric Algebra with Applications in Engineering*, Geometry and Computing, Vol. 4 (Springer, 2009) Habilitation thesis, Christian-Albrechts-Universität zu Kiel.
[15] P. Ramachandran and G. Varoquaux, Computing in Science & Engineering **13**, 40–51 (2011).
[16] D. Hildenbrand, *Foundations of Geometric Algebra Computing*, Geometry and Computing, Vol. 8 (Springer, 2013), pp. xxvii + 196.
[17] S. J. Sangwine and E. Hitzer, Advances in Applied Clifford Algebras 1–20 (2016).
[18] R. Ablamowicz and B. Fauser, "CLIFFORD - A Maple Package for Clifford Algebra Computations," (2015), `http://math.tntech.edu/rafal/` [Online; accessed 05 Sep. 2016].
[19] SymPy Development Team, *SymPy: Python library for symbolic mathematics* (2016).
[20] A. Bromborsky, *Geometric Algebra Module for Sympy* (2016).
[21] E. Hitzer, "Creative Peace License," (2011), `https://gaupdate.wordpress.com/2011/12/14/the-creative-peace-license-14-dec-2011/` [Online; accessed 05 Sep. 2016].